8.851 Course 2013

Soft-Collinear Effective Theory (SCET)

Lecture Notes

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Formalism & Applications

[more Refs online as we go along]

Partial Topic List

(i) Intro, Degrees of Freedom, Scales, expansion of spinors, propagators, power counting see ②, ③

(ii) Construction of $\mathcal{L}_{SCET}$, Currents Multipole Expn, Labels, Zero-bin, I.R. divergences see ②, ③, ⑩

(iii) $SCET_I$, Gauge Symmetry ③, ④, ⑥ Reparameterization Invariance

(iv) Ultrasoft-Collin. Factorization Hard-Collinear Factorization Matchg & Running for Hard Fn's ④, ①, ②, ③

(v) DIS, how SCET p.c. includes twist expansion as special case renormalization with convolutions

(vi) $SCET_{II}$ Soft-Collinear Interactions use of auxillary Lagrangians Power Counting formula, Rapidity Divergences ④, ⑦, ⑩, ⑤ ⑫

(vii) Power Corrections, including $SCET_{II}$ from $SCET_I$

Refs I used

① hep-ph/0005275 (d.o.f.)

② hep-ph/0011336 (d.o.f, $\mathcal{L}$, W, ...)

③ hep-ph/0107001 (hard-collin fact.)

④ hep-ph/0109045 (Gauge Inv. soft-collin)

⑤ hep-ph/0205289 (power counting)

⑥ hep-ph/0204229 (RPI)

⑦ hep-ph/0303156 (Gauge Inv. $O(\lambda^2)$ $\mathcal{L}$)

⑧ hep-ph/0202088 (Hard Scattering)

⑨ hep-ph/0107002 ($B \to D\pi$)

⑩ hep-ph/0605001 (0-bin)

⑪ hep-ph/0211069 ($SCET_I \to SCET_{II}$)

⑫ arXiv: 1202.0814 (rapidity RGE)

Processes: $e^+e^- \to$ jets, $B \to D\pi$, $e^-p \to e^-X$, $pp \to$ Higgs + jets

$B \to \pi \ell \nu$, $\gamma^* \gamma \to \pi^0$, ...

Section 1 Intro, Degrees of Freedom, Coordinates

SCET : an EFT for energetic hadrons $E_H \simeq Q \gg \Lambda_{QCD} \sim M_H$

an EFT for energetic jets $E_J \simeq Q \gg M_J = \sqrt{p_J^2}$

an EFT for massless hard ⇔ collinear ⇔ soft interactions

Why? • "Factorization" Our main probe of short distance physics is hard collisions ($e^+e^- \to$ stuff, $pp \to$ stuff). Disentangling the physics of QCD & other interactions requires a separation of scales → EFT → SCET

• jets, energetic hadrons are very common

• eg. Hard Scattering $e^- p \to e^- X$ (DIS) , $p\bar{p} \to X \overset{\text{Drell Yan}}{\ell^+\ell^-}$, $pp \to HX$

$\gamma^*\gamma \to \pi^0$, $e^+e^- \to$ jets , $e^+e^- \to J/\psi\, X$, ...

jet substructure

eg B-decays $B \to X_s \gamma$, $B \to X_u e \bar{\nu}$, $B \to D\pi$, $B \to \pi \ell \nu$

$B \to \pi\pi$, ...

$m_B = 5.279$ GeV $\gg \Lambda_{QCD}$

• Need to separate perturbative $\alpha_s(Q) \ll 1$ & non-perturbative effects in QCD (eg. hard scattering vs. parton distn's)

• Sum large Sudakov double logs $\sim (\alpha_s \ln^2)^k$

• New EFT tools

Prelude (What Makes SCET different from other EFT's)

- We will have multiple fields for the same particle
  - ξ_n = collinear quark field
  - q_s = soft " "
- We will integrate out offshell modes but not entire d.o.f. (like HQET)
- SCET has convolutions $\sum_i C_i O_i \rightarrow \int d\omega\, C(\omega)\, O(\omega)$
- power counting parameter $\lambda \ll 1$ is not related to mass dimension of fields
- Wilson Lines $P\exp\left(ig\int ds\; n\cdot A(ns)\right)$ appear everywhere, subtle & interesting gauge symmetry structure
- $1/\epsilon^2$ divergences at 1-loop that require UV counterterm

Degrees of freedom for SCET:

eg 1 $B \to D\pi$ hadrons

in B rest frame $P_\pi^\mu = (2.310\text{ GeV}, 0, 0, -2.306\text{ GeV})$

$\simeq Q\, n^\mu$ to good approx.

where $n^\mu = (1, 0, 0, -1)$, $n^2 = 0$ light-like vector

$\uparrow$ 0,1,2,3 basis

$Q \gg \Lambda_{QCD}$

Light-Cone Coordinates

Basis vectors $n^\mu, \bar{n}^\mu$

$$n^2=0, \quad \bar{n}^2=0, \quad n\cdot\bar{n}=2$$

vectors $\quad p^\mu = \frac{n^\mu}{2}\,\bar{n}\cdot p + \frac{\bar{n}^\mu}{2}\,n\cdot p + p_\perp^\mu$

Notation

$$p^+ \equiv n\cdot p, \quad p^- \equiv \bar{n}\cdot p$$

$$p^2 = n\cdot p\,\bar{n}\cdot p + p_\perp^2 = p^+p^- + p_\perp^2 = p^+p^- - \vec{p}_\perp^{\,2}$$

metric $\quad g^{\mu\nu} = \frac{n^\mu\bar{n}^\nu}{2} + \frac{\bar{n}^\mu n^\nu}{2} + g_\perp^{\mu\nu}$

epsilon $\quad \epsilon_\perp^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\,\frac{\bar{n}_\alpha n_\beta}{2}$

- $n^2=0$ requires complementary vector $\bar{n}^\mu$ for decomposition (dual vector for orthogonality)
- choice $n^\mu=(1,0,0,-1)$, $\bar{n}^\mu=(1,\underbrace{0,0}_{\perp},1)$ works

  but other choices do too [eg $n=(1,0,0,-1)$, $\bar{n}=(3,2,2,1)$] (more later)

Constituent Quark & Gluons:

In $B\to D\pi$ the B, D are soft $E_H\sim M_H$, use HQET for their constituents. quark & gluons with $p^\mu\sim\Lambda$

Pion is "collinear" $E_\pi \gg M_\pi$, is highly boosted

- In rest frame $\textcircled{\pi}$ has quark & gluon constituents $p^\mu\sim(\overset{+}{\Lambda},\overset{-}{\Lambda},\overset{\perp}{\Lambda})$
- boost along $\hat{z}$, $K\gg1$ $\leftarrow\textcircled{\pi}$ has constituents $p^\mu\sim\underbrace{\left(\frac{\Lambda^2}{Q}, Q, \Lambda\right)}_{\text{relative scaling}}$

  $p^-\to Kp^-$, $p^+\to\frac{p^+}{K}$

  $p_\perp\to p_\perp$

  fluctuations about $(0,Q,0)=P_\pi^\mu$

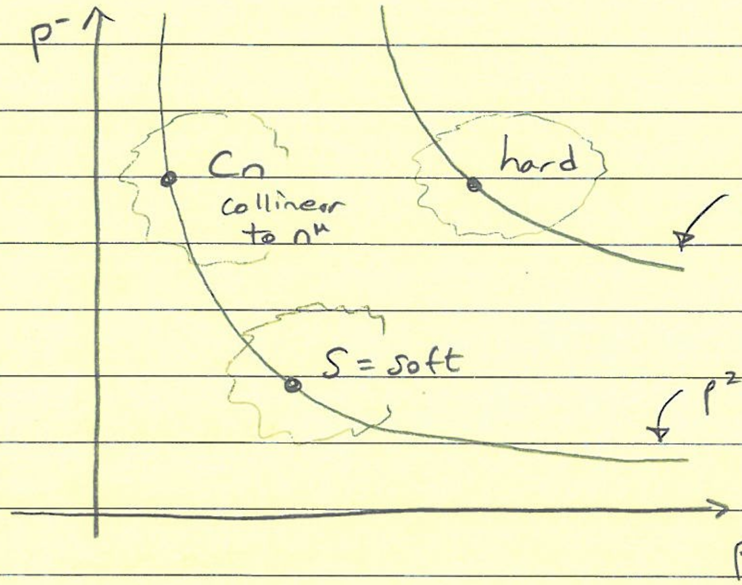
  $p^-\gg p_\perp\gg p^+$ defines collinear

Generically $(p^+, p^-, p^\perp) \sim Q\,(\lambda^2, 1, \lambda)$ is collinear

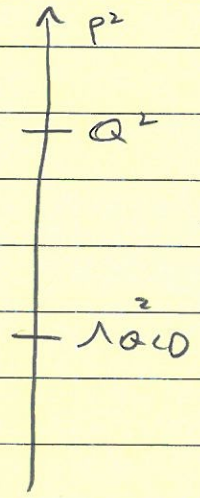
where $\lambda \ll 1$ is small parameter (our eg. has $\lambda = \frac{\Lambda}{Q}$)

Degrees of freedom occupy momentum regions in SCET

SCET$_{II}$



usual EFT

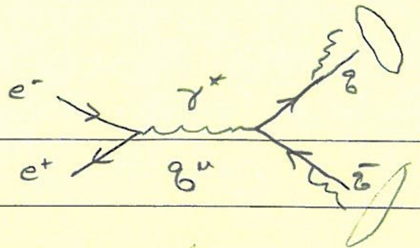


Comments

- $p^2 = p^+p^- - \vec{p}_\perp^{\,2}$, enough to characterize d.o.f in p^+-p^- plane since $\vec{p}_\perp^{\,2} \sim p^+p^-$ for modes that can go on-shell

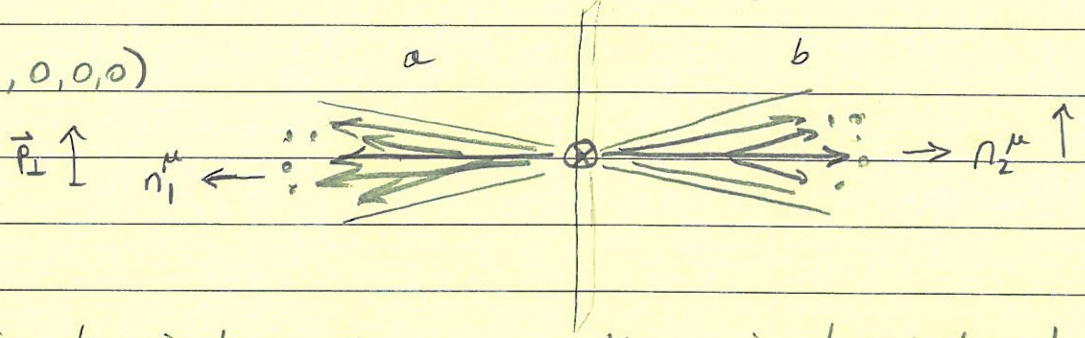
- boundary of regions would be a cutoff in Wilsonian EFT, but we'll use dim. reg to preserve symmetries. Still the correct picture but region overlaps a bit more tricky

- the theory with C_n & S d.o.f. is known as SCET$_{II}$ & it applies for energetic hadron production eg 2. $e^+e^- \to$ dijets



CM frame $q^\mu = (Q, 0, 0, 0)$

back-to-back jets



jet of hadrons in hemisphere a, another in hemisphere b

n_1-collinear jet

jet constituents have $p_\perp \sim \Delta \ll p_- \sim Q$

$$(p^+, p^-, p_\perp) \sim \Big(\underset{\uparrow \text{ fixed by } p^+p^- \sim p_\perp^2}{\frac{\Delta^2}{Q}}, Q, \Delta\Big) \sim Q(\lambda^2, 1, \lambda)$$

collinear

Jet Mass $M_J^2 = \Big(\sum_{i \in a} p_i^\mu\Big)^2 \sim p^- p^+ \sim \Delta^2 \ll Q^2$

(another way to characterize that its a jet)

here $\lambda = \frac{\Delta}{Q} \ll 1$

If $\Delta \sim Q$ we don't have dijets (hadrons in all directions) (inclusive sum over many jets, local OPE region)

$\Delta \sim \Lambda_{QCD}$ we have energetic hadrons, jets are so narrow that all constituents bind into a hadron

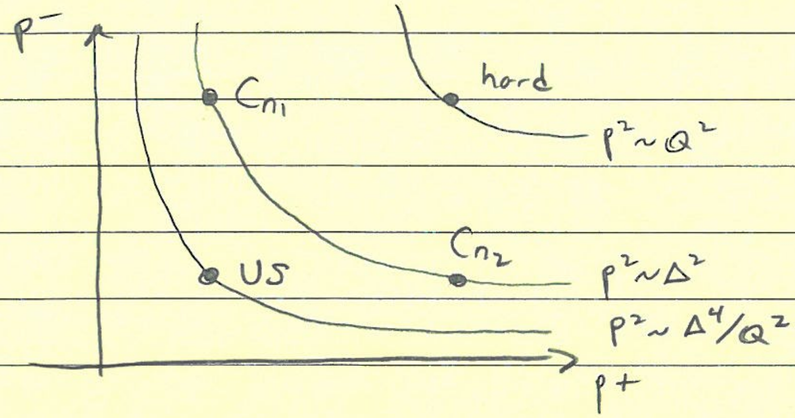
n_2-collinear jet take $n_1 = n$
$n_2 = \bar{n}$ for simplicity

just mirror of above with $+ \leftrightarrow -$

Another important d.o.f. are ultrosoft modes "US" that can communicate between jets

$$p^\mu \sim \Big(\underset{+}{\frac{\Delta^2}{Q}}, \underset{-}{\frac{\Delta^2}{Q}}, \underset{\perp}{\frac{\Delta^2}{Q}} \Big)$$

"communicate" means sharing momenta of a common size



	$(+, -, \perp)$
n-collin	$(\lambda^2, 1, \lambda)\, Q$
$\bar{n}$-collin	$(1, \lambda^2, \lambda)\, Q$
usoft	$(\lambda^2, \lambda^2, \lambda^2)\, Q$

↑ IR degrees of freedom with $p^2 \lesssim Q^2\lambda^2$

[soft $(\lambda, \lambda, \lambda)\, Q$ in this notation]

↑ $\mathrm{SCET}_{\mathrm{I}}$, EFT for energetic jets

<u>Note</u> (Discuss)

(i) multiple modes for IR ↔ needed for p.c. ↔ multiple fields

(ii) we integrate out modes above a given hyperbola in invariant mass (offshell modes)

(iii) important thing is relative scaling of momenta btwn modes (absolute scaling frame dependent, but relative scaling is frame independent) eg 3 1-jet only?

$b \to s \gamma$ $B \to X_s \gamma$

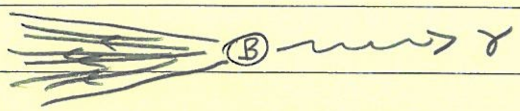
$\uparrow$ ≥ 1 hadron, summed over

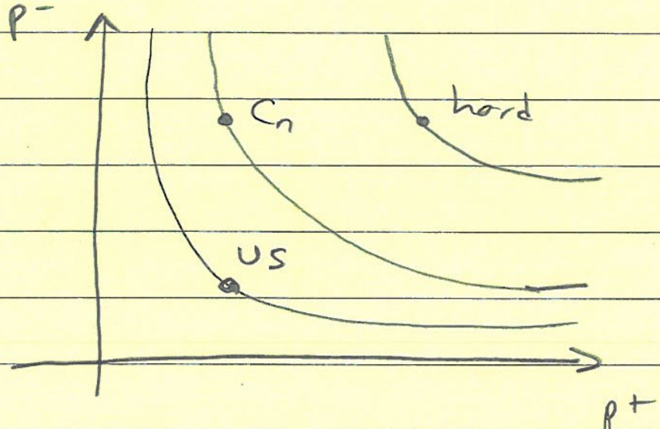
two-body kinematics $E_\gamma = \frac{m_B^2 - m_X^2}{2m_B} \in \left[0, \frac{m_B^2 - m_{K^*}^2}{2m_B} \right]$

for $m_X \in [m_B, m_K^*]$

$\Lambda_{QCD}^2 \ll m_X^2 \ll m_B^2 = Q^2$ gives

Jet





natural case

$p_{us}^2 \sim \Lambda_{QCD}^2 \sim \Delta^4/Q^2$

$\Delta \sim \sqrt{\Lambda_{QCD} Q}$

ultrasoft modes are constituents of B-meson

Collinear Spinors

u_n labelled by direction n

(analog of HQET spinor u_v)

massless QCD spinors
(Dirac Rep.)

$$u(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0}\, u \end{pmatrix}, \quad v(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{p^0}\, v \\ v \end{pmatrix}$$

let $n^\mu = (1,0,0,1)$
$\bar{n}^\mu = (1,0,0,-1)$

expand $\bar{n}\cdot p = p^0 + p^3 \gg p_\perp \gg n\cdot p = p^0 - p^3$

$$\frac{\vec{\sigma}\cdot\vec{p}}{p^0} = \sigma^3$$

$$u_n = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ \sigma^3 u \end{pmatrix} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\} \quad \text{particles}$$

$$v_n = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^3 v \\ v \end{pmatrix} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{antiparticles}$$

$$\not{n} = \begin{pmatrix} \mathbb{1} & -\sigma^3 \\ \sigma^3 & -\mathbb{1} \end{pmatrix} \quad \text{so} \quad \boxed{\not{n}\, u_n = \not{n}\, v_n = 0}$$

$$\frac{\not{n}\bar{\not{n}}}{4} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & \sigma^3 \\ \sigma^3 & \mathbb{1} \end{pmatrix} \quad \text{so} \quad \boxed{\frac{\not{n}\bar{\not{n}}}{4}\, u_n = u_n, \quad \frac{\not{n}\bar{\not{n}}}{4}\, v_n = v_n}$$

$\uparrow$

Projection Operator

Decompose $\mathbb{1} = \frac{\not{n}\bar{\not{n}}}{4} + \frac{\bar{\not{n}}\not{n}}{4}$

$$\mathbb{1}\,\psi^{QCD} = \xi_n + \varphi_{\bar{n}}$$

[← slightly different from above spinors, more later]

At high energy we produce/annihilate the components ξ_n, not the "small" components $\varphi_{\bar{n}}$

Collinear Propagators $\quad p^2 + i0 = \bar{n}\cdot p\, n\cdot p + p_\perp^2 + i0$

$$\sim \lambda^0 * \lambda^2 + \lambda * \lambda \qquad \underline{\text{same}}\ \text{size}$$

$\boxed{\text{Fermions}}$

$$\frac{i\not{p}}{p^2+i0} = \frac{i\not{n}}{2}\,\frac{\bar{n}\cdot p}{p^2+i0} + \underset{\lambda\ \text{suppressed}}{\cdots}$$

$$\xrightarrow[p]{} \qquad = \frac{i\not{n}}{2}\,\frac{1}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p} + i0\,\text{sign}(\bar{n}\cdot p)} + \cdots$$

↑ both particles $\bar{n}\cdot p > 0$ & antiparticles $\bar{n}\cdot p < 0$

from $T\{\xi_n(x), \bar{\xi}_n(0)\}$

Power counting of fields from free kinetic term

$$\mathcal{L} = \int d^4x \quad \bar{\xi}_n \frac{\bar{\not{n}}}{2} \left[in\cdot\partial + \cdots \right] \xi_n$$

$$\lambda^{-4} \quad \lambda^a \quad \left[\lambda^2 + \cdots \right] \quad \lambda^a \quad = \lambda^{2a-2}$$

set $\mathcal{L} \sim \lambda^0$, normalize kinetic term so no λ's

then $\boxed{\xi_n \sim \lambda}$

Note: mass dimension $[\xi_n] = 3/2$

λ dimension $[\xi_n]^\lambda = 1$

Collinear Gluons

consider general covariant gauge

gauge param.

$$\int d^4x \; e^{ik\cdot x} \langle 0 | T A_n^\mu(x) A_n^\nu(0) | 0 \rangle = \frac{-i}{k^2}\left(g^{\mu\nu} - \tau \frac{k^\mu k^\nu}{k^2} \right)$$

as above $k^2 = k^+ k^- + k_\perp^2 \sim \lambda^2$, no expansion

Also $g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$ has two terms of same size

eg. $g_\perp^{\mu\nu} \sim \lambda^0 \sim \frac{k_\perp^\mu k_\perp^\nu}{k^2} \sim \frac{\lambda^2}{\lambda^2}$, $g^{+-} \sim \lambda^0 \sim \frac{k^+ k^-}{k^2} \sim \frac{\lambda^2 \lambda^0}{\lambda^2}$

dot $n_\mu n_\nu$: $g^{++} = 0$, $\frac{(n\cdot k)^2}{k^2} \sim \frac{\lambda^4}{\lambda^2} = \lambda^2$

$d^4x \sim \lambda^{-4} \sim \frac{1}{(k^2)^2}$ so $\underline{A_n^\mu \sim k^\mu} \sim \overset{+,\,-,\,\perp}{(\lambda^2, 1, \lambda)}$

$A_n^\mu = (A_n^+, A_n^-, A_n^\perp) \sim (\lambda^2, 1, \lambda)$

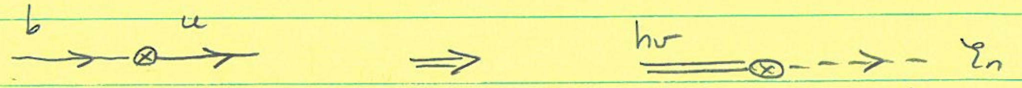
ie $k^\mu + gA^\mu = iD^\mu$ homogeneous covariant derivative

Note: $A_n^- \sim \lambda^0$ no suppression to add A_n^- fields

To see how this has an impact, consider an external weak current

eg. $b \to u e \bar{\nu}$ QCD $J = \bar{u}\Gamma b$ $\Gamma = \gamma^\mu(1-\gamma_5)$

consider heavy b (HQET), energetic u (SCET)



dashed collinear quark

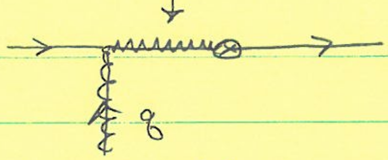
$J_{eff} = \bar{\xi}_n \Gamma h_v$

QCD

$= igT^A\gamma^\mu$

sign convention

k^μ this is far-offshell

Consider 

$\bar n\cdot A_n \sim \lambda^0$

$$k^\mu = m_b v^\mu + \frac{n^\mu}{2}\,\bar n\cdot q + \cdots$$

$$k^2 = m_b^2 + n\cdot v\, m_b\, \bar n\cdot q + \cdots$$

$$k^2 - m_b^2 \sim m_b^2 \quad \text{for} \quad \bar n\cdot q \sim \lambda^0 \sim m_b$$

no power suppression for these gluons

Find

$0 = \not n^2$

$$A_{n\mu}\,\bar\xi_n \Gamma \frac{i(\not k + m_b)}{k^2 - m_b^2}\, igT^A\gamma^\mu h_v = -g A_n^{\mu A}\bar\xi_n\Gamma \frac{\left[\cancel{m_b}(1+\not v) + \cancel{\frac{\not n}{2}\bar n\cdot q}\right]}{n\cdot v\,\cancel{m_b}\,\bar n\cdot q}\frac{\not n}{2}\bar n_\mu T^A h_v$$

$$+ \cdots$$

$$= \frac{-g\,\bar n\cdot A^A}{\bar n\cdot q}\,\bar\xi_n\Gamma T^A\left[\frac{+\frac{\not n}{2}\cancel{(1-\not v)}^{\,0} + \cancel{n\cdot v}}{\cancel{n\cdot v}}\right]h_v$$

$\not v h_v = h_v$

$$= \frac{-g}{\bar n\cdot q}\,\bar\xi_n\Gamma\,\bar n\cdot A\, h_v =$$ 

same order in λ.

Consider 

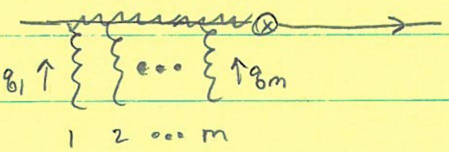
$p - q$ = collinear for

p & q both collinear,

so <u>not offshell</u>

↔ Lagrangian interaction

QCD graph

Consider 

+ crossed gluon graphs →

SCET graph

$$= (-g)^m\, \Gamma \sum_{\substack{\text{perms}\\ \{1,\dots,m\}}} \frac{\bar n^{\mu_m}T^{A_m}\cdots\bar n^{\mu_1}T^{A_1}}{[\bar n\cdot q_1]\,[\bar n\cdot(q_1+q_2)]\cdots\left[\bar n\cdot\sum_{i=1}^{m} q_i\right]}$$

when we write fields for external lines we must be a bit careful

Since SCET vertex is localized with m identical fields

$$\rightarrow \frac{(\bar n\cdot A)^m}{m!}$$

Complete tree level matching is

$$\bar{u}\,\Gamma\, b \;\rightarrow\; \bar{\xi}_n W\, \Gamma\, h_v$$

where $W = \sum_{k} \sum_{\text{perms}} \frac{(-g)^k}{k!} \left(\frac{\bar{n}\cdot A_{q_1} \cdots \bar{n}\cdot A_{q_k}}{[\bar{n}\cdot q_1][\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot \sum_{i=1}^{k} q_i]} \right)$

is momentum space Wilson Line

position space Wilson line is

$$W(0,-\infty) = \underset{\uparrow}{P} \exp\left(ig \int_{-\infty}^{0} ds\; \bar{n}\cdot A_n(\bar{n} s) \right)$$

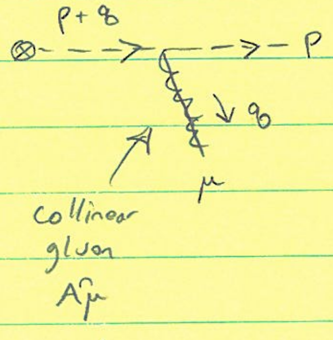
path ordering puts fields with larger arguement to the left $\bar{n}\cdot A_n(\bar{n}s)\, \bar{n}\cdot A_n(\bar{n}s')$

for $s > s'$

Effectively: $\bar{n}\cdot A$ field gets traded for $W[\bar{n}\cdot A]$

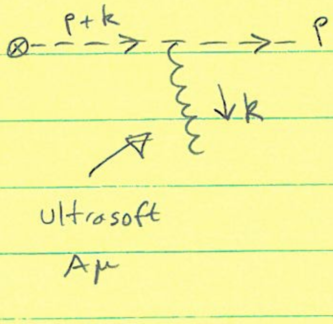
Consider SCET_I, collinear & usoft

$(\lambda^2, 1, \lambda)$ $(\lambda^2, \lambda^2, \lambda^2)$



propagator $= \dfrac{\bar{n}\cdot(q+p)}{n\cdot(q+p)\,\bar{n}\cdot(q+p) + (q_\perp + p_\perp)^2 + i0}$

$q^\mu \sim p^\mu$ so nothing dropped in denominator



here $k^\mu \sim \lambda^2$ $\bar{n}\cdot k \ll \bar{n}\cdot p \sim \lambda^0$

$k_\perp^\mu \ll p_\perp^\mu \sim \lambda$

$n\cdot k \sim n\cdot p$

propagator $= \dfrac{\bar{n}\cdot p}{n\cdot(k+p)\,\bar{n}\cdot p + p_\perp^2 + i0} + \underset{\text{higher order terms}}{\underset{\downarrow}{\cdots}}$

SCET Collinear Quark Lagrangian

Should:
- yield propagator & have interactions with both collinear gluons and usoft gluons
- have both quarks and antiquarks
- must yield LO propagator for different situations (without requiring an additional expansion)
- should be setup so we do not have to revist LO result when formulating power corrections

[we'll meet & resolve some technical hurtles along the way]

Step 1: Start with $\mathcal{L}_{QCD} = \bar\psi \, i \not{D} \, \psi$

Write $\psi = \xi_n + \varphi_{\bar n}$ where $\xi_n = \frac{\not{n}\not{\bar n}}{4}\psi$, $\varphi_{\bar n} = \frac{\not{\bar n}\not{n}}{4}\psi$ $\qquad \frac{\not{n}\not{\bar n}}{4}\xi_n = \xi_n$, $\frac{\not{\bar n}\not{n}}{4}\varphi_{\bar n} = \varphi_{\bar n}$

$$\mathcal{L} = (\bar\varphi_{\bar n} + \bar\xi_n)\left(i\frac{\not{\bar n}}{2} n\cdot D + i\frac{\not{n}}{2}\bar n\cdot D + i\not{D}_\perp\right)(\xi_n + \varphi_{\bar n})$$

$$= \bar\xi_n \frac{\not{\bar n}}{2} in\cdot D\, \xi_n + \bar\varphi_{\bar n}\, i\not{D}_\perp \xi_n + \bar\xi_n\, i\not{D}_\perp \varphi_{\bar n} + \bar\varphi_{\bar n}\frac{\not{n}}{2} i\bar n\cdot D\, \varphi_{\bar n} \qquad \textcircled{*}$$

other terms are zero eg. $\bar\xi_n\, i\not{D}_\perp \xi_n = \bar\xi_n\, i\not{D}_\perp \frac{\not{n}\not{\bar n}}{4}\xi_n = \bar\xi_n \underbrace{\frac{\not{n}\not{\bar n}}{4}}_{0} i\not{D}_\perp \xi_n$

So far this $\mathcal{L}$ is just QCD written in terms of $\xi_n, \varphi_{\bar n}$ vars.

- $\varphi_{\bar n}$ corresponds to subleading spinor components. We will not consider a source for $\varphi_{\bar n}$ in the path integral $\therefore$ we can do path integral over $\varphi_{\bar n}$

e.o.m. $\frac{\delta}{\delta \bar{\varphi}_{\bar n}}$: $\frac{\not{n}}{2}\, i\bar{n}\cdot D\, \varphi_{\bar n} + i\not{D}_\perp \xi_n = 0$

$$i\bar{n}\cdot D\, \varphi_{\bar n} + \frac{\not{\bar n}}{2}\, i\not{D}_\perp \xi_n = 0$$

$$\varphi_{\bar n} = \frac{1}{i\bar n\cdot D}\, i\not{D}_\perp \frac{\not{\bar n}}{2}\, \xi_n \quad , \quad \psi = \left(1 + \frac{1}{i\bar n\cdot D}\, i\not{D}_\perp \frac{\not{\bar n}}{2}\right)\xi_n$$

Plug back into ⊛: already used / satisfied 2nd & 4th terms, 1st & 3rd give

$$\mathcal{L} = \bar{\xi}_n \left(i n\cdot D + i\not{D}_\perp \frac{1}{i\bar n\cdot D}\, i\not{D}_\perp \right) \frac{\not{\bar n}}{2}\, \xi_n \qquad (**)$$

← insert (107.5) Aside

We're not yet done. We still need to:

② separate collinear & usoft gauge fields

③ " " " " momenta

④ expand and put pieces together

Step ②: $A_n^\mu \sim (\lambda^2, 1, \lambda) \sim p_n^\mu$, $A_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2) \sim k_{us}^\mu$

write $A^\mu = A_n^\mu + A_{us}^\mu + \dots$

like a classical background field to ξ_n, A_n^μ

$P_{us}^2 \sim Q^2\lambda^4 \ll P_c^2 \sim Q^2\lambda^2$

↑ long wavelength

there are some more terms that will matter for power corrections (& are fixed by gauge invariance). Ignore them for now.

Power counting

$\bar n\cdot A_n \sim \lambda^0 \gg \bar n\cdot A_{us}$

$A_{\perp n}^\mu \sim \lambda \gg A_{us}^\perp$

$n\cdot A_n \sim \lambda^2 \sim n\cdot A_{us}$

} so $A_{us}^\perp$ & $\bar n\cdot A_{us}$ can be dropped at leading order

What does $\frac{1}{i\bar{n}\cdot\partial}$ mean ?

Its the analog of how you define $\frac{1}{\hat{r}}$ in quantum mechanics, you use the eigenbasis:

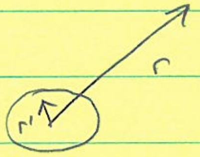
$$\frac{1}{i\bar{n}\cdot\partial}\phi(x) = \frac{1}{i\bar{n}\cdot\partial}\int d^4p\, e^{-ip\cdot x}\varnothing(p) = \int d^4p\, e^{-ip\cdot x}\frac{1}{\bar{n}\cdot p}\varnothing(p)$$

Step ③ We had a λ-expansion for a propagator carrying collinear & soft momenta

$$\frac{1}{(p_n+k_{us})^2} = \underset{\sim \lambda^{-2}}{\frac{1}{p_n^-(p_n^+ + k_{us}^+) + p_{n\perp}^2}} - \underset{\sim \lambda^{-1}}{\frac{2\, k_{us}^\perp \cdot p_n^\perp}{\left[p_n^-(p_n^+ + k_{us}^+) + p_{n\perp}^2\right]^2}} + \ldots$$

There must be Feyn. Rules in SCET to reproduce 2nd term too, so when we expand $k_{us}^\perp \ll p_n^\perp$, $k_{us}^- \ll p_n^-$ we can't just ignore $k_{us}^{\perp,-}$. We need a systematic (gauge invariant) <u>multipole</u> <u>expansion</u>.

[Recall E&M

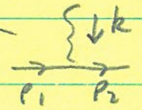


$r' \ll r$

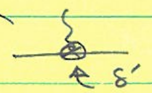
$$V(r) = \frac{1}{r}\int d^3r'\, \rho + \frac{1}{r^2}\int r'\cos\theta\, \rho\, d^3r' + \ldots$$
]

<u>Position Space</u> (1-dim), consider

- $\int dx\ \bar{\psi}(x)\, A(0)\, \psi(x) = \int dx \int dp_1\, dp_2\, dk\ e^{ip_1\cdot x}\, e^{-ik\cdot 0}\, e^{-ip_2\cdot x}\, \bar{\psi}(p_1) A(k) \psi(p_2)$
  $= \int dp_1 dp_2 dk\ \delta(p_1-p_2)\ \bar{\psi}(p_1)\, A(k)\, \psi(p_2)$
- $\int dx\ \bar{\psi}(x)\ x\cdot i\partial A(0)\, \psi(x)$
  $= \int dp_1 dp_2 dk\ \delta'(p_1-p_2)\ k\ \bar{\psi}(p_1) A(k)\, \psi(p_2)$



k gets dropped [monetum not conserved]



must int. by parts...

We will carry out the multipole expansion in <u>momentum space</u>

- more directly get mom. space Feyn. Rules
- simplifies formulation of gauge transformations
- mom. expansion sits in propagators rather than vertices

eg. $\dfrac{k_{us}^\perp \cdot p_n^\perp}{[\cdots]^2} \sim$  propagator insertion

Call $\xi_n(x)$ field from

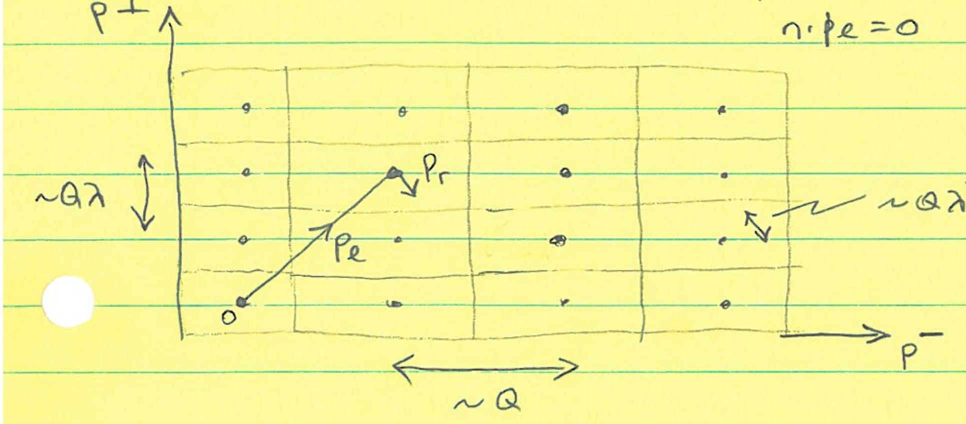
Eq. (**) pg. (107) → $\hat{\xi}_n(x)$. [Consider only quark part, $a^s_{\vec{p}}$, to start.]

Let $\tilde{\xi}_n(p) = \int d^4x \; e^{ip\cdot x} \, \hat{\xi}_n(x)$

Analogy HQET: $p^\mu = \overset{\text{label}}{m v^\mu} + \overset{\text{residual}}{k^\mu}$

SCET: $p^\mu = \underset{\substack{\uparrow \\ (p_\ell^-, p_\ell^\perp) \sim (1,\lambda) \\ \text{only}}}{p_\ell^\mu} + \underset{\substack{\uparrow \\ p_r^\mu \sim (\lambda^2,\lambda^2,\lambda^2)}}{p_r^\mu}$

$n\cdot p_\ell = 0$



p_ℓ^μ discrete grid points

p_r^μ continuous

$p^\mu = p_\ell^\mu + p_r^\mu$ unique for given grid

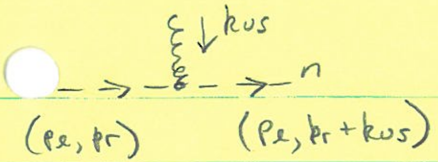
$\int d^4p = \sum_{p_\ell \neq 0} \int d^4 p_r$ for collinear p
[$p_\ell = 0$ is not collinear]

$\int d^4p = \int d^4 p_r$ for usoft p [usoft has $p_\ell = 0$]

Write: $\tilde{\xi}_n(p) \to \tilde{\xi}_{n,p_\ell}(p_r)$

Note: We have separate conservation of label & residual momenta

$$\int d^4x \; e^{i(p_\ell - q_\ell)\cdot x} \, e^{i(p_r - q_r)\cdot x} = \delta_{p_\ell, q_\ell} \, \delta^4(p_r - q_r) \, (2\pi)^4$$



"non-conservation" of momenta is replaced by two separate conservations where some fields don't carry label momenta.

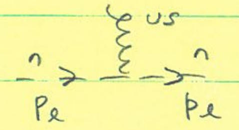
Final Step

since all fields carry residual momenta the conservation law just corresponds to locality with respect to Fourier transform $p_r \to x$

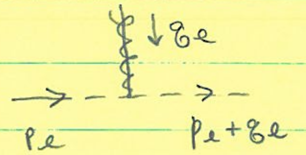
$$\xi_{n,p_\ell}(x) = \int \frac{d^4 p_r}{(2\pi)^4} e^{-ip_r\cdot x}\, \tilde{\xi}_{n,p_\ell}(p_r)$$

↑ build action from these fields

- usoft gluons leave labels conserved



- collinear gluons change labels



- label n for collinear direction always preserved by usoft & collinear gluons only a hard interaction can couple fields with different n's eg

back-to-back production by hard γ^*

All together

$$\hat{\xi}_n(x) = \int d^4p\, e^{-ip\cdot x}\, \tilde{\xi}_n(p) = \sum_{p_\ell \neq 0} \int d^4 p_r\, e^{-ip_\ell\cdot x}\, e^{-ip_r\cdot x}\, \tilde{\xi}_{n,p_\ell}(p_r)$$

$$= \sum_{p_\ell\neq 0} e^{-ip_\ell\cdot x}\, \xi_{n,p_\ell}(x)$$

Define <u>two</u> derivative operators:

$$i\partial^x_\mu \, \xi_{n,p_\ell}(x) \sim \lambda^2 \, \xi_{n,p_\ell}(x) \qquad \text{residual}$$

$$\mathcal{P}_\mu \, \xi_{n,p_\ell}(x) \equiv p_\ell^\mu \, \xi_{n,p_\ell}(x) \sim (\overset{+}{0}, \overset{-}{1}, \overset{\perp}{\lambda}) \, \xi_{n,p_\ell}(x)$$

$$\Rightarrow \quad i\bar{n}\cdot\partial \ll \bar{\mathcal{P}} = \bar{n}\cdot\mathcal{P} \,, \quad i\partial_\perp^\mu \ll \mathcal{P}_\perp^\mu$$

implements multipole expansion
similar structure to expansion for
gauge fields → gauge symmety easier

Notation is friendly:

$$\hat{\xi}_n(x) = \sum_{p_\ell \neq 0} e^{-ip_\ell\cdot x} \, \xi_{n,p_\ell}(x) = e^{-i\mathcal{P}\cdot x} \sum_{p_\ell\neq 0} \xi_{n,p_\ell}(x)$$

$$\equiv e^{-i\mathcal{P}\cdot x} \underbrace{\xi_n(x)}_{\sum_{p_\ell\neq 0} \xi_{n,p_\ell}(x)}$$

← suppress labels if we don't need them explicitly

Field products

$$\hat{\xi}_n(x)\,\hat{\xi}_n(x) = e^{-i\mathcal{P}\cdot x} \, \xi_n(x)\,\xi_n(x)$$

↑ acts on both fields & just gives <u>label</u> conservation

Last Step is to consider anti-quarks & gluons

Mode Expn

$$\psi(x) = \int \bar{d}^4p \;\; \delta(p^2)\,\theta(p^0)\,\left[u(p)\,a(p)\,e^{-ip\cdot x} + v(p)\,b^\dagger(p)\,e^{ip\cdot x}\right]$$

$$= \psi^+ + \psi^-$$

QCD

Write $\psi^+(x) = \sum_{p_\ell\neq 0} e^{-ip_\ell\cdot x}\,\xi^+_{n,p_\ell}(x)$

$\psi^-(x) = \sum_{p_\ell\neq 0} e^{ip_\ell\cdot x}\,\xi^-_{n,p_\ell}(x)$

both have $\theta(p^0) = \theta(\bar{n}\cdot p)$

$\not{n}\,\xi^\pm_{n,p_\ell} = 0$

Define $\xi_{n,p_\ell}(x) \equiv \xi^+_{n,p_\ell}(x) + \xi^-_{n,-p_\ell}(x)$ any p_ℓ signs

$\bar{n}\cdot p_\ell > 0$ particles destroy $\bar{\xi}_{n,p_\ell}$ $\bar{n}\cdot p_\ell > 0$ part. create

$\bar{n}\cdot p_\ell < 0$ antiparticles create $\bar{n}\cdot p_\ell < 0$ anti, destroy

p_ℓ carries same sign as mom. flow along fermion # $\xrightarrow[p_\ell]{}$

then $\hat{\xi}_n(x) = e^{-i\mathcal{P}\cdot x}\,\xi_{n,p_\ell}(x)$ as before

Collinear Gluons $A^\mu_{n,q_\ell}(x)$, $\left[A^\mu_{n,q_\ell}(x)\right]^* = A^\mu_{n,-q_\ell}(x)$

$q_\ell > 0$ destroy

$q_\ell < 0$ create $\hat{A}_n(x) = e^{-i\mathcal{P}\cdot x}A_n(x)$

$\uparrow \sum_{q_\ell} A_{n,q_\ell}(x)$

General Results

sign on † fields

$$\mathcal{P}^\mu\left(\phi^\dagger_{q_1}\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots\right) = \left(p_1^\mu + p_2^\mu + \cdots - q_1^\mu - q_2^\mu - \cdots\right)\left(\phi^\dagger_{q_1}\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots\right)$$

eigenvalue eqtn

$$i\partial^\mu \sum_p e^{-ip\cdot x}\,\phi_{n,p}(x) = \sum_p e^{-ip\cdot x}\left(\mathcal{P}^\mu + i\partial^\mu\right)\phi_{n,p}(x)$$

$$= \underbrace{e^{-i\mathcal{P}\cdot x}}\left(\mathcal{P}^\mu + i\partial^\mu\right)\phi_n(x)$$

later we'll suppress this & recall that labels are conserved

Step ④ Expand $\mathcal{L} = \bar{\hat{\xi}}_n(x)\left[in\cdot D + i\not{D}_\perp \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \right]\frac{\bar{\not{n}}}{2}\,\hat{\xi}_n(x)$

$$iD^\mu = \mathcal{P}^\mu + gA_n^\mu + i\partial^\mu + gA_{us}^\mu + \cdots$$

$$in\cdot D = in\cdot\partial + g\,n\cdot A_n + g\,n\cdot A_{us} \qquad (\text{exact, all} \sim \lambda^2)$$

$$iD_\perp = \underbrace{(\mathcal{P}_\perp + gA_n^\perp)}_{\lambda} + \underbrace{(i\partial_\perp + gA_\perp^{us})}_{\lambda^2} + \cdots$$

$$i\bar{n}\cdot D = \underbrace{(\bar{\mathcal{P}} + g\bar{n}\cdot A_n)}_{\lambda^0} + \underbrace{(i\bar{n}\cdot\partial + g\bar{n}\cdot A_{us})}_{\lambda^2} + \cdots$$

From before $\hat{\xi}_n(x) \sim \lambda \overset{\text{so}}{\sim} \xi_n(x)$

$$d^4x\, \underbrace{e^{-ix\cdot\mathcal{P}}} \sim \lambda^{-4}$$

$\mathcal{O}(1)$ phases implies $x^- \sim 1/p^+$, $x^+ \sim \frac{1}{p^-}$
$x^\perp \sim 1/p_\perp$

<u>Leading Order $\mathcal{L}$ is $\mathcal{O}(\lambda^4)$</u>

$$\mathcal{L}^{(0)}_{\xi\xi} = e^{-ix\cdot\mathcal{P}}\,\bar{\xi}_n\left[in\cdot D + i\not{D}_\perp^n \frac{1}{i\bar{n}\cdot D_n}\, i\not{D}_\perp^n\right]\frac{\bar{\not{n}}}{2}\,\xi_n$$

where $\left.\begin{array}{l} iD_\perp^{n\mu} = \mathcal{P}_\perp^\mu + gA_n^{\perp\mu} \\ i\bar{n}\cdot D_n = \bar{\mathcal{P}} + g\bar{n}\cdot A_n \end{array}\right\}$ collinear cov. derivatives

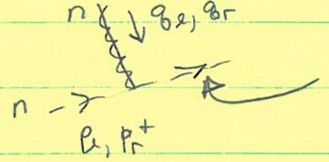
Note:

- both terms $\sim \lambda\cdot\lambda^2\cdot\lambda \sim \lambda^4$
- all fields at x, derivatives $i\partial \sim \lambda^2$, action is explicitly <u>local</u> at $Q\lambda^2$ scale

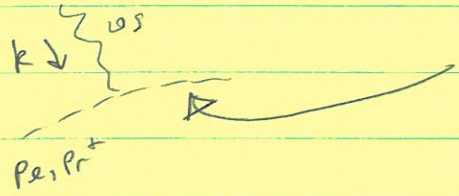
  also local at $Q\lambda$ too ($D_\perp^n$ in numerator, momentum space version of locality)

  <u>only</u> non-local at $\sim Q$ from $\frac{1}{\bar{n}\cdot p}$ factors

- Collinear propogators



$$\frac{\bar n\cdot(q_\ell+p_\ell)}{\bar n\cdot(q_\ell+p_\ell)\, n\cdot(q_r+p_r)+(q_\ell^\perp+p_\ell^\perp)^2+i0}$$

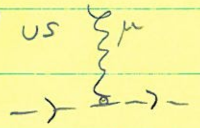


$$\frac{\bar n\cdot p_\ell}{\bar n\cdot p_\ell\, n\cdot(p_r+k)+(p_\ell^\perp)^2+i0}$$

because no $i\bar n\cdot\partial$ or $i\partial^\perp$ in $\mathcal{L}_{\xi\xi}^{(0)}$

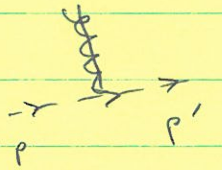
$\mathcal{L}_{\xi\xi}^{(0)}$ knows how to give LO propogator in both situations <u>without</u> further expansions

<u>Feyn. Rules</u>



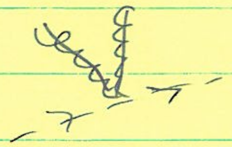
$$= ig\frac{\bar n\!\!\!/}{2}n^\mu T^A$$

only $n\cdot A_{us}$ gluons



$$= igT^A\frac{\bar n\!\!\!/}{2}\left[n^\mu+\frac{\gamma_\perp^\mu \not p_\perp}{\bar n\cdot p}+\frac{\not p_\perp'\,\gamma_\perp^\mu}{\bar n\cdot p'}-\frac{\not p_\perp'\not p_\perp}{\bar n\cdot p'\,\bar n\cdot p}\,\bar n^\mu\right]$$

all 4 components couple



$= \ldots$

terms with ≥ 2 gluons also exist but have at most 2 $\perp$ gluons & rest $\bar n\cdot A$

trade $\bar{n}\cdot A_n \Leftrightarrow W$

Wilson Line Eqtns

$$i\bar{n}\cdot D_x \, W(x,-\infty) = 0$$

equivalent def'n to position space W-line is momentum space W_n

$$i\bar{n}\cdot D_n \, W_n = 0$$

$$(\bar{\mathcal{P}} + g\bar{n}\cdot A_n)\, W_n = 0$$

$$i\bar{n}\cdot D_n \, W_n \, \mathcal{O} = W_n \bar{\mathcal{P}} \, \mathcal{O}$$

some operator

so $\boxed{i\bar{n}\cdot D_n \, W_n = W_n \bar{\mathcal{P}}}$

as operator equation

and since $(W(x,-\infty))^\dagger W(x,-\infty) = 1$

$$W_n^\dagger W_n = 1$$

we have $\boxed{i\bar{n}\cdot D_n = W_n \bar{\mathcal{P}} W_n^\dagger}$ $\quad \bar{\mathcal{P}} = W_n^\dagger \, i\bar{n}\cdot D_n W_n$

$$\frac{1}{\bar{\mathcal{P}}} = W_n^\dagger \frac{1}{i\bar{n}\cdot D} W_n \,, \quad \frac{1}{i\bar{n}\cdot D_n} = W_n \frac{1}{\bar{\mathcal{P}}} W_n^\dagger$$

(easy to check that these are inverses)

$$\mathcal{L}^{(0)}_{\xi\xi} = e^{-ix\cdot\wp}\, \bar{\xi}_n \frac{\bar{n}\!\!\!/}{2} \left[in\cdot D + iD\!\!\!\!/_{n\perp} W_n \frac{1}{\bar{\mathcal{P}}} W_n^\dagger iD\!\!\!\!/_{n\perp} \right] \xi_n$$

Collinear Gluon Lagrangian

QCD $$\mathcal{L} = \underbrace{-\frac{1}{2}\,\mathrm{tr}\{G^{\mu\nu}G_{\mu\nu}\}}_{\text{standard } -\frac{1}{4}G^{\mu\nu}_A G^{\mu\nu A}} + \underbrace{\tau\,\mathrm{tr}\{(i\partial_\mu A^\mu)^2\}}_{\text{gen. cov. gauge fixing}} + \underbrace{2\,\mathrm{tr}\{\bar c\, i\partial_\mu iD^\mu c\}}_{\text{gen. cov. ghost}}$$

gen. cov. ghost < adjoint, scalar, fermi statistics

$$G^{\mu\nu} = G^{\mu\nu}_A T^A = \frac{i}{g}[D^\mu, D^\nu]$$

SCET: same steps as for quark action

Let $$i{}^{\circ}\mathcal{D}^\mu = \frac{n^\mu}{2}(\bar{\mathcal{P}} + g\,\bar n\cdot A_n) + ({}^{\circ}\mathcal{P}_\perp^\mu + g A_{n\perp}^\mu) + \frac{\bar n^\mu}{2}(in\cdot\partial + g\,n\cdot A_n + g\,n\cdot A_{us})$$

$\boxed{iD^\mu \to i{}^{\circ}\mathcal{D}^\mu}$ at LO

$$i{}^{\circ}\mathcal{D}_{us}^\mu = \frac{n^\mu}{2}\bar{\mathcal{P}} + \mathcal{P}_\perp^\mu + \frac{\bar n^\mu}{2}(in\cdot\partial + g\,n\cdot A_{us})$$

recall A^μ_{us} behaves like background to A^μ_n. Maintaining gauge inv. for the background even in the A_n^μ gauge fixing term requires

$\boxed{i\partial^\mu \to i{}^{\circ}\mathcal{D}^\mu_{us}}$ at LO In SCET this needed so collinear gauge fixing term does not break the usoft gauge inv.

$$\mathcal{L}_{cg}^{(0)} = \frac{1}{2g^2}\,\mathrm{tr}\{([i{}^{\circ}\mathcal{D}^\mu, i{}^{\circ}\mathcal{D}^\mu])^2\} + \tau\,\mathrm{tr}\{([i{}^{\circ}\mathcal{D}^\mu_{us}, A_{n\mu}])^2\} + 2\,\mathrm{tr}\{\bar c_n[i{}^{\circ}\mathcal{D}_\mu^{us}, [i{}^{\circ}\mathcal{D}^\mu, c_n]]\}$$

$$\boxed{\mathcal{L}^{(0)}_{SCET_1} = \mathcal{L}^{(0)}_{\xi\xi} + \mathcal{L}^{(0)}_{cg} + \underbrace{\mathcal{L}^{(0)}_{q} + \mathcal{L}^{(0)}_{A}}}$$

full QCD actions for usoft quark q_{us} and for US gluon A^μ_{us}. These have no collinear fields

Analysis so far was tree level. To go further we need symmetries & power counting

① Gauge Symmetry ⎤ very useful
② Reparameterization Invariance ⎦
③ Spin Symmetry ?

Lets first consider ③ :

revisit spinors $\psi(x) = e^{-ix\cdot\tilde{p}}\left(1 + \frac{1}{i\bar{n}\cdot D_n} i\not{D}_{\perp n}\frac{\not{\bar{n}}}{2}\right)\xi_n(x)$

so $u(p) = \left(1 + \frac{1}{\bar{n}\cdot p}\not{p}_\perp\frac{\not{\bar{n}}}{2}\right)u_n$, $u_n = \frac{\not{n}\not{\bar{n}}}{4}u$

$\left[\not{n}\,u_n = 0,\ \frac{\not{n}\not{\bar{n}}}{4}u_n = u_n\right]$

- consider $\sum_s u_n^s\,\bar{u}_n^s = \frac{\not{n}\not{\bar{n}}}{4}\sum_s \bar{u}^s u^s\frac{\not{\bar{n}}\not{n}}{4} = \frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4} = \frac{\not{n}}{2}\bar{n}\cdot p$

→ quantized ξ_n field does give collinear propogator, including numerator.

- u_n is not equal to expanded spinor $\sqrt{p^-/2}\begin{pmatrix}u\\ \sigma^3 u\end{pmatrix}$, $u = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ even though it obeys the same relations

Instead

$$u_n = \frac{1}{2}\begin{pmatrix}\mathbb{1} & \sigma^3\\ \sigma^3 & \mathbb{1}\end{pmatrix}\sqrt{p^0}\begin{pmatrix}u\\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0}u\end{pmatrix} = \frac{\sqrt{p^0}}{2}\begin{pmatrix}\left(1 + p_3/p_0 - \frac{(i\vec{\sigma}\times\vec{p}_\perp)_3}{p^0}\right)u\\ \sigma_3\left(1 + \frac{p_3}{p^0} - \frac{(i\vec{\sigma}\times\vec{p}_\perp)_3}{p^0}\right)u\end{pmatrix}$$

$$= \sqrt{\frac{p^-}{2}}\begin{pmatrix}\tilde{u}\\ \sigma^3\tilde{u}\end{pmatrix}$$

Here $\tilde{u} \equiv \sqrt{\frac{p^0}{2p^-}}\left(1 + \frac{p_3}{p^0} - \frac{(i\vec{\sigma}\times\vec{p}_\perp)_3}{p^0}\right)u$ is two-componet spinor

$$\sum_s \tilde{u}^s\,\tilde{u}^{\dagger s} = \mathbb{1}_{2\times 2}.$$

The extra terms in $\tilde{u}$ compared to u ensure proper structure under ② RPI. (In particular projetors $P_n' = \frac{\not{n}\not{\bar{n}}}{4} + \frac{\not{n}}{2}$, $P_{\bar{n}}' = \frac{\not{\bar{n}}\not{n}}{4} - \frac{\not{n}}{2}$ would give $\sqrt{\frac{p^-}{2}}\begin{pmatrix}u\\ \sigma^3 u\end{pmatrix}$ but are not RPI-III invariant.)

Spin Symmetry easiest to analyze in two-component form

$\xi_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_n \\ \sigma^3 \varphi_n \end{pmatrix}$ where $\dim \xi_n = \dim \varphi_n$

$$\mathcal{L}^{(0)} = \varphi_n^\dagger \left\{ i n\cdot D + i D_{n\perp}^\mu \frac{1}{i\bar{n}\cdot D} i D_{n\perp}^\nu \left(g_{\mu\nu}^\perp + i \epsilon_{\mu\nu}^\perp \sigma_3 \right) \right\} \varphi_n$$

not $SU(2)$

just $U(1)$ helicity $h = \frac{i \epsilon_\perp^{\mu\nu}}{4} [\gamma_\mu, \gamma_\nu] \sim \sigma_3$ generator,

spin along the direction of collinear motion n

- broken by masses
- broken by non-perturbative effects
- useful in perturbation theory
- related to chiral rotation $\gamma_5 \xi_n = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_n \\ \sigma^3 \varphi_n \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma^3 \varphi_n \\ \varphi_n \end{pmatrix}$

  ie $\varphi_n \to \sigma_3 \varphi_n$

---

① Gauge Symmetry $U(x) = \exp\left[i\, \alpha^A(x) T^A \right]$

Need to consider U's which leave us within EFT

eg. $i\partial^\mu \alpha^A \sim Q \alpha^A$ then $\xi_n' = U(x)\xi_n$ would no longer have $p^2 \lesssim Q^2 \lambda^2$.

global $U = e^{i\alpha^A T^A}$

collinear $U_c(x)$ $\quad i\partial^\mu U_c(x) \sim Q(\lambda^2, 1, \lambda)\, U_c(x) \leftrightarrow A_n^\mu$

usoft $U_u(x)$ $\quad i\partial^\mu U_u(x) \sim Q(\lambda^2, \lambda^2, \lambda^2)\, U_u(x) \leftrightarrow A_{us}^\mu$

- two classes of gauge trnsfm for two gauge fields

- in label momentum space we have $\xi_{n,p_\ell}^{(x)} \to \sum_q (U_c)_{p_\ell - q_\ell}^{(x)} \xi_{n,q_\ell}^{(x)}$

  (analog of $\psi(x) \to U(x)\psi(x)$

  $\tilde{\psi}(p) \to \int dq\, \tilde{U}(p-q)\, \tilde{\psi}(q)$)

Let $(U_c)_{p_\ell - q_\ell} = \overset{\text{matrix}}{(U_c)_{p_\ell, q_\ell}}$ ie $\{p_\ell, q_\ell\}$'th entry is number $(U_c)_{p_\ell - q_\ell}$

For A_n^μ we let its U_c transformation be that of quantum gauge trnsfm of a quantum field in a A_{us}^μ background (in manner homogeneous in p.c.)

$U_c(x)$

* $\xi_n(x) \to U_c(x)\, \xi_n(x)$ using a matrix notation

* $A_n^\mu \to U_c \left(A_n^\mu + \frac{i}{g}\, {}^{\sigma}\mathcal{D}_{\underbrace{us}_{\text{adjoint}}}^{\mu} \right) U_c^\dagger$

* Also $q_{us} \xrightarrow{U_c} q_{us}$ since otherwise we
$A_{us}^\mu \xrightarrow{U_c} A_{us}^\mu$ give large momentum to usoft field

For $U_{us}(x)$ the fields ξ_n, A_n^μ transform like quantum fields under background gauge trnsfm. That is, they transform like matter fields of appropriate rep.

$U_{us}(x)$

* $\xi_n(x) \to \underset{\substack{\uparrow \\ \text{one number for} \\ \text{all } \xi_{n,p} \text{ "vector"} \\ \text{components}}}{U_{us}(x)}\, \xi_n(x)$, $A_n^\mu \to U_{us} A_n^\mu U_{us}^\dagger$

* $q_{us} \to U_{us}\, q_{us}$, $A_{us}^\mu \to U_{us} \left(A_{us}^\mu + \frac{i\partial^\mu}{g} \right) U_{us}^\dagger$

↑ ↗ usual gauge transformations

These transformations are fundamental, they are not corrected by power corrections.

U_c, U_{us}

Gauge transformations are homogeneous in λ

no mixing of term of different orders

eg. recall our heavy-to-light current

$$\bar\xi_n \Gamma h_v^{us} \xrightarrow{U_c} \bar\xi_n U_c^\dagger \Gamma h_v^{us} \quad \text{is \underline{not} gauge inv!}$$

BUT recall offshell propagators generated Wilson line

$\bar W(x,-\infty)$

In general $\bar W(x,y) \to U(x)\, \bar W(x,y)\, U^\dagger(y)$. To avoid double counting with U_{global}, we will take $U_c^\dagger(-\infty) = 1$

$\bar W(x,-\infty) \to U_c(x)\, \bar W(x,-\infty)$

Momentum Space $$W = \sum_{m=0}^{\infty} \sum_{\text{perms}} \sum_{q_i} \frac{(-g)^m}{m!} \frac{\bar n \cdot A_{n,q_1}^{a_1}(x) \cdots \bar n \cdot A_{n,q_m}^{a_m}(x)\, T^{a_m} \cdots T^{a_1}}{\bar n \cdot q_1\ \bar n \cdot (q_1+q_2) \cdots \bar n \cdot (\Sigma q_i)}$$

$$W(x) = \left[\sum_{\text{perms}} \exp\left(\frac{-g}{\bar{\mathcal{P}}} \bar n \cdot A_n(x) \right) \right]$$

the dependence on x encodes residual momenta in Wilson line. For $x = 0$ the Fourier transform w.r.t p_e^- gives the line $\bar W(\cancel{x}, -\infty)$ where $\cancel{x}$ is conjugate p_e^-.

* $W(x) \xrightarrow{U_c} U_c(x)\, W(x)$ in label matrix space.

* $W(x) \xrightarrow{U_{us}} U_{us}(x)\, W(x)\, U_{us}^\dagger(x)$ from transformation of A_n directly.

$\bar\xi_n W \Gamma h_v \xrightarrow{U_c} \bar\xi_n \cancel{U_c^\dagger} \cancel{U_c} W \Gamma h_v = \bar\xi_n W \Gamma h_v$ <u>invariant</u>

$\bar\xi_n W \Gamma h_v \xrightarrow{U_{us}} \bar\xi_n \cancel{U_{us}^\dagger} \cancel{U_{us}} W \cancel{U_{us}^\dagger} \Gamma \cancel{U_{us}} h_v = \bar\xi_n W \Gamma h_v$ 〃

- the Wilson line carries n-collinear gluons, which in full QCD combine with attachments to ξ_n $--\!\!\rightarrow\!\!--$ to give gauge invariant answers.
- usoft can be taken to include global, and connects all fields.

Gauge Symmetry ties together

$$in\cdot D = in\cdot\partial + g\,n\cdot A_n + g\,n\cdot A_{us}$$

$$iD_{n\perp}^{\mu} = \mathcal{P}_\perp^{\mu} + g\,A_{n\perp}^{\mu}$$

$$i\bar{n}\cdot D_n = \bar{\mathcal{P}} + g\,\bar{n}\cdot A_n$$

$iD_{us}^{\mu} = i\partial^{\mu} + g\,A_{us}^{\mu}$ acting on usoft fields

Is Power Counting & Gauge Invariance enough to fix $\mathcal{L}_{\xi\xi}^{(0)}$?

$in\cdot D \sim \lambda^2$, $\frac{1}{\bar{\mathcal{P}}}(iD_\perp)^2 \sim \lambda^2$ ← no other $\mathcal{O}(\lambda^2)$ operators with correct mass dimension

but so far nothing rules out $\bar{\xi}_n\, iD_{n\perp}^{\mu}\, \frac{1}{i\bar{n}\cdot D_n}\, iD_{n\perp\mu}\, \frac{\bar{n}\!\!\!/}{2}\, \xi_n$.

② Reparameterization Invariance (RPI)

$n, \bar{n}$ break Lorentz Invariance (c.f. v^μ in HQET)

generators $\underbrace{n^\mu M_{\mu\nu},\ \bar{n}^\mu M_{\mu\nu}}_{5\text{ total}}$ ($M_{\mu\nu}$ usual 6 antisymm SO(3,1) generators)

only $\epsilon_\perp^{\mu\nu} M_{\mu\nu}$, rotations about $\vec{n}$ axis are preserved

3 types of RPI that keep $n^2=0$, $\bar{n}^2=0$, $n\cdot\bar{n}=2$

inf $\Delta_\perp$	inf $\epsilon_\perp$	finite α (simpler)
I. $n \to n + \Delta_\perp$	II. $n \to n$	III. $n \to e^{\alpha} n$
$\bar{n} \to \bar{n}$	$\bar{n} \to \bar{n} + \epsilon_\perp$	$\bar{n} \to e^{-\alpha}\bar{n}$

Power counting : $\Delta_\perp \sim \lambda$ eg: $n\cdot p \to n\cdot p + \Delta_\perp\cdot p_\perp \sim \lambda^2$

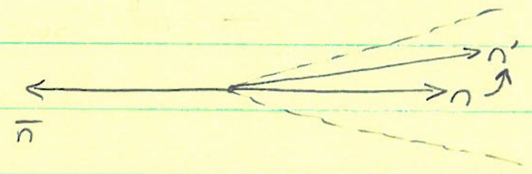
$$\left.\begin{array}{l}\epsilon_\perp \sim \lambda^0 \\ \alpha \sim \lambda^0\end{array}\right\}\ \text{unconstrained}$$

$\boxed{\text{type III}}$ simple, just implies for any operator with $\bar{n}^\mu$ in numerator there must be another n^μ in numerator, or $\bar{n}$ in denominator

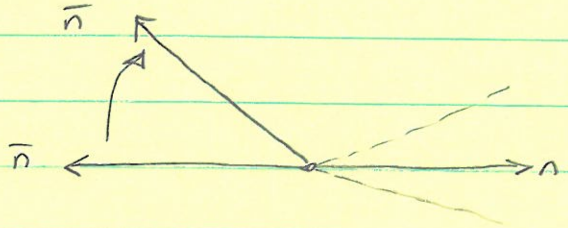
eg in $\mathcal{L}_{qq}^{(0)}$: had $\not{\bar{n}} \frac{1}{i\bar{n}\cdot D}$, $\not{\bar{n}}\, n\cdot D$ ✓

[no $\not{\bar{n}}\ \bar{n}\cdot D$]

$\boxed{\text{Type I \& II}}$



type-I



type-II

We only can care about restoring Lorentz Inv, for the set of fluctuations described by SCET

Vector $p^\mu = \frac{n^\mu}{2}\bar{n}\cdot p + \frac{\bar{n}^\mu}{2} n\cdot p + p_\perp^\mu$ is invariant to choice for decomposition

→ implies transformations for $p_\perp^\mu$ to compensate $n, \bar{n}$'s.

Find

type-I	type-II
$n \to n + \Delta_\perp$	$\bar{n} \to \bar{n} + \epsilon_\perp$
$n\cdot D \to n\cdot D + \Delta_\perp\cdot D_\perp$	$n\cdot D \to n\cdot D$
$D_\perp^\mu \to D_\perp^\mu - \frac{\Delta_\perp^\mu}{2}\bar{n}\cdot D - \frac{\bar{n}^\mu}{2}\Delta^\perp\cdot D^\perp$	$D_\perp^\mu \to D_\perp^\mu - \frac{\epsilon_\perp^\mu}{2} n\cdot D - \frac{n^\mu}{2}\epsilon_\perp\cdot D^\perp$
$\bar{n}\cdot D \to \bar{n}\cdot D$	$\bar{n}\cdot D \to \bar{n}\cdot D + \epsilon_\perp\cdot D_\perp$
$\xi_n \to \left[1 + \frac{\not{\Delta}^\perp \not{\bar{n}}}{4}\right]\xi_n$	$\xi_n \to \left[1 + \frac{\not{\epsilon}^\perp}{2}\frac{1}{i\bar{n}\cdot D} i\not{D}_\perp\right]\xi_n$
$W \to W$	$W \to \left[\left(1 - \frac{1}{i\bar{n}\cdot D} i\epsilon^\perp\cdot D^\perp\right) W\right]$

[I write D^μ everywhere, but your free to think of it as $\mathcal{P}^\mu$ or $i\partial^\mu$ with appropriate gauging from symmetry ①] eg. $\delta^{(I)}\left(\bar\xi_n\, i\slashed{D}_{n\perp}\frac{1}{i\bar n\cdot D_n}\, i\slashed{D}_{n\perp}\frac{\bar{\slashed{n}}}{2}\xi_n\right) = -\,\bar\xi_n\, i\Delta^\perp\cdot D^\perp\,\frac{\bar{\slashed{n}}}{2}\xi_n$

$\delta^{(I)}\left(\bar\xi_n\, in\cdot D\,\frac{\bar{\slashed{n}}}{2}\xi_n\right) = \underbrace{+\,\bar\xi_n\, i\Delta^\perp\cdot D^\perp\,\frac{\bar{\slashed{n}}}{2}\xi_n}$

sum = 0, so connected by RPI, no non-trivial Wilson coefficient btwn them

type-II rules out the $\bar\xi_n\, iD_{n\perp}^\mu\frac{1}{i\bar n\cdot D_n}\, iD_{n\perp\mu}\frac{\bar{\slashed{n}}}{2}\xi_n$ operator in $\mathcal{L}_{\xi\xi}^{(0)}$.

So

$$\boxed{\mathcal{L}_{\xi\xi}^{(0)} = \bar\xi_n\left[in\cdot D + i\slashed{D}_{\perp n}\frac{1}{i\bar n\cdot D_n}\, i\slashed{D}_{n\perp}\right]\frac{\bar{\slashed{n}}}{2}\xi_n}$$

is unique LO $\mathcal{L}$ for ξ_n by p.c., gauge inv, RPI

MORE RPI: Freedom in the label + residual decomposition

$\bar n\cdot(p_\ell + p_r)$, $p_{\ell\perp}^\mu + p_{r\perp}^\mu$

$\mathcal{P}_\mu \to \mathcal{P}_\mu + \beta_\mu$, $i\partial_\mu \to i\partial_\mu - \beta_\mu$ with $n\cdot\beta = 0$

$\xi_{n,p}(x) \to e^{i\beta\cdot x}\,\xi_{n,p+\beta}(x)$

Connects: $\mathcal{P}^\mu + i\partial^\mu$ ie leading & subleading Wilson coefficients in $\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \cdots$ and in operators $C^{(0)}O^{(0)} + C^{(1)}O^{(1)} + \cdots$

Gauge This $n\cdot\mathcal{P} = 0$, so just $in\cdot\partial \to in\cdot D$

$in\cdot D \to U_c\, in\cdot D\, U_c^\dagger$, $U_u\, in\cdot D\, U_u^\dagger$ (with our gauge trnsfms)

Also

$iD_{n\perp}^\mu \to U_c\, iD_{n\perp}^\mu\, U_c^\dagger$ or $U_u\, iD_{n\perp}^\mu\, U_u^\dagger$

$i\bar n\cdot D_n \to U_c\, i\bar n\cdot D_n\, U_c^\dagger$ or $U_u\, i\bar n\cdot D_n\, U_u^\dagger$

$iD_{us}^\mu \to iD_{us}^\mu$ or $U_u\, iD_{us}^\mu\, U_c^\dagger$

∴ simplest idea $\left.\begin{array}{l} iD_{n\perp}^\mu + iD_{us\perp}^\mu \\ i\bar n\cdot D_n + i\bar n\cdot D_{us}\end{array}\right\}$ doesn't work due to lack of trnsfm of iD_{us}^μ under U_c

The object that can compensate is $W \to U_c W$.
The unique result that gauges $\mathcal{P}^\mu + i\partial^\mu$ (with our strictly LO, homogeneous transfm) is

$$i D_{n\perp}^{\mu} + W \, i D_{\perp}^{us\,\mu} W^{+} \equiv i D_{\perp}^{\mu}$$
$$i\bar{n}\cdot D_n + W \, i\bar{n}\cdot D_{us} W^{+} \equiv i\bar{n}\cdot D$$

} combined result of RPI & gauge inv.

↖ the extra terms from W, W^{+} induce the $+\cdots$ in our earlier $A^\mu = A_n^\mu + A_{us}^\mu + \cdots$ expression

eg

from the $\bar{\xi}_n \, i\cancel{D}_{\perp n} \frac{1}{i\bar{n}\cdot D_n} i\cancel{D}_{\perp n} \frac{\bar{\cancel{n}}}{2} \xi_n$ term in $\mathcal{L}_{\xi\xi}^{(0)}$ we get

$$\mathcal{L}_{\xi\xi}^{(1)} = (\bar{\xi}_n W) \, i\cancel{D}_{\perp}^{us} \frac{1}{\bar{\mathcal{P}}} (W^{+} i\cancel{D}_{\perp n} \xi_n) + (\bar{\xi}_n \, i\cancel{D}_{\perp n} W) \frac{1}{\bar{\mathcal{P}}} \, i\cancel{D}_{\perp}^{us} (W^{+} \xi_n)$$

which is U_c & U_{us} gauge invariant & has no Wilson Coeff.

Like HQET, RPI also connects Wilson Coeff of leading & λ-suppressed external currents

Extension to more collinear fields — for >1 energetic hadron or >1 energetic jet

$$\sum_n \mathcal{L}_n^{(0)} = \sum_n \left[\mathcal{L}_{\xi_n \xi_n}^{(0)} + \mathcal{L}_{A_n}^{(0)} \right]$$

the sum is over inequivalent RPI equivalence classes

For $n_1, n_2, n_3, \ldots$ the collinear modes are distinct only if $n_i \cdot n_j \gg \lambda^2$ for $i \neq j$

eg. $p_2 = Q n_2$ $\quad n_1 \cdot p_2 = Q\, n_1 \cdot n_2 \sim \lambda^2$ if $n_1 \cdot n_2 \sim \lambda^2$

but then P_2 is n_1-collinear. So n_2 is within RPI equivalence class defined by $[n_1]$

All the things we derived with 1-collinear direction get repeated when we have more than one

- Collinear Gauge trnsfmo: $\mathcal{U}_{n_1}, \mathcal{U}_{n_2}, \ldots$
- RPI: separate invariance for $\{n_1, \bar{n}_1\}$
  $\{n_2, \bar{n}_2\}$ etc

here there is no simple connection to overall Lorentz Trnsfm (more like a type of Lorentz inv in each $[n_i]$ sector)

- Matching calculations generate Wilson lines

eg. $e^+e^- \to \gamma^* \to$ two-jets

$$J^\mu = \bar{\psi}\gamma^\mu\psi \quad \to \quad J^\mu_{SCET} = \underbrace{\underbrace{(\bar{\xi}_{n_1} W_{n_1})}_{n_1 \text{ gauge inv}} \gamma^\mu \underbrace{(W^\dagger_{n_2} \xi_{n_2})}_{n_2 \text{ gauge inv}}}_{\text{usoft gauge inv}}$$

$$\text{here} \quad \left.\begin{aligned} W_{n_1} &= W_{n_1}[\bar{n}_1 \cdot A_{n_1}] \\ W_{n_2} &= W_{n_2}[\bar{n}_2 \cdot A_{n_2}] \end{aligned}\right\} \text{generated by integrating out offshell } p^2 \sim Q^2 \text{ lines}$$

Final Comment on Discrete Symmetries: $\quad n = (1,0,0,1)$, $\bar{n} = (1,0,0,-1)$

$C^{-1}\xi_{n,p}C = -[\bar{\xi}_{n,-p}\mathcal{C}]^T \qquad p = (p^+, p^-, p^\perp)$

$P^{-1}\xi_{n,p}P^{-1} = \gamma_0\, \xi_{\bar{n},\tilde{p}}(x_P) \qquad \tilde{p} = (p^-, p^+, -p^\perp)$

↑ swaps role $n \leftrightarrow \bar{n}$

$T^{-1}\xi_{n,p}T = T\, \xi_{\bar{n},\tilde{p}}(x_T) \qquad x_P = (x^-, x^+, -x^\perp)$

$x_T = (-x^-, -x^+, x^\perp)$

Study $\mathcal{L}_{\xi\xi}^{(0)}$

① Propagator

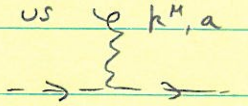
$$\frac{i\bar{n}\!\!\!/}{2}\,\frac{\theta(\bar n\cdot p)}{n\cdot p+\frac{p_\perp^2}{\bar n\cdot p}+i\epsilon}+\frac{i\bar n\!\!\!/}{2}\,\frac{\theta(-\bar n\cdot p)}{+n\cdot p+\frac{p_\perp^2}{\bar n\cdot p}-i\epsilon}=\frac{i\bar n\!\!\!/}{2}\,\frac{\bar n\cdot p}{n\cdot p\,\bar n\cdot p+p_\perp^2+i\epsilon}$$

✓ expn. of QCD

particles $\bar n\cdot p>0$ anti $\bar n\cdot p<0$

② Interactions

- for usoft gluons, only $n\cdot A_{us}$ at LO



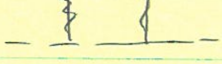
$$= i g T^a n^\mu \frac{\bar n\!\!\!/}{2}$$

& only sees $n\cdot k$ usoft momentum (multipole expn.)

$$\frac{\bar n\cdot p}{\bar n\cdot p\, n\cdot(p+k)+p_\perp^2+i\epsilon}=\frac{\bar n\cdot p}{\bar n\cdot p\, n\cdot k+p^2+i\epsilon}$$

$$\underset{\text{on-shell}}{=}\frac{\bar n\cdot p}{\bar n\cdot p\, n\cdot k+i\epsilon}$$

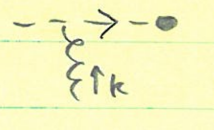
(Compare Collinear Gluon

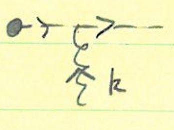


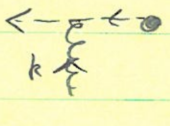
$$\frac{\bar n\cdot(p+q)}{(p+q)^2+i\epsilon}$$)

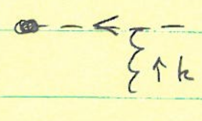
Propagator reduces to eikonal approx when appropriate

$\bar n\cdot p>0$ $\bar n\cdot p<0$





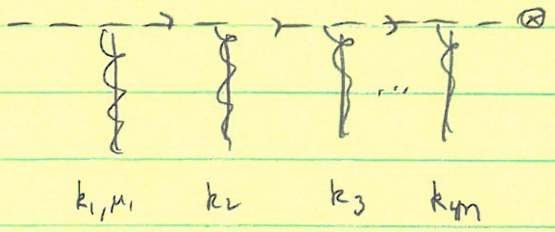




$$\frac{n^\mu}{n\cdot k+i\epsilon}\qquad\frac{n^\mu}{-n\cdot k+i\epsilon}\qquad\frac{n^\mu}{-n\cdot k-i\epsilon}\qquad\frac{n^\mu}{n\cdot k-i\epsilon}$$

Usoft - Collinear Factorization

Consider



$$= \Gamma \sum_m \sum_{\text{perms}} \frac{(-g)^m \, n\cdot A^{a_1} \cdots n\cdot A^{a_m} T^{a_m} \cdots T^{a_1}}{n\cdot k_1 \; n\cdot(k_1+k_2) \cdots n\cdot(\Sigma k_i)} * u_n$$

on-shell so $\dfrac{1}{n\cdot k + \frac{p^2}{\bar{n}\cdot p}} \to \dfrac{1}{n\cdot k}$

Motivates us to consider a field redefinition

$$\xi_{n,p}(x) = Y(x)\, \xi_{n,p}^{(0)}(x) \qquad A_{n,p} = Y A_{n,p}^{(0)} Y^\dagger$$

↑ adjoint version

$$Y(x) = P \exp\left(ig \int_{-\infty}^{0} ds \; n\cdot A_{us}^{a}(x+ns)\, T^a \right)$$

$$n\cdot D\, Y = 0 \quad , \quad Y^\dagger Y = 1 \qquad \text{find} \quad W = Y W^{(0)} Y^\dagger$$

$$\mathcal{L}_{\xi\xi}^{(0)} = \bar{\xi}_{n,p'} \frac{\bar{n}\!\!\!/}{2} \left[in\cdot D + \cdots \right] \xi_{n,p}$$

$$= \bar{\xi}_{n,p'}^{(0)} \frac{\bar{n}\!\!\!/}{2} \left[Y^\dagger in\cdot D_{us} Y + Y^\dagger (Y g n\cdot A_n Y^\dagger) Y + \cdots \right] \xi_{n,p}$$

$$= \bar{\xi}_{n,p'}^{(0)} \frac{\bar{n}\!\!\!/}{2} \left[\underbrace{in\cdot\partial + g n\cdot A_n}_{in\cdot D_c} + \cdots \right] \xi_{n,p}$$

↑ all $n\cdot A_{us}$'s disappear!

True for gluon action too

$$\mathcal{L}(\xi_{n,p}, A_{n,q}^{\mu}, n\cdot A_{us}) = \mathcal{L}(\xi_{n,p}^{(0)}, A_{n,q}^{(0)\mu}, 0)$$

Interactions don't disappear, but are moved out of L.O. $\mathcal{L}$ and into currents eg 1 $J_1 = \bar{\xi}_n W \Gamma h_v = \bar{\xi}_n^{(0)} Y^\dagger Y W^{(0)} Y^\dagger \Gamma h_v$

$= (\bar{\xi}_n^{(0)} W^{(0)}) \Gamma (Y^\dagger h_v)$

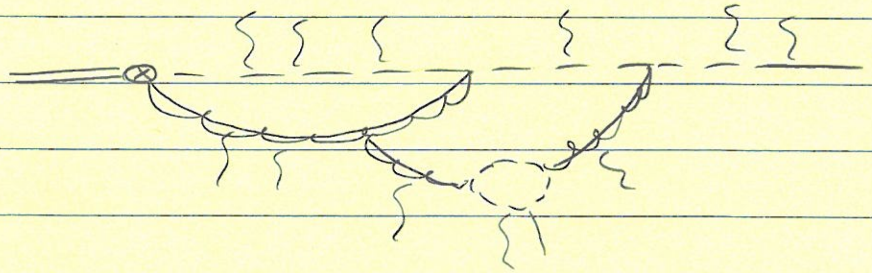
eg 2 $J_2 = (\bar{\xi}_{n_1} W_{n_1}) \Gamma (W_{n_2}^\dagger \xi_{n_2}) = (\bar{\xi}_{n_1}^{(0)} W_{n_1}^{(0)}) (Y_{n_1}^\dagger Y_{n_2}) \Gamma (W_{n_2}^{(0)\dagger} \xi_{n_2}^{(0)})$

eg 3 collinear parts are global color singlet set $n_1 = n_2$ above

$J_3 = (\bar{\xi}_n W_n) \Gamma (W_n^\dagger \xi_n) = (\bar{\xi}_n^{(0)} W_n^{(0)}) (\cancel{Y}^\dagger \cancel{Y}) \Gamma (W_n^{(0)\dagger} \xi_n^{(0)})$

Quite powerful, this "BPS field redefinition" sums an ∞ class of diagrams

in eg 1



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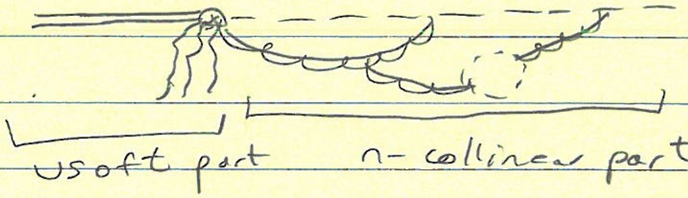
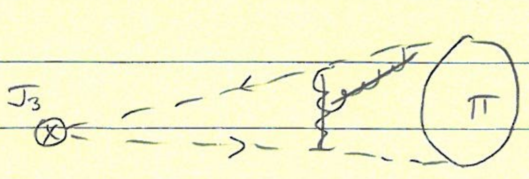


in eg 2 usoft gluons decouple at LO from any graph
This "color transparency"



- usoft gluons decouple from energetic partons in a color singlet state
- they just "see" overall color singlet due to the multipole expansion

127.5

U_c & U_{us} transformations post field redefinition

Our Logic with the $\xi_n \to Y \xi_n$ and $A_n \to Y A_n Y^\dagger$ field redefinitions is that they allow us to express in a simple way some of the consequences of dynamics <u>within</u> the EFT

Nevertheless, after the field redefinition we see that all operators become products of collinear & usoft blocks of fields, so its natural to ask about gauge symmetries that act seperately within those blocks. They can be derived in the following way:

<u>Original</u> U_c

$\xi_n \to U_c \xi_n$

$A_n^\mu \to U_c \left(A_n^\mu + \frac{i}{g} n\cdot\mathcal{D}_{us}^\mu \right) U_c$

$q_{us} \to q_{us}$

$A_{us}^\mu \to A_{us}^\mu$

U_{us}

$\xi_n \to U_{us} \xi_n$

$A_n \to U_{us} A_n U_{us}^\dagger$

$q_{us} \to U_{us} q_{us}$

$A_{us}^\mu \to U_{us} \left(A_{us}^\mu + \frac{i}{g} \partial^\mu \right) U_{us}^\dagger$

Consider $U_c^{(0)}$ defined by $U_c^{(0)}(x) = Y^\dagger(x)\, U_c(x)\, Y(x)$

$U_c = Y(x)\, U_c^{(0)}(x)\, Y^\dagger(x)$

- $\xi_n(x) = Y(x)\, \xi_n^{(0)}(x) \xrightarrow{U_c} U_c \xi_n = U_c Y(x) \xi_n^{(0)} = Y(x) U_c^{(0)} \xi_n^{(0)}$

  $\xrightarrow{U_{us}} U_{us} \xi_n = U_{us} Y(x) \xi_n^{(0)}$

  taking $Y(x) \to U_{us} Y(x)$ (so $U_{us}(-\infty) = 1$, distinguished from global)

  we find $\xi_n^{(0)} \xrightarrow{U_c^{(0)}} U_c^{(0)} \xi_n^{(0)}$

  $\xi_n^{(0)} \xrightarrow{U_{us}} \xi_n^{(0)}$.

- similarly

  acts in adjoint rep: $n\cdot\mathcal{D}_{us}^\mu Y_{adj} = 0$

  $$U_c \left(A_n^\mu + \frac{i}{g} n\cdot\mathcal{D}_{us}^\mu \right) U_c^\dagger = Y(x)\, U_c^{(0)} Y^\dagger(x) \left[Y(x) A_n^{(0)} Y^\dagger(x) + \frac{i}{g} n\cdot\mathcal{D}_{us}^\mu \right] Y(x) U_c^{(0)} Y^\dagger(x)$$

  $$= Y(x) \left[U_c^{(0)} \left[A_n^{(0)\mu} + \frac{i}{g} \partial^\mu \right] U_c^{(0)\dagger} \right] Y^\dagger(x)$$

  so $A_n^{(0)\mu} \xrightarrow{U_c^{(0)}} U_c^{(0)} \left[A_n^{(0)\mu} + \frac{i}{g} \partial^\mu \right] U_c^{(0)\dagger}$

  $A_n^{(0)\mu} \xrightarrow{U_{us}} A_n^{(0)\mu}$

But <u>note</u> that this teaches us nothing new

What about Wilson Coefficients?

have $C(\bar{P},\mu)$ ie depend on large momenta picked out by label operator $\bar{P} \sim \lambda^0$

eg. $C(-\bar{P},\mu)\,(\bar{\xi}_n W)\,\Gamma h_v = (\bar{\xi}_n W)\,\Gamma h_v\, C(\bar{P}^+)$

<u>must</u> act on product $(\bar{\xi} W)$ since only momentum of this combination is gauge invariant

Write $(\bar{\xi} W)\Gamma h_v C(\bar{P}^+) = \int d\omega\, C(\omega,\mu)\,[(\bar{\xi}W)\,\delta(\omega-\bar{P}^+)\,\Gamma h_v]$

$$= \int d\omega\, \underset{\uparrow}{C(\omega,\mu)}\ \underset{\uparrow}{O(\omega,\mu)}$$

convolution (as promised) of "C" and collinear "O"

Hard-Collinear Factorization

Recall defn of W, $i\bar{n}\cdot D_c W = 0$, $W^\dagger W = 1$

as operator
$$i\bar{n}\cdot D_c\, W = W\bar{P}$$
$$i\bar{n}\cdot D_c = W\bar{P}W^\dagger$$
$$(i\bar{n}\cdot D_c)^k = W\bar{P}^k W^\dagger$$

$$f(i\bar{n}\cdot D_c) = \underset{\uparrow}{W}\ \underset{\uparrow}{f(\bar{P})}\ \underset{\uparrow}{W^\dagger}$$

trades $\bar{n}\cdot A \to W$

Part of collin Op. $p^2 \sim \lambda^2 Q^2$

hard coefficient

$$= \int d\omega\ f(\omega)\ W\,\delta(\omega-\bar{P})\,W^\dagger$$

In general we can define a convenient set of (collinear gauge invariant) <u>building blocks</u> for operators:

- $\chi_n \equiv (W_n^\dagger \xi_n)$ "quark jet-field"

  $\chi_{n,\omega} \equiv \delta(\omega - \bar{\mathcal{P}})(W_n^\dagger \xi_n)$

  operators $\int d\omega_1\, d\omega_2\, C(\omega_1,\omega_2)\, \bar\chi_{n,\omega_1} \Gamma \chi_{n,\omega_2}$ etc.

- $g\,\mathcal{B}_{n\perp}^\mu = \Big[\frac{1}{\bar{\mathcal{P}}} W_n^\dagger [i\bar n\cdot D_n, iD_{n\perp}^\mu] W_n \Big] = g A_{n\perp}^\mu + \cdots$

  $\uparrow$ derivatives act only inside $[\cdots]$

  "gluon jet-field" for two physical gluon-pol.

  $\mathcal{B}_{n\perp,\omega}^\mu = [\, \mathcal{B}_{n\perp}^\mu\, \delta(\omega - \bar{\mathcal{P}}^\dagger)\,]$

  $\uparrow$ convention/choice, acts left inside $[\cdots]$

<u>Building Blocks</u>

All operators can be constructed solely from $\{\chi_n, \mathcal{B}_{n\perp}^\mu, \mathcal{P}_\perp^\nu\}$ + usoft fields & D_{us}^μ.

① Let $i\mathcal{D}_n^\mu = W_n^\dagger\, iD_n^\mu\, W_n$ where iD_n^μ has $\left\{\begin{matrix}\bar{\mathcal{P}}\\ \mathcal{P}_\perp\\ in\cdot\partial\end{matrix}\right\} + g\left\{\begin{matrix} n\cdot A_n \\ A_n^\perp \\ \bar n\cdot A_n\end{matrix}\right\}$

$\bar n\cdot i\mathcal{D}_n = \bar{\mathcal{P}}$

$i\mathcal{D}_n^{\perp\mu} = \mathcal{P}_\perp^\mu + g\,\mathcal{B}_{n\perp}^\mu$, $in\cdot\mathcal{D}_n = in\cdot\partial + \underbrace{g\,n\cdot\mathcal{B}_n}_{\text{analogous to defn } \mathcal{B}_{n\perp}^\mu}$

derivatives $\bar{\mathcal{P}}\,\chi_{n,\omega} = \omega\,\chi_{n,\omega}$ can be absorbed in coefficients

$in\cdot\partial\,\chi_n = -(g\,n\cdot\mathcal{B}_n)\chi_n - i\mathcal{D}\!\!\!/_{n\perp}\frac{1}{\bar{\mathcal{P}}}\, i\mathcal{D}\!\!\!/_{n\perp}\chi_n$ equation of motion for χ_n

$\uparrow$ remove $in\cdot\partial$'s

$in\cdot\partial\,\mathcal{B}_{n\perp}^\mu = \ldots$ eqtn of motion

Note: $i\mathcal{D}_{n\perp}^\mu = W_n^\dagger(\mathcal{P}_\perp^\mu + gA_{n\perp}^\mu)W_n = \mathcal{P}_\perp^\mu + [W_n^\dagger iD_\perp^\mu W_n] = \mathcal{P}_\perp^\mu + \Big[\frac{1}{\bar{\mathcal{P}}}\bar{\mathcal{P}}\, W^\dagger iD_\perp^\mu W\Big]$

$= \mathcal{P}_\perp^\mu + \Big[\frac{1}{\bar{\mathcal{P}}} W_n^\dagger i\bar n\cdot D_n D_{n\perp}^\mu W_n\Big] = \mathcal{P}_\perp^\mu + \Big[\frac{1}{\bar{\mathcal{P}}} W_n^\dagger [i\bar n\cdot D_n, iD_\perp^\mu] W_n\Big]$

(2) $\omega\,(g\,\bar n\cdot \mathcal{B}_n)_\omega = 2\,P^\perp_\nu\, g\, \mathcal{B}^\nu_{\perp,\omega} + \dots$

also part of gluon e.o.m.

<u>All</u> other collinear operators, $W_n^\dagger\,[iD^\mu_{n\perp}, iD^\nu_{n\perp}]\,W_n, \dots$
are reducible to $\{\chi_n, \mathcal{B}_\perp^\mu, P_\perp^\nu\}$

(3) Do need usoft derivatives, Field strengths, q_{us}, etc
Statement of RPI becomes

$i\mathcal{D}^\mu_{n\perp} + iD^\mu_{us\perp}\,, \quad \bar{\mathcal{P}}_n + i\bar n\cdot D_{us}$

(equiv. to earlier, but W's around collinear D^μ here, rather than usoft)

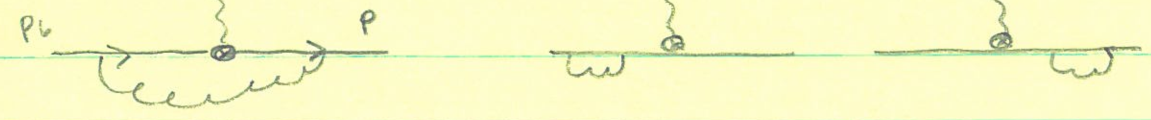
<u>Loops, IR divergences, Matching & Running</u>

Consider heavy-to-light current for $b\to s\gamma$

$J^{QCD} = \bar s\,\Gamma\, b \qquad \Gamma = \sigma^{\mu\nu} P_R F_{\mu\nu}$

$J^{SCET}_{LO} = (\bar\xi W)\,\Gamma\, h_v\, C(\bar P^\dagger)$ [pre Y-field redefn]

QCD graphs at one-loop, take $p^2 \neq 0$ to regulate IR of collin quark
use Feyn. Gauge



$$= -\bar u_s \Gamma u_b \frac{\alpha_s C_F}{4\pi}\left[\ln^2\left(\frac{-p^2}{m_b^2}\right) + 2\ln\left(\frac{-p^2}{m_b^2}\right) + \dots\right]$$

$f(p\cdot p_b/m_b^2)$, IR finite

$$Z_{\psi b} = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} + 3\ln\frac{\mu^2}{m_b^2} + \dots\right]$$

full z's (not $\overline{MS}$) match for S-matrix

$$Z_{\psi s} = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} - \ln \frac{p^2}{\mu^2}\right]$$

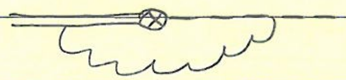
$$Z_{tensor} = 1 + \frac{\alpha_s C_F}{4\pi}\,\frac{1}{\epsilon}$$ ← Tensor current in QCD not conserved

$$\text{sum} = \bar{u}_s \Gamma u_b \left[1 - \frac{\alpha_s C_F}{4\pi} \left\{ \ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2} \ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}} + \cdots \right\} \right]$$

$\boxed{\text{SCET}_I}$

Use Feyn. Gauge everywhere

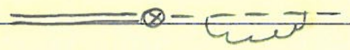
Usoft - loops



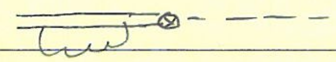
$$\int \frac{\bar{d}^d k \quad n\cdot v}{(v\cdot k + i0)(k^2 + i0)(n\cdot k + p^2/\bar{n}\cdot p + i0)}$$

$$= -\bar{u}_n \Gamma u_v \frac{\alpha_s C_F}{4\pi} \left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} \ln\left(\frac{\mu\, \bar{n}\cdot p}{-p^2 - i0}\right) + 2 \ln^2\left(\frac{\mu\, \bar{n}\cdot p}{-p^2 - i0}\right) + \frac{3\pi^2}{4} \right]$$

Note: $p^2/\bar{n}\cdot p \sim \lambda^2$ so logs $\mathcal{O}(1)$ for $\mu \sim \lambda^2$ usoft scale

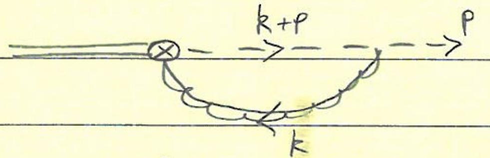


$\propto n_\mu n^\mu = 0$ in Feyn Gauge , $Z_\xi^{us} = 0$



$$Z_{HQET} = 1 + \frac{\alpha_s C_F}{4\pi} \left[\frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}} \right]$$

Collinear graphs



$$= \bar{u}_n \Gamma u_v \sum_{\substack{k_\ell \neq 0 \\ k_\ell \neq -p_\ell}} \int \frac{\bar{d}^d k_r \quad n\cdot\bar{n} \quad \bar{n}\cdot(p+k)}{(\bar{n}\cdot k)(k^2)(k+p)^2}$$

all +i0

$n\cdot k_r,\ \bar{n}\cdot k_\ell,\ k_\ell^\perp$

$n\cdot p_r,\ \bar{n}\cdot p_\ell,\ p_\ell^\perp$

each momentum has $k = (k_\ell, k_r)$, label & residual

Label & residual ensure we have LO piece (important for mixed collinear & usoft graphs)

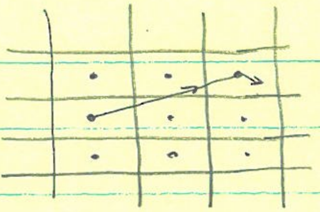
- But now we want to turn $\sum_{k_\ell} \int \bar{d}^d k_r$ back into $\int \bar{d}^d k_\ell$ to do loop integration

Claim if we ignore $\begin{matrix} k_\ell \neq 0 \\ k_\ell \neq -p_\ell \end{matrix}$ restrictions we just get

$$\int \frac{\bar{d}^d k \quad n\cdot\bar{n} \quad \bar{n}\cdot(p+k)}{(\bar{n}\cdot k)(k^2)(k+p)^2}$$

k_r^+ is only + loop momentum. Worry about: $k_\ell^\perp, k_r^\perp$ & k_ℓ^-, k_r^-

call grid



was like Wilsonian EFT (with finite edges)

Continuum EFT: each grid point specifies an ∞-space of residual momenta ($k_r^\mu \in \mathbb{R}$), subject to rules

Ignore $k_\ell \neq 0$, $k_\ell \neq -p_\ell$,

i) $\sum_{k_\ell} \int dk_r = \int dk_\ell$ for $-$ & $\perp$ momenta (use 1-dim notation for simplicity)

ii) $\sum_{k_\ell} \int dk_r \, \underset{\uparrow}{F(k_\ell)} = \sum_{k_\ell} \int dk_r \, F(k_\ell + k_r) = \int dk_\ell \, F(\underset{\uparrow}{k_\ell})$

constant throughout each box

continuous dummy var.

- This is the (simplified version of) main rule for obtaining $\int dk_\ell$. For each *label* loop momentum k_ℓ, there will *always* be a corresponding k_r that we can absorb in this fashion.
- Recall that grid facilitated multipole expansion. For a purely collinear loop there is often no physical p_r^+, p_r^- flowing through it. In this case answer *must* reduce to what we get from considering $\int d^d k_n$

iii) $\sum_{k_\ell} \int dk_r \, (k_r)^j \, F(k_\ell) = 0$ for $j > 0$ integer

dim-reg type rule which maintains Lorentz-Inv in residual Space

iv) Ultrasoft external particles or loops give non-trivial ℓ_r^μ & hence residual momenta that we *can not* absorb

eg. $\sum_{k_\ell} \int dk_r \int d\ell_r \, F(k_\ell, \underset{\uparrow}{\ell_r}) = \int dk_\ell \int d\ell_r \, F(k_\ell, \ell_r)$

usoft propagator (say)

ignoring restrictions

$$\sum_{k_\ell} \int \frac{d^d k_r \; n\cdot\bar n \; \bar n\cdot(p_\ell + k_\ell)}{\bar n\cdot k_\ell \,(k_\ell^- k_r^+ + k_\ell^{\perp 2})\,\left((k_\ell^- + p_\ell^-)(k_r^+ + p_r^+) + (k_\ell^\perp + p_\ell^\perp)^2\right)}$$

$$= \int \frac{d^d k \; n\cdot\bar n \; \bar n\cdot(p+k)}{\bar n\cdot k \; k^2 \, (k+p)^2}$$

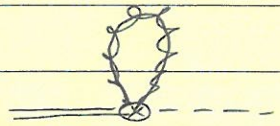
do as standard loop integral

$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-p^2} + \ln^2\left(\frac{\mu^2}{-p^2}\right) + 2\ln\left(\frac{\mu^2}{-p^2}\right) + 4 - \frac{\pi^2}{6}\right]$$

logs minimized for $\mu^2 \sim p^2$, collinear scale

collinear w.fn. renormalization, same as massless QCD

$$Z_\xi = 1 + \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \ln\frac{\mu^2}{-p^2}\right]$$



$\propto \bar n^2 = 0$ (Feyn.) ,

scaleless power-divergent $\rightarrow 0$

(cancels unphysical sing. for $\bar n\cdot(p+k) \rightarrow 0$, $k_\perp$ fixed in )

<u>Matching</u> Compare QCD & SCET, kinematics in $b \rightarrow s\gamma$ sets $p^- = m_b$

$$(\text{sum QCD})^{ren} = -\frac{\alpha_s C_F}{4\pi}\left[\underline{\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}}} + 2\ln\left(\frac{\mu^2}{m_b^2}\right) + \cdots\right]$$

$$(\text{sum SCET})^{bare} = -\frac{\alpha_s C_F}{4\pi}\left\{\underline{\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}}}\right.$$

$$\underbrace{-\frac{1}{\epsilon^2} - \frac{5}{2\epsilon} - \frac{2}{\epsilon}\ln\left(\frac{\mu}{m_b}\right)} \; \underbrace{- 2\ln^2\left(\frac{\mu^2}{m_b^2}\right) - \frac{3}{2}\ln\frac{\mu}{m_b}} + \cdots$$

— match these IR divergences

want to take care of this with UV renormalyzation of SCET,

Have to know $1/\epsilon$'s are UV.

difference gives matching for one-loop $C(m_b, \mu)$

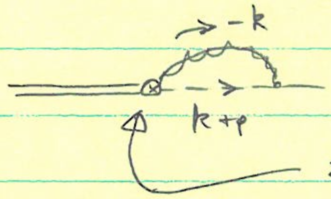
discuss later

Note ① $\sum_{\omega} C(\omega,\mu)\ \underbrace{\bar{\chi}_{n,\omega}}\ \Gamma\, h_v$

$(\bar{\xi}_n W)\ \delta_{\omega,\bar{p}^+}$ total momentum of $\bar{\xi}_n$ & W fixed as ω

so its always $\omega = p^-$ above

- non-trivial example



sum $= \bar{n}\cdot(k+p) + \bar{n}\cdot(-k) = \bar{n}\cdot p$

② Should be careful with $k_\ell \neq 0$, $k_\ell \neq -p_\ell$ (Zero-Bin's)
Collinear momenta have non-zero labels
When $k_\ell = 0$ gluon is usoft ($k_\ell = -p_\ell$ quark is usoft)
These restrictions avoid double counting in SCET fields and hence also in results for loop integrations

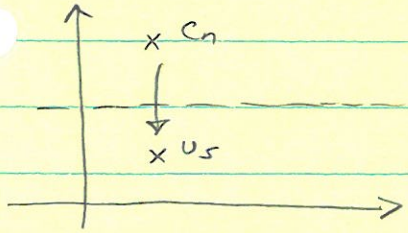
Rule ii) above with restrictions (encoded via propagators) is

$$\sum_{k_\ell \neq 0} \int dk_r\, F(k_\ell) = \sum_{k_\ell} \int dk_r\, F(k_\ell) - \int dk_r\, F^{k_\ell \to 0}(0)$$
$$= \sum_{k_\ell} \int dk_r\, F(k_\ell + k_r) - \int dk_r\, F^{k_\ell \to 0}(k_r)$$
$$= \int dk \left[F(k) - \underset{\uparrow}{F^{k_\ell \to 0}(k)} \right]$$

zero-bin subtraction term

$F^{k_\ell \to 0}(k)$ is defined by taking scaling limit $k_n^\mu \to k_{us}^\mu$
ie $k_n^\mu \sim \lambda^2$

and expanding to keep all subtractions that are same order in λ (dropping power suppressed terms, a "minimal subtraction")



subtraction ensures "c_n" has no non-trivial support in ultrasoft "us" region r eg. 

$$\int d^dk \left[\frac{n\cdot\bar n \;\; \bar n\cdot(k+p)}{\bar n\cdot k\,(k+p)^2\,k^2} - \frac{n\cdot\bar n \;\; \bar n\cdot p}{\bar n\cdot k\,(\bar n\cdot p\, n\cdot k + p^2)\,k^2} \right]$$

↑ scaling limit

$$= \frac{i}{16\pi^2}\left[\left(\frac{2}{\epsilon_{IR}\epsilon_{UV}} + \frac{2}{\epsilon_{IR}}\ln\frac{\mu^2}{-p^2} + \ln^2\frac{\mu^2}{-p^2} + \left(\frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}}\right)\ln\frac{\mu}{\bar n\cdot p} + \cdots \right) \right.$$

$$\left. - \left(\underbrace{\left(\frac{2}{\epsilon_{IR}} - \frac{2}{\epsilon_{UV}}\right)}_{\text{zero in pure-dim reg.}} \left(\frac{1}{\epsilon_{UV}} + \ln\frac{\mu^2}{-p^2} - \ln\frac{\mu}{\bar n\cdot p} \right) \right) \right]$$

subtraction :
- cancels $\bar n\cdot q \to 0$ IR singularity of first term,
- UV. divergences come from $\bar n\cdot q \to \infty$ & are indep. of IR regulator
- here $\epsilon_{IR} = \epsilon_{UV}$ and ignoring subtraction gives correct answer

- for other less inclusive calculations (eg. jet algorithms) or other regulators (eg. $\Omega_\perp^2 \le k_\perp^2 \le \Lambda_\perp^2$, $\Omega_-^2 \le (k^-)^2 \le \Lambda_-^2$) the subtraction is crucial to avoid double counting (get correct IR structure) & have UV div. indep of IR regulator.

Renormalization in SCET & Summing Sudakov Logs

our eg.

$$C^{bare} = C + (Z_C - 1)\,C$$

need counter term $\quad Z_C = 1 - \frac{\alpha_s(\mu)\,C_F}{4\pi}\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu}{\omega} + \frac{5}{2\epsilon} \right)$

where current with Wilson Coeff was

$$\int d\omega\, C(\omega)\, O(\omega) = \int d\omega\, C(\omega)\, \underbrace{\bar\chi_{n,\omega}}_{(\bar\xi W_n)\,\delta(\omega - \bar P^+)}\, \Gamma\, h_v$$

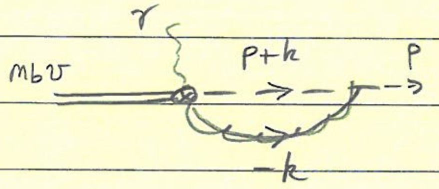
Running

In general we must be careful with integral over ω, which is the momentum of the product $(\bar{\xi}_n W)$.

But in our example ω is fixed by external Kinematics

- it does not involve loop momenta

non-trivial example



$$C(\bar{n}\cdot(p+k)+\bar{n}\cdot(-k)) = C(\bar{n}\cdot p)$$

- $m_b v = p_\gamma + p = E_\gamma \bar{n} + p$ so $\bar{n}\cdot p = m_b = \omega$

Anom dim $\quad \mu \, d/d\mu \, C^{\text{bare}} = 0 \Rightarrow \mu \frac{d}{d\mu} C(\omega,\mu) = \gamma_C(\omega,\mu)\, C(\omega,\mu)$

where

$$\gamma_C = -Z_C^{-1}\mu\frac{d}{d\mu} Z_C = \mu\frac{d}{d\mu}\frac{C_F\alpha_s(\mu)}{4\pi}\left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\ln\frac{\mu}{\omega}+\frac{5}{2\epsilon}\right)$$

$$= \frac{C_F\alpha_s(\mu)}{4\pi}\Big(\underbrace{-\frac{\cancel{2}}{\cancel{\epsilon}} - 4\ln\frac{\mu}{\omega} - 5}_{\text{from } \mu\, d/d\mu\, \alpha_s = -2\epsilon\alpha_s + \mathcal{O}(\alpha_s^2)} + \frac{\cancel{2}}{\cancel{\epsilon}}\Big)$$

$$= -\frac{\alpha_s(\mu)}{4\pi}\Big(\underset{\substack{\uparrow \\ \text{LL from} \\ \text{"cusp anom. dim"}}}{4C_F\ln\frac{\mu}{\omega}} + \underset{\substack{\uparrow \\ \text{part of NLL}}}{5C_F}\Big)$$

LL RGE

$$\mu\, d/d\mu\, C(\mu,\omega) = -\frac{\alpha_s(\mu)C_F}{\pi}\ln\left(\frac{\mu}{\omega}\right)C(\mu,\omega)$$

or

$$\frac{d\ln C(\mu,\omega)}{d\ln\mu} = -\frac{\alpha_s(\mu)C_F}{\pi}\ln\left(\frac{\mu}{\omega}\right)$$

Soln take boundary condition $C(\mu=\omega,\omega)=1$

"QED" α_s = fixed, $C_F=1$,

$$C(\mu,\omega) = \exp\left[\frac{-\alpha}{2\pi}\ln^2\left(\frac{\mu}{\omega}\right)\right]$$ Sudakov Exponential

related to restrictions we've placed on radiation with our operators (& to probability of evolving without branching in a parton shower)

QCD $$d\ln\mu = \frac{d\alpha_s}{\beta[\alpha_s]} = \frac{-2\pi}{\beta_0}\frac{d\alpha_s}{\alpha_s^2} + \ldots$$

$$\ln(\mu/\omega) = \frac{-2\pi}{\beta_0}\int_{\alpha_s(\omega)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\alpha_s^2}$$

$$\ln C(\mu,\omega) = -\frac{C_F}{\pi}\left(\frac{2\pi}{\beta_0}\right)^2\int_{\alpha_s(\omega)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\alpha_s^2}\,\alpha_s\int_{\alpha_s(\omega)}^{\alpha_s}\frac{d\alpha}{\alpha^2}$$

$$C(\mu,\omega) = \exp\left[\frac{-4\pi C_F}{\beta_0^2\,\alpha_s(\omega)}\left(\frac{1}{z}-1+\ln z\right)\right], \quad z=\frac{\alpha_s(\mu)}{\alpha_s(\omega)}$$

↑ running coupling effects

To discuss the order we're working to, look at series in

$$\ln C(\mu,\omega) \sim \underset{LL}{\alpha_s^k \ln^{k+1}} + \underset{NLL}{\alpha_s^k \ln^k} + \underset{NNLL}{\alpha_s^k \ln^{k-1}} + \ldots$$

What Coefficients do we need to compute?

	tree-level	one-loop	2-loop	3-loop
LL	matching	$1/\epsilon^2$	—	—
NLL	matching	$1/\epsilon$	$1/\epsilon^2$	—
NNLL		matching	$1/\epsilon$	$1/\epsilon^2$

↑ differs from our earlier single log resummation case

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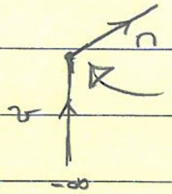
- Where is the "cusp" in "cusp anomalous dimension"?

$$J_{SCET} = (\bar{\xi}_n W_n)\, \Gamma\, (Y_n^\dagger h_v)$$

Here h_v has $\mathcal{L}_{HQET} = \bar{h}_v\, i v\cdot D\, h_v$ and coincides with a timelike Wilson line. $h_v = Y_v h_v^{(0)}$, $\mathcal{L}_{HQET} = \bar{h}_v^{(0)}\, i v\cdot\partial\, h_v^{(0)}$

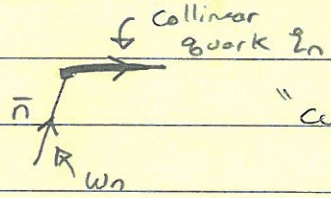
$Y_n^\dagger Y_v$ is



cusp is kink in Wilson line path

With light-like particle give a single $\ln(\mu/\omega)$ in anom. dimensions.

Also

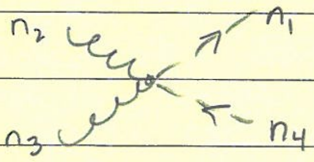


"cusp" where one part is not a Wilson line

- When will ω's be fixed by external kinematics?

If our operator only involves one building block (χ_n or ${}^{\sigma}\!B_{\perp n}^{\mu}$) for each collinear direction

eg $$\int d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4\;\; \bar{\chi}_{n_1,\omega_1}\, \Gamma\, {}^{\sigma}\!B^{\mu}_{\perp n_2,\omega_2}\, {}^{\sigma}\!B^{\perp}_{n_3,\omega_3\,\mu}\, \chi_{n_4,\omega_4}\;\; C(\omega_1,\dots,\omega_4)$$



again ω_i's only involve momenta external to collinear loops

eg. where its not true, same n in two χ_n's

$$\int d\omega_1\, d\omega_2\;\; \bar{\chi}_{n,\omega_1}\, \frac{\bar{n}\!\!\!/}{2}\, \chi_{n,\omega_2}\;\; C(\omega_1,\omega_2)$$

Here the ω_i's will involve loop momenta [one combination is not fixed by momentum conservation] and we'll get anom. dimension equations with integrals

$$\mu \frac{d}{d\mu} C(\mu, \omega) = \int d\omega' \gamma(\mu, \omega, \omega') \, C(\mu, \omega')$$

Indeed, the above operator is responsible for several classic evolution equations

DIS	Altarelli-Parisi (DGLAP) evolution for PDF
$\gamma^* \pi^0 \to \pi^0$	Brodsky - Lepage "
$\gamma^* p \to \gamma p'$	Deeply Virtual Compton Scattering

Lets see how this works for the parton dist'n

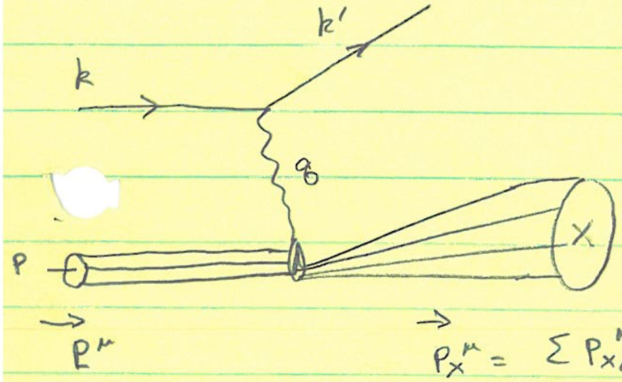
First we'll prove its the right operator by studying DIS factorization

(there is
no page (139) in
my notes)

$\boxed{\text{DIS}}$ A rich subject, only aspects related to QCD factorization are covered here using SCET

Refs: § 1.8 of Heavy Quark physics
Aneesh M.'s review: hep-ph/9204208
Bob J.'s review: hep-ph/9602236
paper: hep-ph/0202088 (for material below)

$e^- p \to e^- X$



$\vec{P}_X^\mu = \sum_{\substack{\text{all} \\ \text{hadrons}}} P_{X_i}^\mu$

$Q^2 \gg \Lambda^2$

$q^2 = -Q^2 \,, \quad x = \dfrac{Q^2}{2P\cdot q}$

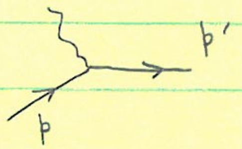
$P_X^\mu = P^\mu + q^\mu$

$P_X^2 = \dfrac{Q^2}{x}(1-x) + M_p^2$

regions

P_X^2	$(\frac{1}{x}-1)$	
$\sim Q^2$	~ 1	inclusive OPE
$\sim Q\Lambda$	$\sim \Lambda/Q$	endpt. region
$\sim \Lambda^2$	$\sim \Lambda^2/Q^2$	resonance region $e^- p \to e^- p'$ $\uparrow$ eg. excited state

Parton Variables



struck quark carries some fraction ξ of proton momentum

$\bar{n}\cdot p = \xi\, \bar{n}\cdot P$ ← we'll see how to formulate ξ in QCD

$p'^2 \simeq Q^2\left(\dfrac{\xi}{x} - 1\right)$

$\boxed{\text{Frames}}$

Breit Frame

$$q^\mu = \frac{Q}{2}\left(\bar{n}^\mu - n^\mu\right)$$

$$P^\mu = \frac{n^\mu}{2}\,\bar{n}\cdot P + \frac{\bar{n}^\mu M_p^2}{2\,\bar{n}\cdot P} = \frac{n^\mu}{2}\,\frac{Q}{x} + \dots \qquad \text{collinear}$$

$$P_X^\mu = \frac{\bar{n}^\mu}{2}\,Q + \frac{n^\mu}{2}\,\frac{Q(1-x)}{x} + \dots \qquad \text{hard}$$

Proton is made of collinear quarks and gluons

Rest Frame

$$P^\mu = \frac{m_p}{2}\left(n^\mu + \bar{n}^\mu\right) \qquad \text{soft}$$

$$q^\mu = \frac{\bar{n}^\mu}{2}\,\frac{Q^2}{m_p x} - \frac{n^\mu}{2}\,m_p x + \dots$$

$$P_X^\mu = \text{sum} \qquad \text{"collinear"} \quad P_X^2 \sim Q^2$$

Like $B \to X_c e \nu$ we can write cross-section in terms of leptonic & hadronic tensors

$$d\sigma = \frac{d^3k'}{2|\vec{k}'|}\,\frac{e^4}{sQ^4}\,L^{\mu\nu}(k,k')\,W_{\mu\nu}(P,q)$$

we'll look at spin-avg. case

$$W_{\mu\nu} = \frac{1}{\pi}\,\text{Im}\,T_{\mu\nu}$$

$$T_{\mu\nu} = \frac{1}{2}\sum_{\text{spin}} \langle p|\,\hat{T}_{\mu\nu}(q)\,|p\rangle$$

$$\hat{T}_{\mu\nu} = i\int d^4x\; e^{iq\cdot x}\; T\left[J_\mu(x)\,J_\nu(0)\right]$$

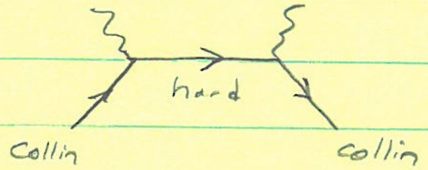
$\uparrow$

e.m. currents

$$T_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(x,Q^2) + \left(P_\mu + \frac{q_\mu}{2x}\right)\left(P_\nu + \frac{q_\nu}{2x}\right) T_2(x,Q^2)$$

satifies current conservation, P, C, T, etc.

Want imaginary part of forward scattering



First Match onto SCET ops.

at L.O. :

$$\hat{T}^{\mu\nu} = \frac{g_\perp^{\mu\nu}}{Q}\left(O_1^{(i)} + \frac{O_1^g}{Q}\right) + \frac{(n^\mu+\bar{n}^\mu)(n^\nu+\bar{n}^\nu)}{Q}\left(O_2^{(i)} + \frac{O_2^g}{Q}\right)$$

$O(\lambda^2)$ operators

flavor $= u, d, \ldots$

$$O_j^{(i)} = \bar{\xi}^{(i)}_{n,p'} W \frac{\bar{n}\!\!\!/}{2} C_j^{(i)}(\bar{P}_+, \bar{P}_-) W^\dagger \xi^{(i)}_{n,p}$$

$$O_j^{(g)} = \mathrm{tr}\left[W^\dagger B_\perp^\lambda W \, C_j^g(\bar{P}_+, \bar{P}_-) W^\dagger B_{\perp\lambda} W\right]$$

$$\text{where} \quad ig B_\perp^\lambda \equiv [i\bar{n}\cdot D_c, iD_{\perp c}^\lambda] \sim \lambda \sim \xi_n$$

$$\bar{P}_\pm = \bar{P}^\dagger \pm \bar{P}$$

$O_j^{(i)}$ will lead to quark, anti-quark p.d.f.'s

O_j^g " " gluon p.d.f.'s

Quark contribution in detail:

$$O_j^{(i)} = \int d\omega_1 d\omega_2 \, C_j^{(i)}(\omega_+, \omega_-) \left[\underset{\substack{\uparrow \\ \delta(\omega_1 - \bar{P}^\dagger)}}{(\bar{\xi}_n W)_{\omega_1}} \frac{\bar{n}\!\!\!/}{2} \underset{\substack{\uparrow \\ \delta(\omega_2 - \bar{P})}}{(W^\dagger \xi_n)_{\omega_2}} \right]$$

$$\omega_\pm = \omega_1 \pm \omega_2$$

coord space $f_{i/p}(z) = \int dy \; e^{-i2z\bar{n}\cdot p y} \langle p| \bar{\xi}(y)\, W(y,-y)\, \bar{n}\!\!\!/\, \xi(y) \,|p\rangle$

parton distn for quark i in proton p

$\bar{f}_{i/p}(z) = -f_{i/p}(-z)$ for anti-quark

mom. space $\langle p_n| (\bar{\xi}_n W)_{\omega_1} \bar{n}\!\!\!/ (W^\dagger \xi_n)_{\omega_2} |p_n\rangle = 4\bar{n}\cdot p \int_0^1 d\xi \; \delta(\omega_-)$

$$* \left[\delta(\omega_+ - 2\xi\bar{n}\cdot p) f_{i/p}(\xi) - \delta(\omega_+ + 2\xi\bar{n}\cdot p) \bar{f}_{i/p}(\xi) \right]$$

recall positive $\omega_1=\omega_2$ gives particles

negative $\omega_1=\omega_2$ gives anti-particles

$\underbrace{(\bar{\xi}_n W)_\omega}_{\text{a parton}} \bar{n}\!\!\!/ (W^\dagger \xi_n)$ is a number operator for collinear quarks with momentum ω

[If we tried to couple usoft or soft gluons to this op. its a singlet so they decouple, more later]

Charge Conjugation

$$C_j^{(i)}(-\omega_+, \omega_-) = -C_j^{(i)}(\omega_+, \omega_-)$$

$\omega_1 \Leftrightarrow -\omega_2$

- relates Wilson-Coeff for quarks & anti-quarks at operator level
- Only need matching for quarks
- δ-functions set $\omega_- = 0$, $\omega_+ = 2\xi\bar{n}\cdot p = 2Q\frac{\xi}{x}$

Relate basis

$$\frac{1}{\pi}\,\text{Im}\,T_1 = \int [d\omega] \;\; \frac{-1}{Q}\left(\frac{1}{\pi}\,\text{Im}\,C_1(\omega)\right) \langle O^{(i)}(\omega)\rangle$$

$$\frac{1}{\pi}\,\text{Im}\,T_2 = \int [d\omega] \;\left(\frac{4x}{Q}\right)^2 \frac{1}{Q}\,\frac{1}{\pi}\,\text{Im}\left(C_2(\omega) - \frac{C_1(\omega)}{4}\right) \;\; \langle O^{(i)}(\omega)\rangle$$

Define $H_j(z) = \frac{\text{Im}}{\pi}\, C_j(\underset{\omega_+}{2Qz}, \underset{\omega_-}{0}, Q^2, \mu^2)$ $\qquad z > 0$

(use charge conj for $H_j(z<0)$) $\qquad$ do $\omega_\pm$ with δ-functions

$$T_1(x, Q^2) = \frac{-1}{x}\int_0^1 d\xi \;\; H_1^{(i)}\left(\frac{\xi}{x}\right)\left[f_{i/p}(\xi) + \bar{f}_{i/p}(\xi)\right]$$

$$T_2(x, Q^2) = \frac{4x}{Q^2}\int_0^1 d\xi \left(4 H_2^{(i)}\left(\frac{\xi}{x}\right) - H_1^{(i)}\left(\frac{\xi}{x}\right)\right)\left[f_{i/p}(\xi) + \bar{f}_{i/p}(\xi)\right]$$

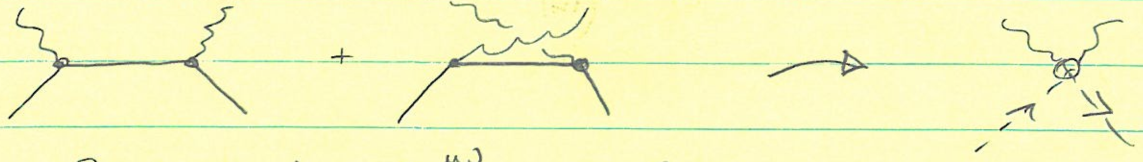
- this is factorization for DIS (to all order in α_s) into
  computable coefficients H_i
  universal non-pert. functions $f_{i/p}$, $\bar{f}_{i/p}$
  (show up in many processes)
- Coefficients C_j were dimensionless and can only have $\alpha_s(\mu)\ \ln(\mu/Q)$ dependence on Q
  → Bjorken scaling

[Analysis valid to LO in $\frac{\Lambda^2}{Q^2}$]

- $H_i(\mu)$ $f_{i/p}(\mu)$ traditionally this μ-dependence is called the "factorization-scale" $\mu = \mu_F$
  & one also has "renorm. scale" $\alpha_s(\mu = \mu_R)$

In SCET the μ is just the ren. scale in SCET. We have new UV. divergences associated with running of p.d.f., along with running for $\alpha_s(\mu)$.

- Tree Level Matching
  (upon which a lot of intuition is based)



find just $g_\perp^{\mu\nu}$ ie $C_2 = 0$

↳ Callan-Gross relation that $W_1/W_2 = Q^2/4x^2$

$$C_1(\omega_+) = \underset{\substack{\uparrow \\ \text{charges}}}{2e^2 Q_i^2} \left[\frac{Q}{(\omega_+ - 2Q) + i\epsilon} - \frac{Q}{-(\omega_+ + 2Q) + i\epsilon} \right]$$

$$H_1\left(\frac{\xi}{x}\right) = -e^2 Q_i^2 \; \delta\left(\frac{\xi}{x} - 1\right)$$

gives parton-model interpretation $\xi = x$

One-Loop Renormalization of PDF

one δ-function ↘ ↙ proton state, momentum P_n^-

$$f_q(\xi) = \langle P_n | \bar{\chi}_n(0) \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega}(0) | P_n \rangle \quad \text{where} \quad \xi = \frac{\omega}{P_n^-}$$

mass dimension $-1 \quad + \frac{3}{2} \quad + \frac{3}{2} \quad -1 \quad -1 \quad = 0$

λ dimension $-1 \quad +1 \quad +1 \quad -1 \quad = 0$

$$\frac{d^3p}{2E_p} = \frac{dp^-}{2p^-} d^2p_\perp \qquad \text{states:} \quad \underbrace{\langle P_n(p)|}_{\lambda^{-1}} \underbrace{P_n(p')\rangle}_{\lambda^{-1}} = \underbrace{2p^- \delta(p^- - p'^-)}_{\lambda^0} \underbrace{\delta^2(p_\perp - p_\perp')}_{\lambda^{-2}}$$

$$p^- = \underbrace{(p_\perp^2 + p_z^2)^{1/2}}_{E_p} + p_z$$

Loops can change ω (or ξ). $f_q(\xi)$ mixes with $f_q(\xi')$ which in general is what we expect for operators with same quantum #'s. Loops also mix parton types $i = q, g$

$$\underset{\uparrow \mu \text{ independent}}{f_i^{\text{bare}}(\xi)} = \int d\xi' \underset{\uparrow \frac{1}{\epsilon_{UV}} \,\&\, \alpha_s(\mu) \text{ in } \overline{MS}}{Z_{ij}(\xi, \xi')} \underset{\uparrow \text{UV finite, but IR div. encodes } \Lambda_{QCD} \text{ effects}}{f_j(\xi', \mu)}$$

gives

$$\mu \, d/d\mu \; f_i(\xi, \mu) = \int d\xi' \; \gamma_{ij}(\xi, \xi') \; f_j(\xi', \mu)$$

$$\gamma_{ij} \equiv - \int d\xi'' \; Z_{ii'}^{-1}(\xi, \xi'') \, \mu \frac{d}{d\mu} Z_{i'j}(\xi'', \xi')$$

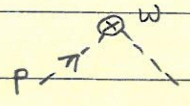
like matrix product in ξ vars. too.

1-loop: $Z_{ii'}^{-1}(\xi, \xi'') = \delta_{ii'} \, \delta(\xi - \xi'')$

$$\gamma_{ij}^{1\text{-loop}} = - \mu \, d/d\mu \left[Z_{ij}(\xi, \xi') \right]^{1\text{-loop}}$$

Calculations

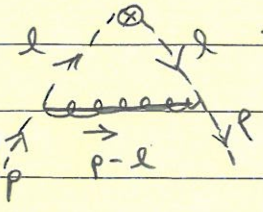
tree level



$$= \sum_{\text{spin}} \bar{u}_n \frac{\bar{n}\!\!\!/}{2} u_n \, \delta(\omega - p^-) = p^- \delta(\omega - p^-) = \delta(1 - \omega/p^-)$$

one-loop, use offshellness $p^2 = p^+ p^- \neq 0$ to regulate IR

ⓐ



$$= -i g^2 C_F \int \frac{d^d\ell \;\; p^- (d-2) \, \ell_\perp^2}{[\ell^2 + i0]^2 \, [(\ell - p)^2 + i0]} \, \delta(\ell^- - \omega) \, \mu^{2\epsilon} \frac{e^{\epsilon \gamma_E}}{(4\pi)^\epsilon} \quad \Leftarrow \text{after simplifying numerator}$$

$$= \frac{2 g^2 C_F}{(4\pi)^2} (1-\epsilon)^2 \, \Gamma(\epsilon) \, e^{\epsilon \gamma_E} \, (1-z) \, \theta(z) \, \theta(1-z) \left(\frac{A}{\mu^2} \right)^{-\epsilon}$$

$$= \frac{\alpha_s C_F}{\pi} (1-z) \theta(z) \theta(1-z) \left[\frac{1}{2\epsilon} - 1 - \frac{1}{2} \ln \frac{A}{\mu^2} \right], \qquad A \equiv -p^+ p^- z(1-z), \quad z = \omega/p^-$$

① 

\+ symmetric graph

two contractions

$$\bar\xi_n\, \underbrace{W\, \overset{\substack{\text{cut}\\ \downarrow}}{\delta}\, \overbrace{W^\dagger \xi_n}^{\text{virtual}}}_{\text{real}}$$

$$= 2ig^2C_F \int \frac{\bar{d}^d l\ \ \bar u_n \frac{\not{\bar n}}{2}\frac{\not n}{2}\,\bar n\cdot l\,\not{\bar n}\, u_n}{(l^- - p^-)\,(l^2)\,(l-p)^2}\ \Big[\overset{\text{real}}{\delta(l^- - \omega)} - \overset{\text{virtual}}{\delta(p^- - \omega)}\Big]$$

$$= \frac{C_F\alpha_s(\mu)}{\pi}\, e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\left[\frac{z\,\theta(z)\,\theta(1-z)}{(1-z)^{1+\epsilon}}\left(\frac{-p^-p^+z - i0}{\mu^2}\right)^{-\epsilon} - \delta(1-z)\left(\frac{-p^+p^- - i0}{\mu^2}\right)^{-\epsilon}\frac{\Gamma(2-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}\right]$$

Distribution Identity

$$\frac{\theta(1-z)}{(1-z)^{1+\epsilon}} = -\frac{\delta(1-z)}{\epsilon} + \mathcal{L}_0(1-z) - \epsilon\,\mathcal{L}_1(1-z) + \cdots$$

plus-functions $\quad \mathcal{L}_n(x) = \left[\frac{\theta(x)\ln^n x}{x}\right]_+$

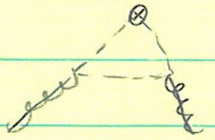
$$\int_0^1 dx\ \mathcal{L}_n(x) = 0\ , \qquad \int_0^1 dx\ \mathcal{L}_n(x)\,g(x) = \int_0^1 dx\ \frac{\ln^n x}{x}\,[g(x) - g(0)]$$

- $1/\epsilon^2$ terms in real & virtual terms cancel
- remaining $1/\epsilon$ is UV

$$= \frac{C_F\alpha_s(\mu)}{\pi}\left[\Big\{\delta(1-z) + z\,\theta(z)\,\mathcal{L}_0(1-z)\Big\}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^+p^-z - i0}\right) - z\,\mathcal{L}_1(1-z)\,\theta(z) + \delta(1-z)\left(2 - \frac{\pi^2}{6}\right)\right]$$

\+ symm.

$$= \delta(1-z)\,(Z_\psi - 1) = \frac{\alpha_s C_F}{\pi}\left[-\frac{1}{4\epsilon} - \frac{1}{4} - \frac{1}{4}\ln\left(\frac{\mu^2}{-p^+p^- - i0}\right)\right]\delta(1-z)$$

[We'll ignore  which mixes $\mathcal{O}^f_{\text{glue}}$ & $\mathcal{O}^f_{\text{quark}}$

this mixing is needed if $\mathcal{O}_{\text{quark}}$ is flavor singlet, but not for non-singlet like $\bar\xi_n(\cdots)\xi_n$

$$\text{sum of SCET graphs} = f^{\text{bare}}_{q/q}(z)^{\text{up to 1-loop}} = \delta(1-z)$$

$$+ \frac{C_F\,\alpha_s(\mu)}{\pi}\left[\left\{\frac{3}{4}\delta(1-z) + z\,\theta(z)\,\mathcal{L}_0(1-z) + \frac{(1-z)}{2}\theta(z)\theta(1-z)\right\}\left(\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{-p^+p^-}\right)\right) + \text{finite function of } z\right]$$

$$= \delta(1-z) + \frac{C_F\,\alpha_s(\mu)}{\pi}\left[\frac{1}{2}\left(\frac{1+z^2}{1-z}\right)_+ \frac{1}{\epsilon} + \cdots\right]$$

$$= \int d\xi'\; Z_{qq}(\xi,\xi')\; f_j(\xi',\mu)$$

$$= \underbrace{\int d\xi'\; \frac{1}{\xi'}\, Z_{qq}\left(\frac{\xi}{\xi'}\right)}_{\text{RPI-III invariant (ratios)}}\; f_j(\xi',\mu) \qquad \xi = z \text{ for quark state}$$

& indep of proton momentum p^- (renormalization indep. of state)

$$= \delta(1-z) + \int \frac{d\xi'}{\xi'}\left[Z^{(1)}_{qq}\left(\frac{\xi}{\xi'}\right) f^{(0)}_q(\xi',\mu) + Z^{(0)}_{qq}\left(\frac{\xi}{\xi'}\right) f^{(1)}_q(\xi',\mu)\right]$$

$$= \delta(1-z) + \underbrace{Z^{(1)}_{qq}(z)}_{1/\epsilon\text{ part}} + \underbrace{f^{(1)}_q(z,\mu)}_{\text{rest}}$$

$$\gamma_{qq}(\xi,\xi') = -\mu\frac{d}{d\mu}\,\frac{1}{\xi'}\,\frac{C_F\,\alpha_s(\mu)}{2\pi}\left(\frac{1+z^2}{1-z}\right)_+ , \qquad \mu\frac{d}{d\mu}\alpha_s = -2\epsilon\,\alpha_s + \cdots$$

$$= \frac{C_F\,\alpha_s(\mu)}{\pi}\,\frac{\theta(\xi'-\xi)\,\theta(1-\xi')}{\xi'}\left(\frac{1+z^2}{1-z}\right)_+ \qquad z = \frac{\xi}{\xi'}$$

Quark Splitting Function, One-loop PDF anom. dim.

$\boxed{\text{SCET}_{\text{I}}}$	hard	$p^\mu \sim (Q, Q, Q)$
	collin	$(Q\lambda^2, Q, Q\lambda)$
	usoft	$(Q\lambda^2, Q\lambda^2, Q\lambda^2)$

↑ non-trivial communication between sectors

$\boxed{\text{SCET}_{\text{II}}}$ (still to come)

hard	(Q, Q, Q)
hard-collin	$(Q\lambda, Q, \sqrt{Q\lambda})$
collin	$(Q\lambda^2, Q, Q\lambda)$
soft	$(Q\lambda, Q\lambda, Q\lambda)$

Results for observables which tie d.o.f. together are "<u>Factorization</u> <u>Theorems</u>"

They can involve convolutions between objects defined by different degrees of freedom (hard, soft, jet, hadron dist'n functions) as long as they have same power counting for the convoluted momenta

Processes

- $\gamma^* \gamma \to \pi^0$ π-γ form factor at $Q^2 \gg \Lambda^2$ for γ^*

Breit frame $q^\mu = \frac{Q}{2}(n^\mu - \bar{n}^\mu)$, $p_\gamma^\mu = E\,\bar{n}^\mu$

$$p_\pi^{\ \mu} = \frac{Q}{2} n^\mu + \underbrace{\left(E - \frac{Q}{2}\right)}_{m_\pi^2/2Q} \bar{n}^\mu$$

pion = collinear in n-direction (SCET$_{\text{II}}$)

- $\gamma^* M \to M'$ M-M' (meson) form factor $Q^2 \gg \Lambda^2$ for γ^*

M = collinear in n

M' = " " $\bar{n}$ (say) (SCET$_{\text{I}}$)

- $B \to D\pi$ Matrix Elt. of 4-quark operators

$Q = \{m_b, m_c, E_\pi\} \gg \Lambda$

B, D are soft $p^2 \sim \Lambda^2$, π-collinear (SCET$_{\text{II}}$)

- DIS Structure Functions at $Q^2 \gg \Lambda^2$

$e^- p \to e^- X$ and $1 - x \gg \Lambda/Q$ (ie not near endpts in Bjorken x)

Breit frame: proton n-collinear , X - hard

(SCET$_{\text{II}}$ or SCET$_{\text{I}}$)

- Drell-Yan $\frac{d\sigma}{dQ^2}$ Q^2 = inv. mass of $\ell^+\ell^-$ $\gg \Lambda^2$

$p\bar{p} \to \ell^+\ell^- X$

p - n-collin, $\bar{p}$ - $\bar{n}$-collin, X - hard

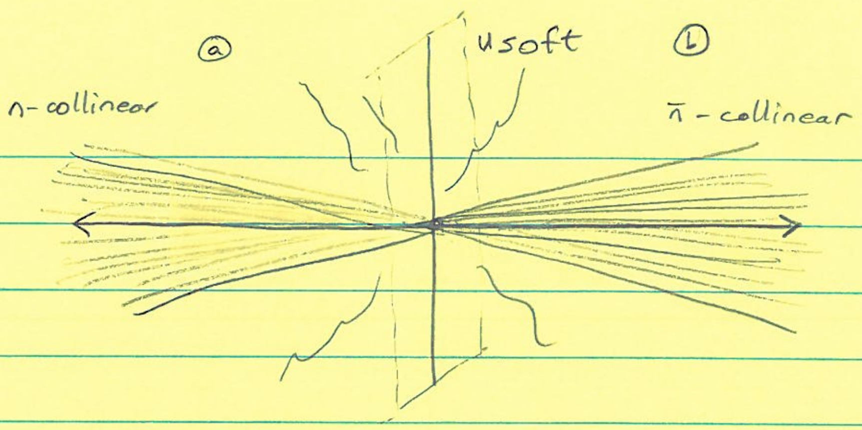
- $e^+e^- \to$ jets

$\bar{p} \to$ jets • depends on observable we formulate

$pp \to$ jets eg two jets n-collin jet

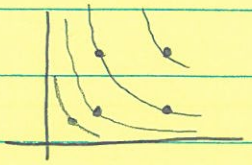
$\bar{n}$-collin jet

etc.



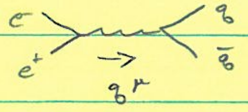
$\boxed{e^+e^- \to \text{dijets}}$

SCET$_I$



$e^+e^- \to \gamma^* \text{ or } Z^* \to X_n X_{\bar n} X_{usoft}$ $\quad$ (e^+e^-) cm frame

<u>Scales</u>



- $q^2 = Q^2$ $\quad$ hard $\quad$ $\mu_h \sim Q$

- Hemisphere invariant mass $\quad$ divide $\quad$ $P_X^\mu = P_{Xa}^\mu + P_{Xb}^\mu$

$$M^2 \equiv (P_{Xa}^\mu)^2 = \Big(\sum_{i\varepsilon a} (p_i^\mu)\Big)^2 \quad , \quad \bar M^2 = \Big(\sum_{i\varepsilon b} p_i^\mu\Big)^2$$

jet $\to$ $M^2 \ll Q^2$

n-collinear $\quad Q(\lambda^2, 1, \lambda)$ $\quad$ $\mu_J \sim M$

n̄-collinear $\quad Q(1, \lambda^2, \lambda)$ $\quad$ $\lambda = M/Q$

- Usoft Radiation
  - uniform in space
  - communication btwn jets
  - eikonal

energy $\sim Q\lambda^2 = M^2/Q$ $\nearrow$ $M^2/Q \gg \Lambda_{QCD}$ perturbative

"tail region"

$\mu_h \gg \mu_J \gg \mu_s \gg \Lambda_{QCD}$

$\mu_s \sim M^2/Q$ $\searrow$ $M^2/Q \sim \Lambda_{QCD}$ non-perturbative

"peak region"

$\mu_h \gg \mu_J \gg \mu_s \sim \Lambda_{QCD}$

In tail region we have power corrections

$$\left(\frac{\Lambda_{QCD}}{\mu_s}\right)^k \ll 1 .$$ Leading order cross-section <u>perturbative</u>.

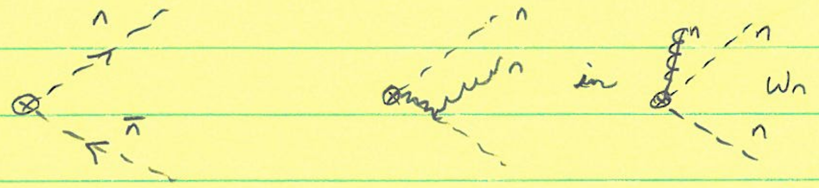
In peak region $\quad (\Lambda_{QCD}/\mu_s)^k \sim 1$ $\quad$ (any k) $\quad \to$ non-pert. soft function

Other Power Corrections

- μ_S/μ_J "Kinematic" expansion of kinematic variables
- Λ_{QCD}/μ_h hard power corr. (Hwk)
- $\Lambda_{QCD}/\mu_J = \frac{\Lambda_{QCD}}{\mu_S}\frac{\mu_S}{\mu_J}$ not independent

<u>Current</u> $\overset{QCD}{J^\mu = \bar\psi\,\Gamma^\mu\,\psi} \rightarrow (\bar\xi_n W_n)_\omega \overset{\text{color singlet}}{\Gamma^\mu} (W_{\bar n}^\dagger \xi_{\bar n})_{\bar\omega}$

$= (\bar\xi_n W_n)_\omega \Gamma^\mu (Y_n^\dagger Y_{\bar n})_{\bar\omega} (W_{\bar n}^\dagger \xi_{\bar n})$ field redefn



Kinematics $q^\mu = p_{X_n}^\mu + p_{X_{\bar n}}^\mu + p_s^\mu$

large $\bar n\cdot q = Q = \bar n\cdot p_{X_n} + \overset{\text{small}}{\cdots}$ $\omega = Q$

$n\cdot q = Q = n\cdot p_{X_{\bar n}} + \cdots$ $\bar\omega = Q$

momentum conservation is strong enough that there are no convolutions in $\omega, \bar\omega$

<u>Factorize the Cross-Section</u>

QCD $\sigma = \sum_X^{res} (2\pi)^4\, \delta^4(q - p_X)\, L_{\mu\nu} \langle 0 | J^{\nu\dagger}(0) | X\rangle\langle X | J^\mu(0) | 0\rangle$

↑ restricted to dijet X states

SCET allows us to move restrictions into operators

$|X\rangle = |X_n\rangle\,|X_{\bar n}\rangle\,|X_s\rangle$

$\bar 3$ rep 3-rep

$$\sigma = N_0 \sum_{\vec n} \sum_{X_n, X_{\bar n}, X_s}^{res'} (2\pi)^4\, \delta^4(q - p_{X_n} - p_{X_{\bar n}} - p_s) \langle 0 | \overline{Y}_{\bar n}\, Y_n | X_s\rangle\langle X_s | Y_n^\dagger\, \overline{Y}_{\bar n}^{\,\dagger} | 0\rangle$$

$$* \; |C(Q)|^2 \langle 0 | \not{\bar n}\, \chi_{n,Q} | X_n\rangle\langle X_n | \bar\chi_n | 0\rangle \langle 0 | \bar\chi_{\bar n, Q} | X_{\bar n}\rangle\langle X_{\bar n} | \not{n}\, \chi_{\bar n} | 0\rangle$$ all orders in α_s

$+ \ldots$ ← "other" power corr.

res' : we must still measure enough things about X to ensure its a dijet state

Measure hemisphere masses $M^2, \bar{M}^2$

$$1 = \int dM^2 \, d\bar{M}^2 \;\; \delta\big(M^2 - (\overset{\uparrow}{p_n} + \overset{\uparrow}{k_s^a})^2\big) \;\; \delta\big(\bar{M}^2 - (p_{\bar{n}} + k_s^b)^2\big)$$

soft momenta

n-collinear total mom. in hemisphere (a)

expand $\delta\big(M^2 - p_n^2 - p_n^-(k_s^a)^+ + \cdots\big) = \delta\big(M^2 - Q(p_n^+ + k_s^{a+})\big) + \cdots$

$\frac{d\sigma}{dM^2 d\bar{M}^2}$ has these δ's under $\sum_X$

- factor measurements:

eg. $$\delta\big(M^2 - Q(p_n^+ + k_s^{a+})\big) = \int dk^+ \, dl^+ \; \delta\big(M^2 - Q(k^+ + l^+)\big) \underbrace{\delta(k^+ - p_n^+)}_{\substack{\text{with} \\ \text{n-collinear} \\ \text{matrix elt}}} \underbrace{\delta(l^+ - k_s^{a+})}_{\substack{\text{with} \\ \text{soft}}}$$

- factor $\delta^4(q - p_{X_n} - p_{X_{\bar{n}}} - p_s)$ too

- write δ's in Fourier space $$\delta(k^+ - p_n^+) = \int \frac{dx^-}{2} \, e^{ix^- k^+/2} \underbrace{e^{-ix^- p_n^+/2}}_{\substack{\text{shifts field} \\ \text{to } \chi_{n,Q}(x^-)}}$$

etc

After some work we get factorized result

$$\frac{d\sigma}{dM^2 d\bar{M}^2} = \sigma_0 \, |C(Q)|^2 \int dk^+ dl^+ dk^- dl^- \; \delta\big(M^2 - Q(k^+ + l^+)\big) \, \delta\big(\bar{M}^2 - Q(k^- + l^-)\big)$$

$$* \sum_{X_n} \frac{1}{2\pi} \int d^4x \; e^{ik^+ x^-/2} \; \mathrm{tr} \langle 0 | \frac{\bar{n}\!\!\!/}{4N_c} \chi_{n,Q}(x) | X_n \rangle \langle X_n | \bar{\chi}_n(0) | 0 \rangle$$

$$* \sum_{X_{\bar{n}}} \frac{1}{2\pi} \int d^4y \; e^{ik^- y^+/2} \; \mathrm{tr} \langle 0 | \bar{\chi}_{\bar{n},Q}(y) | X_{\bar{n}} \rangle \langle X_{\bar{n}} | \frac{n\!\!\!/}{4N_c} \chi_{\bar{n}}(0) | 0 \rangle$$

$$* \underbrace{\sum_{X_s} \frac{1}{N_c} \delta(l^+ - k_s^{a+}) \, \delta(l^- - k_s^{b-}) \; \mathrm{tr} \langle 0 | \bar{Y}_{\bar{n}} Y_n | X_s \rangle \langle X_s | Y_n^\dagger \bar{Y}_{\bar{n}}^\dagger | 0 \rangle}$$

<u>Matrix Elements</u> • $S_{hemi}(l^+, l^-)$ soft function

encodes both momentum scales

$l^\pm \sim \frac{M^2}{Q}$ and $l^\pm \sim \Lambda_{QCD}$

- $$\sum_{X_n} \text{tr} \langle 0| \frac{\bar{n}\!\!\!/}{4N_c} \chi_{n,Q}(x) |X_n\rangle\langle X_n| \bar{\chi}_n(0)|0\rangle = Q\int \frac{d^4r}{(2\pi)^3}\, e^{-ir\cdot x}\, J(Qr^+)$$

$$= \underbrace{\delta(x^+)\,\delta^2(x_\perp)}_{\text{due to collinear multi-pole expn}} \int dr^+\, e^{-ir^+x^-/2}\, \underbrace{J(Qr^+)}_{\text{jet function}}$$

- Same for $\sum_{X_{\bar{n}}} \text{tr}$ ---
- $H(Q) \equiv |C(Q)|^2$ hard function

All Together

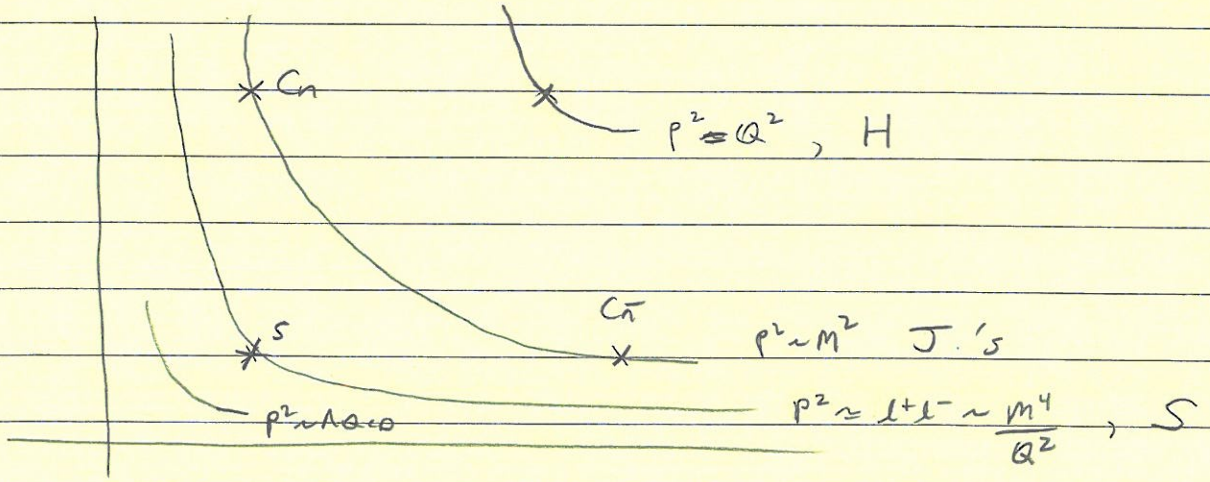
$$\frac{d\sigma}{dm^2 d\bar{m}^2} = \sigma_0\, H(Q) \int dl^+ dl^-\, J(m^2 - Ql^+)\, J(\bar{m}^2 - Ql^-)\, S(l^+, l^-)$$

using renormalized objects on RHS ($C_i^{\text{bare}}\,\mathcal{O}^{\text{bare}} = C(\mu)\,\mathcal{O}(\mu)$)

$$= \sigma_0\, H(Q,\mu) \int dl^+ dl^-\, J(m^2 - Ql^+, \mu)\, J(\bar{m}^2 - Ql^-, \mu)\, S(l^+, l^-, \mu)$$

dijet factorization theorem for hemisphere masses

Note



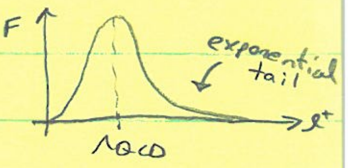
The functions H, J, S have α_s expansions without large logs only if each is evaluated at different scale μ

Soft Function OPE

$$S_{hemi}(\ell^+,\ell^-) = \int d\ell'^{\pm}\ S^{pert}_{hemi}(\ell^+-\ell^{+\prime},\ell^--\ell^{-\prime})\ F(\ell^{+\prime},\ell^{-\prime})$$

$\uparrow$ power tail $\dfrac{(\ln \ell^+/\mu)^k}{\ell^+}$

$\sim\Lambda_{QCD}$ effects



Thrust $\quad T = \max\limits_{\hat t} \dfrac{\sum_i |\vec p_i \cdot \hat t|}{\sum_i |\vec p_i|}$ $\qquad \frac{1}{2} \le T \le 1$

$0 \le \tau \le \frac{1}{2}$

$\tau = 1 - T$

for dijets $\quad \tau = \dfrac{M^2 + \bar M^2}{Q^2}$ $\leftarrow$ symmetric projection $\downarrow$ $\searrow$

$$\frac{d\sigma}{d\tau} = \sigma_0\, H(Q,\mu)\, Q \int d\ell\ J_\tau(Q^2\tau - Q\ell,\mu)\ S_T(\ell,\mu)$$

$p^2 \sim Q^2$ $\qquad$ jet $p^2 \sim Q^2\tau$ $\qquad$ usoft $p^2 \sim Q^2\tau^2$

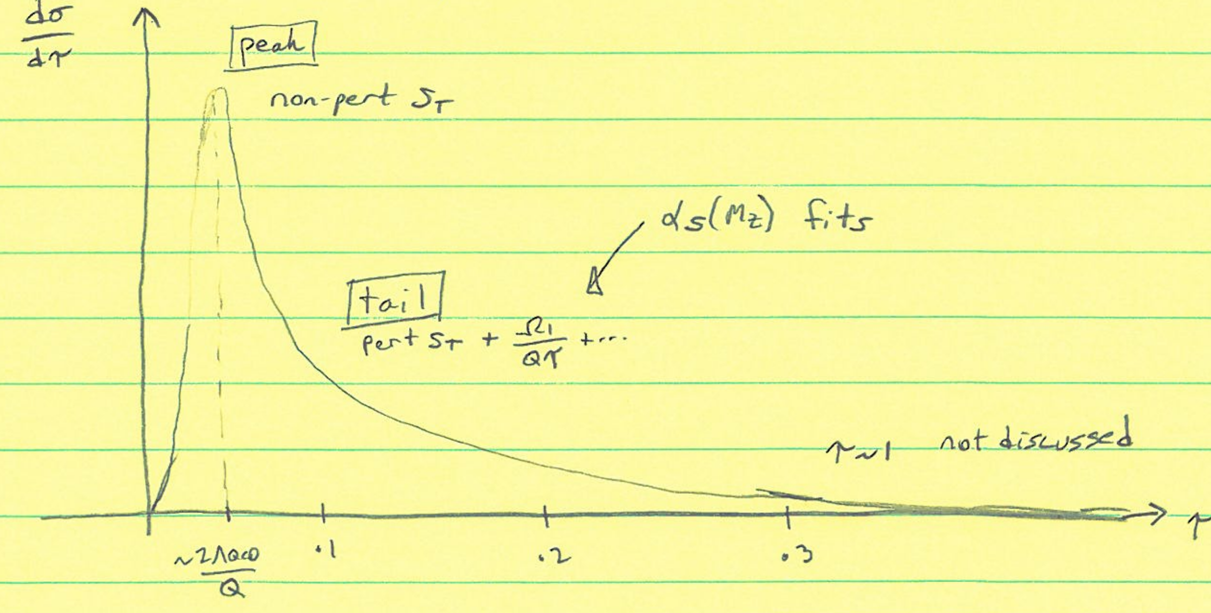
$$Q^2 \gg Q^2\tau \gg Q^2\tau^2$$

$$\mu_H^2 \gg \mu_J^2 \gg \mu_S^2 \ \overset{\gg}{\underset{\sim}{\text{or}}}\ \Lambda_{QCD}^2$$

schematically: $\dfrac{d\sigma}{d\tau} \sim \sum\limits_{n,m} \dfrac{\alpha_s^n \ln^m \tau}{\tau}$ + non-perturbative effects in F

$\swarrow$ sum logs

+ power corrections



Pert. Results

- match quark form factor

$$C(Q,\mu) = 1 + \frac{C_F \alpha_s(\mu)}{4\pi}\left[3 \ln^2\left(\frac{-Q^2}{\mu^2}\right) - \ln\left(\frac{-Q^2}{\mu^2}\right) - 8 + \frac{\pi^2}{6}\right]$$

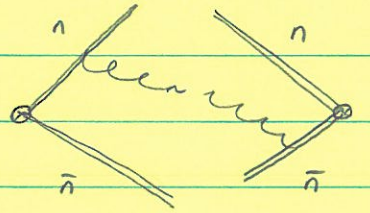
$$H = |C|^2$$

(Renormalized)

- Jet Function

$$J_n(s,\mu) = \delta(s) + \frac{\alpha_s(\mu) C_F}{4\pi}\left[\# \delta(s) + \# \left[\frac{\mu^2 \theta(s)}{s}\right]_+ + \# \left[\frac{\mu^2 \ln(\mu^2/s)\theta(s)}{s}\right]_+\right]$$

- Pert. Soft Fn



etc.

$$S^{pert}(\ell^+,\ell^-) = \Big\{ \delta(\ell^+) + \frac{\alpha_s C_F}{4\pi}\left[\# \delta(\ell^+) + 0 \left[\frac{\mu}{\ell^+}\theta(\ell^+)\right] + \# \left[\frac{\mu}{\ell^+}\ln\left(\frac{\ell^+}{\mu}\right)\right]_+\right] \Big\}$$

$$* \Big\{ \delta(\ell^-) + \frac{\alpha_s C_F}{4\pi}\left[\text{ditto } \ell^+ \to \ell^- \right] \Big\}$$

C renormalizes multiplicatively $\quad C^{bare} = Z_C C = C + (Z_C - 1) C$

$$\mu \, d/d\mu \; C(Q,\mu) = \gamma_C(Q,\mu) \, C(Q,\mu)$$

J, S renormalize like PDF, with convolutions

eg. $$J_n^{bare}(s) = \int ds' \, Z_J(s-s') \, J_n(s',\mu)$$

$$\mu \, d/d\mu \; J_n(s,\mu) = \int ds' \, \gamma_J(s-s') \, J_n(s',\mu)$$

↑ invariant mass evolution

$$\text{Coefficient Renormalyation} = \left(\begin{matrix}\text{Operator} \\ \text{Renormalization}\end{matrix}\right)^{-1}$$

"consistency conditions"

$$|Z_C|^2 \, \delta(s)\,\delta(\bar{s}) = \int ds' \, d\bar{s}' \; Z_J^{-1}(s-s') \, Z_J^{-1}(\bar{s}-\bar{s}') \, Z_S^{-1}\left(\frac{s'}{Q}, \frac{\bar{s}'}{Q}\right)$$

$\boxed{\text{RGE}}$

$$\gamma_J(s,\mu) = -2\,\Gamma^{cusp}[\alpha_s] \, \frac{1}{\mu^2}\left[\frac{\mu^2\theta(s)}{s}\right]_+ + \gamma[\alpha_s]\,\delta(s)$$

all order structure

(γ_S similar, two variables factorize)

Fourier Transform $\quad y = y - i0$

$$\gamma_J(y) = \int ds \; e^{-isy} \, \gamma_J(s)$$

$$J(y) = \int ds \; e^{-isy} \, J(s)$$

$$\mu \, d/d\mu \; J(y,\mu) = \gamma_J(y,\mu) \, J(y,\mu)$$

simple

$$\gamma_J(y,\mu) = 2\,\Gamma^{cusp}[\alpha_s] \, \ln(iy\mu^2 e^{\gamma_E}) + \gamma[\alpha_s]$$

$$\left[\frac{\ln^k(s/\mu)}{s}\right]_+ \longleftrightarrow \ln^{k+1}(iy\mu^2 e^{\gamma_E})$$

$$d\ln\mu = \frac{d\alpha_s}{\beta[\alpha_s]} \quad , \quad \ln \mu/\mu_0 = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta[\alpha_s]}$$

All orders Solution

$$\ln\left[\frac{J(s,\mu)}{J(s,\mu_0)}\right] = \omega(\mu,\mu_0)\,\ln(iy\mu_0^2 e^{\gamma_E}) + K(\mu,\mu_0)$$

← same structure for H, J, S

$$\omega = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \, \frac{2\,\Gamma^{cusp}[\alpha_s]}{\beta[\alpha_s]}$$

$$K = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta[\alpha_s]} \, \gamma[\alpha_s] + \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]} \, 2\,\Gamma^{cusp}[\alpha] \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta[\alpha']}$$

determine ω, K order by order

$y \leftrightarrow \tau$

$$\ln \frac{d\sigma}{dy} = \underset{LL}{(\ln y)(\alpha_s \ln)^k} + \underset{NLL}{(\alpha_s \ln)^k} + \underset{NNLL}{\alpha_s (\alpha_s \ln)^k} + \cdots$$

Momentum Space Answer with resummation

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = H(Q,\mu_Q)\, U_H(Q,\mu_Q,\mu_J)\, J_T(Q^2\tau - s') \otimes U_J(s' - Q\ell, \mu_S, \mu_J)$$
$$\otimes\, S_T^{pert}(\ell - \ell', \mu_S) \otimes F(\ell')$$

where $\mu_Q \sim Q, \quad \mu_J \sim Q\sqrt{\tau}, \quad \mu_S \sim Q\tau$

$$U_J(s,\mu,\mu_0) = \frac{e^K (e^{\gamma_E})^\omega}{\mu_0^2\, \Gamma(-\omega)} \left[\frac{(\mu_0^2)^{1+\omega}\, \theta(s)}{s^{1+\omega}}\right]_+$$

↑ boundary at ∞ rather than 1

Consistency says $\quad \gamma_J[\alpha_s] + \gamma_S[\alpha_s] = -\frac{1}{2}\gamma_H[\alpha_s]$

Soft-Collinear Interactions (SCET$_{II}$)

Recall $q = q_s + q_c \sim Q\,(\overset{+}{\lambda}, \overset{-}{1}, \overset{\perp}{\lambda})$

$q^2 = Q^2 \lambda \gg (Q\lambda)^2$

offshell w.r.t s, c

On-shell modes $q^\mu \sim Q\,(\lambda, 1, \sqrt{\lambda})$ are hard-collinear

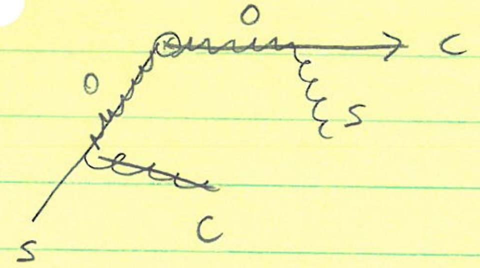
compared to collinear $q^\mu \sim Q(\lambda^2, 1, \lambda)$

Integrating out these fluctuations builds up a soft Wilson line S_n (analogous to $Y(n\cdot A_{us})$ but with soft fields)

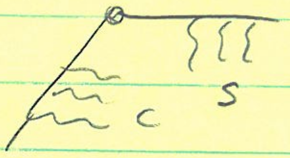
Toy eg. heavy-to-light soft-collin current $\bar{\xi}_n \Gamma h_v$

s = soft, c = collinear

0 = offshell



adding more gives

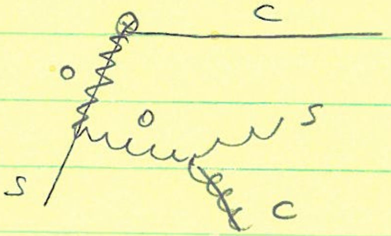


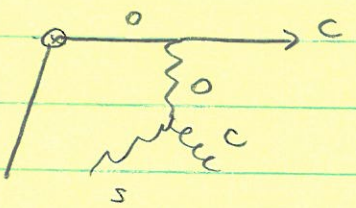
$\bar{\xi}_n S_n^+ \Gamma W h_v$

$S_n^+[n\cdot A_{us}]$

$W[\bar{n}\cdot A_c]$

In QCD need 3-gluon, 4-gluon vertices too; these flip order of S^+ & W





$$\underbrace{(\bar{\xi}_n W)}_{\substack{\text{collinear}\\ \text{gauge}\\ \text{invariant}}} \Gamma \underbrace{(S_n^+ h_v)}_{\substack{\text{soft}\\ \text{gauge}\\ \text{invariant}}}$$

[can be extended to allorders]

this is <u>soft-collinear factorization</u>

Another Method

- construct $SCET_{II}$ operators using $SCET_{I}$

i) Match QCD onto $SCET_I$ usoft $p_u^2 \sim \lambda^2$

collinear $p_c^2 \sim Q\lambda$

ii) Factorize usoft with field redefinition

iii) Match $SCET_I$ onto $SCET_{II}$ soft $p_s^2 \sim \lambda^2$

collin $p_c^2 \sim \lambda^2$

Notes:

- this gives us a simple procedure to construct $SCET_{II}$ ops. (even though they're non-local)
- usoft fields in I are renamed soft for II

eg. i) $J^{I} = (\bar{\xi}_n W)\, \Gamma\, h_v$

ii) $J^{I} = (\bar{\xi}_n^{(0)} W^{(0)})\, \Gamma\, (Y^\dagger h_v)$

iii) $J^{II} = (\bar{\xi}_n W)\, \Gamma\, (S^\dagger h_v)$ as before

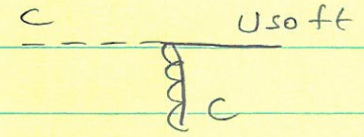
here all T-products in $SCET_I$ & $SCET_{II}$ match up, so matching was trivial

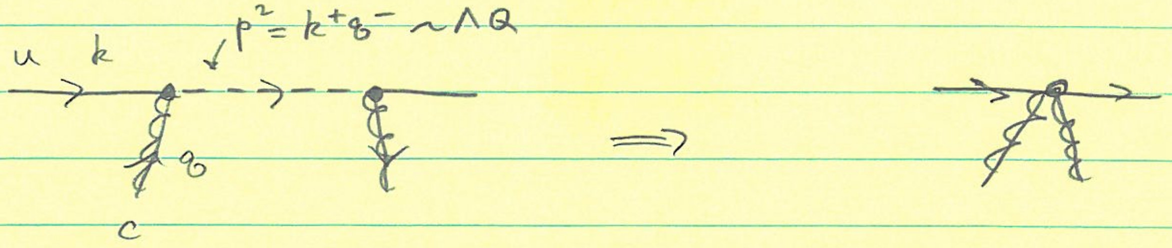
"Thm" • In cases where we have T-products in $SCET_I$ with ≥ 2 operators involving both collin & usoft fields, we can generate a non-trivial coefficient in $SCET_{II}$ (jet-function J)

eg.

$$\int dp_- \, dk_+ \, J(p_-, k_+) \overbrace{(\xi W)_{p_-} \, \Gamma \, (S^\dagger q_s)_{k_+} + (\cdots)}^{p^2 \sim \lambda^2}$$

$SCET_I$ loops $p^2 \sim Q\lambda$ ind's allow k^+ dependence eg. two operators





when we lower offshellness of ext. collin fields the intermediate line still has $p^2 \sim Q\Lambda$ and must really be integrated out

P.C. $\quad T^{I} \sim \lambda^{2K} \quad \Longrightarrow \quad O^{II} \sim \eta^{K+E}$

$$\text{where} \quad \lambda^2 = \eta = \frac{\Lambda}{Q} \quad ,$$

factor $E > 0$ from changing the scaling of ext. fields

eg. $\xi_I \sim \lambda$

$\xi_{II} \sim \eta = \lambda^2$

⟹ No mixed soft-collin $\mathcal{L}$ at leading order

- after field redefn no mixed $\mathcal{L}_I$ ops at LO

- mixed $\mathcal{L}_I^{(1)}$ gives $T\{\mathcal{L}_I^{(1)}, \mathcal{L}_I^{(1)}\} \sim \lambda^2$ matches onto $O_{II} \sim \eta$ or higher

SCET$_I$ $\quad \lambda^\delta$

$$\delta = 4 + \underset{\uparrow\ u=1 \text{ no c., else } u=0}{4u} + \sum_k (k-4) \overset{\text{rest}}{V_k^c} + (k-8) \overset{\text{pure usoft}}{V_k^u}$$

V_K^i = # vertices that are $\mathcal{O}(\lambda^K)$ and type-i

SCET$_{II}$

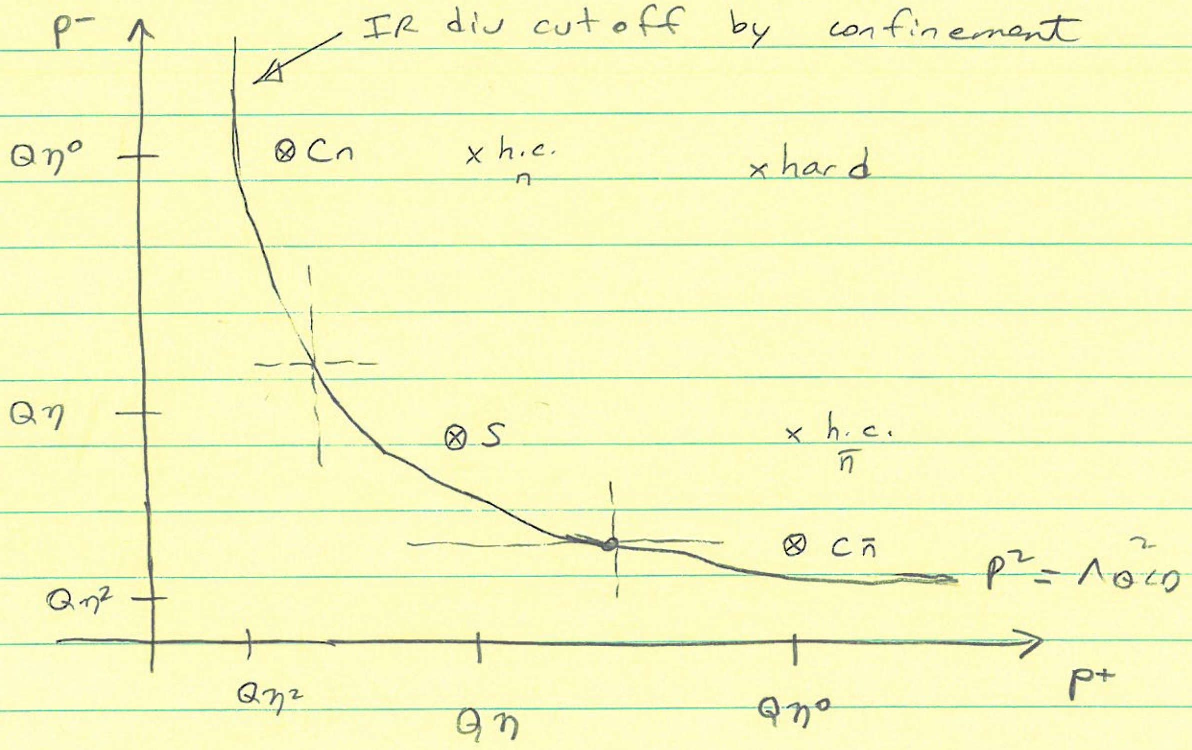
$$\delta = 4 + \sum_k (k-4)\Big(\underset{\substack{\uparrow\\ \text{pure}\\ c}}{V_k^c} + \underset{\substack{\uparrow\\ \text{pure}\\ s}}{V_k^s} + \underset{\substack{\uparrow\\ \text{mixed}}}{V_k^{sc}}\Big) + \underset{\substack{\uparrow\\ p\sim(\eta^2,\eta,\eta)\\ \text{loops}}}{L^{sc}}$$

$$\delta = 5 - \underset{\uparrow}{N_c} - \underset{\uparrow}{N_s} + \sum_k (k-4)\left(V_k^s + V_k^c\right) + (k-3)\, V_k^{sc}$$

\# connected
soft, collin components

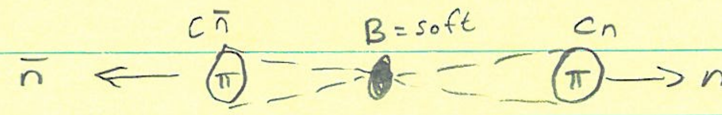
[in eg. SCET$_I$ $\lambda^3\lambda \frac{1}{\lambda^2}\lambda^3\lambda \sim \lambda^{6-4} \sim \lambda^2$ or $\lambda * \lambda \sim \lambda^2$ $\Rightarrow$ $\left(\eta^{3/2}\eta\right)^2 \frac{1}{\eta} = \eta^{4-3} = \eta$]

$$\mathcal{L}_{SCET^{II}} = \mathcal{L}^{(0)}_{soft}[q_s, A_s] + \mathcal{L}^{(0)}_{collin\text{-}n}[q_n, A_n] + \mathcal{L}^{(0)}_{collin\text{-}\bar{n}}[q_{\bar{n}}, A_{\bar{n}}]$$



Non-pert d.o.f in different sectors $B \to \pi\pi$

eg.



Exclusive

eg. $\gamma^* \gamma \to \pi^0$ hard-collin factorization

[Breit frame: soft modes have no active role so this does not really probe difference between $SCET_I$ & $SCET_{II}$]

QCD has

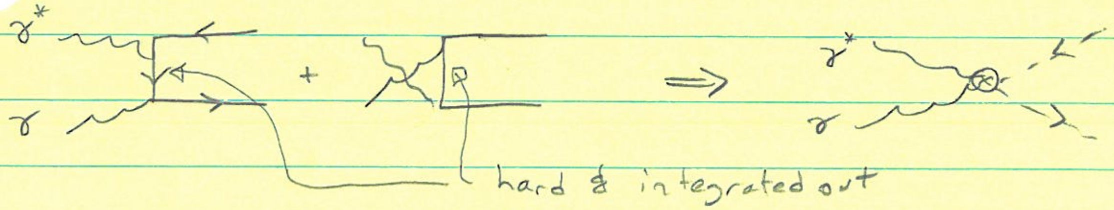
$$\langle \pi^0(p_\pi) | J_\mu(0) | \gamma(p_\gamma, \epsilon) \rangle = ie\, \epsilon^\nu \int d^4z\, e^{-ip_\gamma \cdot z} \langle \pi^0(p_\pi) | T\, J_\mu(0) J_\nu(z) | 0 \rangle$$
$$= -ie\, F_{\pi\gamma}(Q^2)\, \epsilon_{\mu\nu\rho\sigma}\, p_\pi^\nu \epsilon^\rho q^\sigma$$

e.m. current $J^\mu = \bar\Psi \hat{Q} \gamma^\mu \Psi$, $\hat{Q} = \frac{\tau_3}{2} + \frac{\mathbb{1}}{6} = \begin{pmatrix} 2/3 & \\ & -1/3 \end{pmatrix}$

For $Q^2 \gg \Lambda^2$ $F_{\pi\gamma}$ simplifies (a la Brodsky-Lepage)

frame $q^\mu = \frac{Q}{2}(n^\mu - \bar{n}^\mu)$, $p_\gamma^\mu = E\, \bar{n}^\mu$

$p_\pi^\mu = p + p_\gamma = \frac{Q}{2} n^\mu + (E - \frac{Q}{2}) \bar{n}^\mu$



SCET Operator at Leading-order (for T-product) is

$$\mathcal{O} = \frac{i\, \epsilon^\perp_{\mu\nu}}{Q} [\bar{\xi}_{n,p} W]\, \Gamma\, C(\bar{P}, \bar{P}^\dagger, \mu)\, [W^\dagger \xi_{n,p'}]$$

order λ^2 ("twist-2")

- obeys current conservation
- dim analysis fixes $\frac{1}{Q}$ pre-factor for C dimless
- Charge Conj: T{J, J} even so $\mathcal{O}$ even

so $C(\mu, \bar{P}, \bar{P}^\dagger) = C(\mu, -\bar{P}^\dagger, -\bar{P})$

- flavor & spin structure $\quad \Gamma = \underbrace{\bar{n}\!\!\!/ \gamma_5}_{\text{for pion}} \; 3\sqrt{2} \; \underbrace{\hat{Q}^2}_{\text{2nd order e.m.}}$

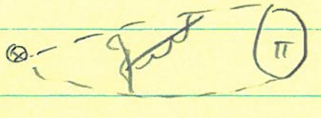
- color singlet, purely collinear (again) so soft gluons decouple

equate $\frac{Q^2}{2} F_{\pi\gamma} = \frac{i}{Q} \langle \pi^0 | (\bar\xi W) \, \Gamma \, \overset{SCET_{II}}{C} \, (W^\dagger \xi) \, |0\rangle$

write $\bar P_\pm = \bar P^\dagger \pm \bar P$

now $\bar P_-$ gives total mom of $(\bar\xi W)\Gamma(W^\dagger\xi)$ operator

ie momentum of pion



$$\bar P_- = \bar n \cdot p_\pi = Q$$

→ total mom.

$$F_{\pi\gamma}(Q^2) = \frac{2i}{Q^2}\int d\omega\, C(\omega,\mu) \, \langle\pi^0|(\bar\xi W)\,\Gamma\,\delta(\omega-\bar P_+)(W^\dagger\xi)|0\rangle$$

<u>Non-perturbative Matrix Elt</u>

finite Wilson line $\left(P\exp i\int_x^y ds\ldots\right)$

Fourier Transform of $\bar n\cdot p$ label

position space

$$\langle\pi^0(p)|\,\bar\xi_n(y)\,\frac{\bar n\!\!\!/\gamma_5\tau^3}{\sqrt2}\,W(y,x)\,\xi_n(x)\,|0\rangle = -i f_\pi\,\bar n\cdot p\int_0^1 dz\, e^{i\bar n\cdot p\,(zy+(1-z)x)}\,\phi_\pi(\mu,z)$$

$$\int_0^1 dz\,\phi_\pi(z) = 1$$

momentum space

$$\langle\pi^0(p)|\,(\bar\xi_{n,p}W)\,\frac{\bar n\!\!\!/\gamma_5\tau^3}{\sqrt2}\,\delta(\omega-\bar P_+)\,(W^\dagger\xi_{n,p_2})\,|0\rangle$$

$$= -i f_\pi\,\bar n\cdot p\int_0^1 dz\;\delta(\omega-(2z-1)\bar n\cdot p)\;\phi_\pi(\mu,z)$$

Plug it into $F_{\pi\gamma}(Q^2)$ and do integral over ω

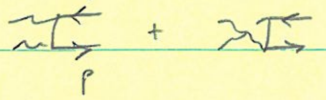
$$F_{\pi\gamma}(Q^2) = \frac{2 f_\pi}{Q^2} \int_0^1 dz\ C\big((2z-1)Q, Q, \mu\big)\ \phi_\pi(z,\mu)$$

- ϕ_π is universal light-cone dist'n for pions

- C is process dependent (all order factorization in α_s)
- one-dim convolution again

Tree Level Matching

expand

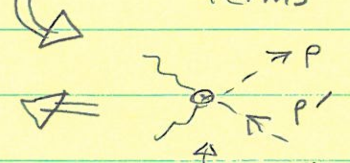


$$i\,\Big(\ \cdots\ + \ \cdots \Big) = \frac{ie}{2}\,\epsilon_{\mu\nu\rho\beta}\,\epsilon^{\nu}\,\bar{n}^{\rho} n^{\beta} \left(\frac{\bar{n}\!\!\!/}{2}\gamma_5\right)\hat{Q}^2 * \left(\frac{1}{\bar{n}\cdot p} - \frac{1}{\bar{n}\cdot p'}\right) + \ldots$$

↖ non-pion terms

So $$C = \frac{1}{6\sqrt{2}}\left(\frac{Q}{\bar{p}^+} - \frac{Q}{\bar{p}}\right)$$

$$C\big(\omega = (2x-1)Q\big) = \frac{1}{6\sqrt{2}}\left(\frac{1}{x} + \frac{1}{1-x}\right)$$



↖ match with $W=1$

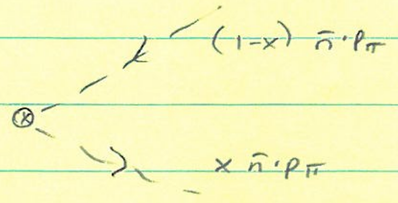
Charge Conj $+1$ for $|\pi^0\rangle$ gives $\phi_\pi(x) = \phi_\pi(1-x)$

So only $\int_0^1 dx\ \frac{\phi_\pi(x,\mu)}{x}$ appears in our prediction

↖ integrate over all x, much different than DIS $\delta(1-\xi/x) \Rightarrow f_{i/p}(x,\mu)$

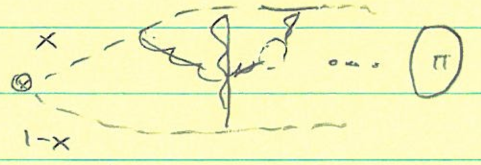
Interpretation:

Naively



mom fraction of quarks in pion

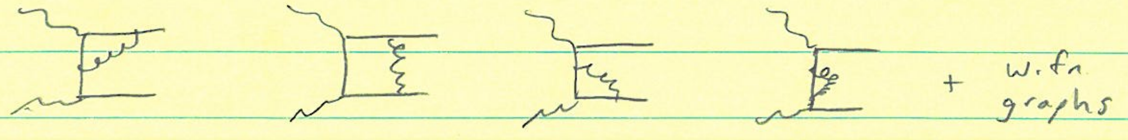
Really



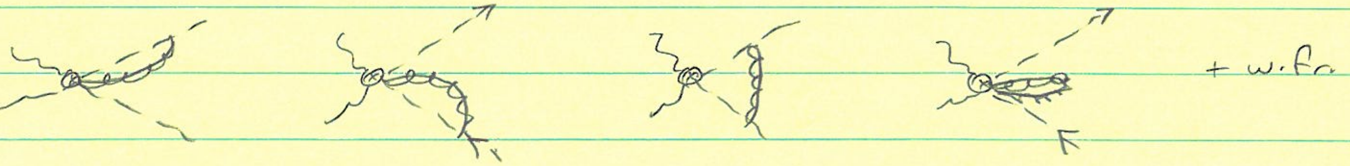
mom. fractions at point where quarks are produced. Hadronization process changes "x" carried by valence quarks which is encoded in $\phi_\pi(x)$

Higher Order Matching

full



SCET



Difference will be IR finite, and gives C at one-loop

Another Exclusive Example

(hep-ph/0107002)

<u>$B \to D\pi$</u> $\qquad \underbrace{m_b,\ m_c,\ E_\pi}_{Q} \gg \Lambda_{QCD}$

QCD Operators at $\mu \simeq m_b$

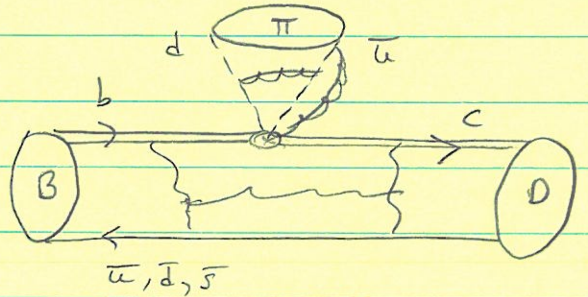
$$H_W = \frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cb} \left[C_0^F O_0 + C_8^F O_8 \right]$$

$$P_L = \frac{1-\gamma_5}{2}$$

Where $$O_0 = [\bar{c}\, \gamma^\mu P_L\, b][\bar{d}\, \gamma_\mu P_L\, u]$$
$$O_8 = [\bar{c}\, \gamma^\mu P_L T^a\, b][\bar{d}\, \gamma_\mu P_L T^a\, u]$$

Want to Factorize $\langle D\pi \,|\, O_{0,8} \,|\, B \rangle$

ie Show at LO



no gluons btwn B, D and quarks in pion

expect $B \to D$ form factor $\xi^{\text{Isgur-Wise}}$

$\phi_\pi(x)$ distn for pion

$$\left.\begin{array}{lll} B, D & \text{soft} & p^2 \sim \Lambda^2 \\ \pi & \text{collinear} & p^2 \sim \Lambda^2 \end{array}\right\} \text{SCET}_{II}$$

Use SCET_I as intermediate step

① Match at $\mu^2 \simeq Q^2$

$$\left.\begin{array}{l} O_0 \\ O_8 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} Q_0^{1,5} = \left[\bar{h}_{v'}^{(c)}\, \Gamma_h^{1,5}\, h_v^{(b)}\right]\left[(\bar{\xi}_n^{(d)} W)\, \Gamma_\ell\, C_0(\bar{P}_+)\, W^\dagger \xi_n^{(u)}\right] \\ Q_8^{1,5} = \left[\ \text{''}\ \ T^A\ \ \text{''}\ \right]\left[\ \text{''}\ \ \text{''}\ C_8(\bar{P}_+) T^a\ \text{''}\ \right] \end{array}\right.$$

$\uparrow$ usoft SCET_I $\qquad\qquad$ $\uparrow$ collinear $p^2 \sim Q\Lambda$

$$\Gamma_h^{1,5} = \frac{\not{n}}{2}\{1, \gamma_5\}$$
$$\Gamma_\ell = \frac{\not{\bar{n}}}{4}(1-\gamma_5)$$

② Field redefinitions $\quad \xi_{n,p} = Y\, \xi_{n,p}^{(0)}\ , \ldots$

in $Q_0^{1,5}$ get $\quad \bar\xi_n^{(0)} W^{(0)}\, \cancel{Y^\dagger Y}\, W^{\dagger(0)} \xi_n^{(0)}$

$Q_8^{1,5}$ get $\quad \bar\xi_n^{(0)} W^{(0)}\, Y^\dagger T^A Y\, W^{\dagger(0)} \xi_n^{(0)}$

$$Y T^a Y^\dagger = \underset{\uparrow \text{adjoint Wilson line}}{\mathcal{Y}^{ba}}\, T^b \qquad\qquad Y^\dagger T^a Y = \mathcal{Y}^{ab}\, T^b$$

$$T^a \otimes Y^\dagger T^a Y = \underset{\uparrow}{Y T^a Y^\dagger} \otimes T^a$$

moves usoft Wilson lines next to h_v fields

③ Match SCET_{I} onto SCET_{II} (trivial here again)

$Y \to S$

$\xi_n^{(0)} \to \xi_n$ in II etc.

$$Q_0^{1,5} = \left[\bar h_{v'}^{(c)} \Gamma_h h_v^{(b)}\right]\left[\bar\xi_n^{(d)} W \Gamma_\ell\, C_0(\bar P_+)\, W^\dagger \xi_{n,p}^{(u)}\right]$$

$$Q_8^{1,5} = \left[\bar h_{v'}^{(c)} \Gamma_h S T^a S^\dagger h_v^{(b)}\right]\left[\bar\xi_n^{(d)} W \Gamma_\ell\, C_8(\bar P_+)\, T^a W^\dagger \xi_{n,p}^{(d)}\right]$$

④ Take Matrix Elements

$$\langle \pi_{n^-} |\, \bar\xi_n W\, \Gamma\, C_0(\bar P_+)\, W^\dagger \xi_n\, | 0 \rangle = \frac{i}{2} f_\pi E_\pi \int_0^1 dx\, C(2E_\pi(2x-1))\, \phi_\pi(x)$$

$$\langle D_{v'} |\, \bar h_{v'}\, \Gamma\, h_v\, | B \rangle = N'\ \underset{\uparrow\ w_0 = v\cdot v'}{\xi(w_0,\mu)}$$

B, D purely soft → no contractions with collinear fields

π " collinear → no " " soft fields

which is why it factors into two matrix elements

F– O_8:

$$\langle D_{v'} |\, \underbrace{\bar h_{v'}\, Y T^a Y^\dagger\, h_v}_{\text{color octet operator}}\, | B_v \rangle = 0$$

between color singlet states

Find   Factorization Formula

$$\langle \pi D | H_W | B \rangle = i \underset{\overset{\uparrow}{\text{prefactors}}}{N} \, \xi(w_0, \mu) \int_0^1 dx \, C(2E_\pi(2x-1), \mu) \, \phi_\pi(x, \mu) + O(\Lambda/Q)$$

- $\xi(w_0, \mu)$ is Isgur-Wise function at max. recoil

  $w_0 = \frac{m_B^2 - m_D^2}{2 m_B}$ (measured in $B \to D e \nu$ recall)

- This applies to type-I (& III) decays

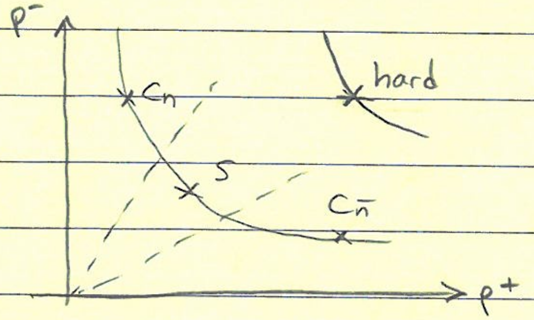
  $\bar{B}^0 \to D^+ \pi^-$ $\bar{B}^0 \to D^{*+} \pi^-$, $\bar{B}^0 \to D^+ \rho^-$, ...

  $B^- \to D^0 \pi^-$ $B^- \to D^{*0} \pi^-$ $B^- \to D^0 \rho^-$, ...

  predicts type-II decays are suppressed by Λ/Q

  $\bar{B}^0 \to D^0 \pi^0$, ... (we could derive fact. thm. for these too)

$SCET_{II}$ & Rapidity Divergences



In $SCET_I$ we had to worry about double counting: $C_n = C_n - \underbrace{C_0}_{\text{zerobin}}$, us

So far in $SCET_{II}$ we have not had to because the overlaps did not generate log divergences

In general $SCET_{II}$ also has 0-bin's: $\underbrace{C_n}_{k^\mu \sim (\lambda^2, 1, \lambda)} - C_{ns}$ ↑ take $k^\mu \sim (\lambda, \lambda, \lambda)$ in collinear integrand & expand

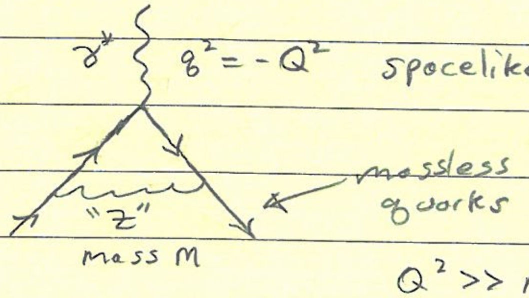
But unlike $SCET_I$ there is another issue. The variable that distinguishes modes is a rapidity Y , $e^{2Y} = \frac{p^-}{p^+}$

$$e^{2Y} \sim \underset{C_n}{\lambda^{-2}} , \underset{S}{\lambda^0} , \underset{C_{\bar{n}}}{\lambda^2}$$

Since all modes live on the same mass hyperbola p^2, divergences that occur when separating modes can not be regulated by dim.reg! (which is a Lorentz Inv. regulator, $p_E^{-2\epsilon}$, which distinguishes hyperbola's)

Lets explore this with a simple example.

Massive Sudakov Form Factor



$J^\mu = \bar{\Psi} \gamma_\mu \Psi$

$Q^2 >> M^2$

$$< q(\bar{p}) \,|\, J^\mu \,|\, q(p) > = F(Q^2, M^2) \; \bar{u}(\bar{p}) \gamma^\mu u(p)$$

$\lambda = \frac{M}{Q}$ Z could be

C_n $Q(\lambda^2, 1, \lambda)$ ← $q(p)$

$C_{\bar{n}}$ $Q(1, \lambda^2, \lambda)$ ← $q(\bar{p})$

S $Q(\lambda, \lambda, \lambda)$

$$p^\mu = p^- \frac{n^\mu}{2} \quad , \quad \bar{p}^\mu = \bar{p}^+ \frac{\bar{n}^\mu}{2} \qquad Q^2 = -(\bar{p}-p)^2 = p^- \bar{p}^+ \underset{\substack{\uparrow \\ \text{frame choice}}}{=} Q \cdot Q$$

Factorize $\quad J^\mu = (\bar{\xi}_{\bar{n}} W_{\bar{n}}) \, S_{\bar{n}}^+ \, S_n \, \gamma^\mu \, (W_n^+ \xi_n)$

$$F(Q^2, m^2) = \mathcal{H} \; C_{\bar{n}} \; \mathcal{S} \; C_n$$

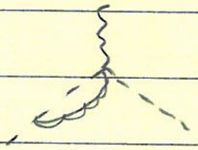
Consider (Scalar) Loop Integral

$$I_{full} = \int d^d k \, \frac{1}{(k^2-m^2)\,(k^2+k^+p^-)\,(k^2+k^-\bar{p}^+)}$$

← terms with most logs have no k's in numerator

UV & IR finite

$$I_{C_n} = \int d^d k \, \frac{1}{(k^2-m^2)\,(k^2+k^+p^-)\,(k^-)\,\bar{p}^+} \; \frac{w^2 \nu^{\eta}}{|k^-|^\eta}$$



$$I_{C_{\bar{n}}} = \int d^d k \, \frac{1}{(k^2-m^2)\,(k^+)(p^-)\,(k^2+k^-\bar{p}^+)} \; \frac{w^2 \nu^\eta}{|k^+|^\eta}$$

$$I_S = \int d^d k \, \frac{1}{(k^2-m^2)\,(k^+)(k^-)\,(\bar{p}^+p^-)} \; \frac{w^2 \nu^\eta}{|2p_z|^\eta}$$

do $d^n k_\perp$ in $I_s \quad \propto \quad \int dk^+ dk^- \, \frac{(k^+k^- - m^2)^{-2\epsilon}}{(k^+)(k^-)} \; \frac{1}{Q^2}$

diverges as $\frac{k^-}{k^+} \to 0$ (towards $C_{\bar{n}}$)

$\gg \infty$ (towards C_n)

Need another regulator. One dim-reg like choice is to regulate Wilson lines

$$S_n = \sum_{perms} \exp\left[\frac{-g}{n\cdot p} \, w \, \frac{\nu^{\eta/2}}{|2p_z|^{\eta/2}} \, n\cdot A_s \right]$$

add red factors above

$p_z = p_- - p_+$ because it does not involve p^0.

(Regulators with p^0 can mess up unitarity / causality)

For collinear W_n, $|2p_z| = |p^-|$ up to power corrections

use $\quad W_n = \sum_{perms} \exp\left[\frac{-g}{\bar{n}\cdot p} \, w^2 \, \frac{\nu^\eta}{|\bar{n}\cdot p|^\eta} \, \bar{n}\cdot A_n \right]$

$$w^{bare} = w(\eta,\nu)\,\nu^{\eta} \quad , \quad \nu\frac{d}{d\nu}w(\eta,\nu) = -\frac{\eta}{2}\,w(\eta,\nu)$$

$$w(0,\nu) \equiv 1$$

$1/\eta$ like $1/\epsilon$ $\quad$ $w(\eta,\nu)$ is dummy coupling to

$\ln\nu$ like $\ln\mu$ $\quad$ facilitate RGE in ν

Note: • $1/\eta$ & η^0 terms are gauge invariant. eg. At one-loop replacing $g^{\mu\nu} \to \left(g^{\mu\nu} + \xi\frac{k^\mu k^\nu}{k^2}\right)$, the $k^\mu k^\nu$ term has no rapidity divergences.

- For any fixed inv. mass we have $1/\eta$ divergences. Proper renormalitation procedure is

  $\eta \to 0$, add $\frac{f(\epsilon)}{\eta}$ counterterm, then $\epsilon \to 0$, find $\frac{1}{\epsilon}$ c.t.'s

For fermion case, including prefactors

$$I_{c_n} = \frac{\alpha_s C_F}{\pi}\left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\frac{\mu}{m}\right)^{2\epsilon}}{2\eta} + \ln\left(\frac{\nu}{p^-}\right)\ln\left(\frac{\mu}{m}\right) + \frac{1}{2\epsilon}\ln\left(\frac{\nu}{p^-}\right) + \frac{1}{2\epsilon} + \ln\left(\frac{\mu}{m}\right) + \text{constant}\right]$$

$I_{c_{\bar{n}}}$ = Same $p^- \to \bar{p}^+$

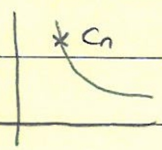
$$I_s = \frac{\alpha_s C_F}{\pi}\left[-\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\mu/m\right)^{2\epsilon}}{\eta} - 2\ln\left(\frac{\nu}{m}\right)\ln\left(\frac{\mu}{m}\right) + \frac{1}{\epsilon}\ln\left(\frac{\mu}{\nu}\right) + \frac{1}{2\epsilon^2} + \ln^2\frac{\mu}{m} + \text{constant}\right]$$

$$I_{c_n} + I_{c_{\bar{n}}} + I_s = \frac{\alpha_s C_F}{\pi}\left[\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu}{Q} + \frac{1}{\epsilon} + \ln^2\frac{\mu}{m} + 2\ln\frac{\mu}{m}\ln\frac{m}{Q} + 2\ln\frac{\mu}{m} + \text{const.}\right]$$

- rapidity divergence cancels between sectors (as expected)
- overall counterterm has only $\ln\mu/Q$, hard scale.

  same for hard match $\mathcal{H}(\mu, Q)$
- logs in I_{c_n} minimized for $\mu \sim m$, $\nu \sim p^- = Q$

  which is precisely



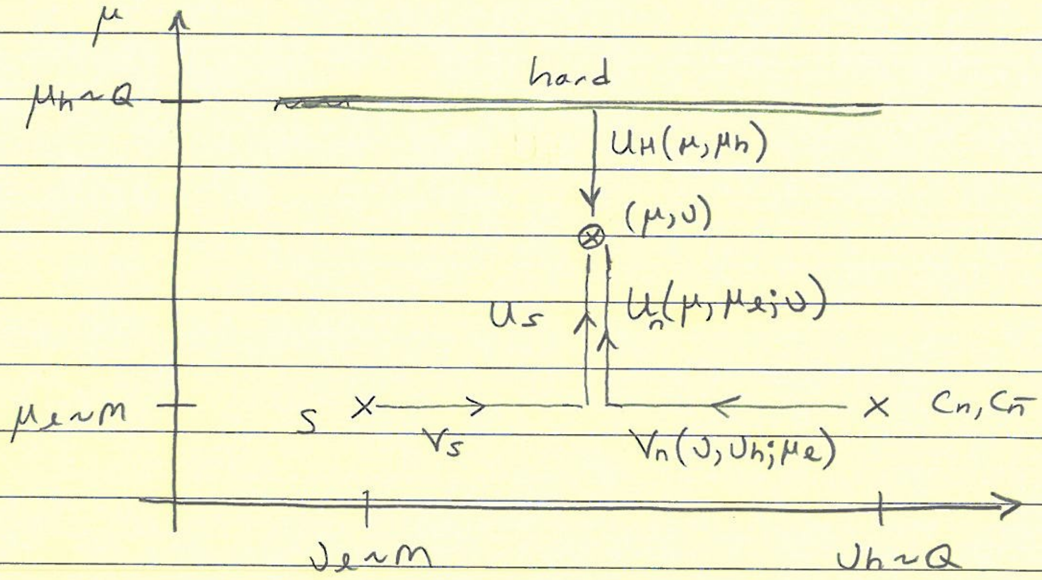
the location of c_n

- likewise need $\mu \sim M$, $\nu \sim \bar{p}^+ = Q$ for $C_{\bar{n}}$
  need $\mu \sim \nu \sim M$ for S

$$F(Q^2, M^2) = \mathcal{H}(Q^2, \mu)\, C_n(M, \mu, \nu/Q)\, C_{\bar{n}}(M, \mu, \nu/Q)\, \mathcal{S}(M, \mu, \nu/\mu)$$

renormalized fact. thm. with 2-cutoffs μ & ν

- Will have a μ-RGE and ν-RGE to sum logs



choice of (μ, ν) arbitrary (just freedom to run coeffs or operators)
eg. pick $(\mu, \nu) = (\mu_e, \nu_h)$ then just evolution kernels
$U_h(\mu_e, \mu_h)\, V_S(\nu_h, \nu_e; \mu_e)$

- Path Independence. μ & ν parameters are independent

$$\mu \frac{d}{d\mu}\, \nu \frac{\partial}{\partial \nu} = \nu \frac{\partial}{\partial \nu}\, \mu \frac{d}{d\mu}$$

- Counterterms & Anom. Dims.

$$C_n(m, \mu, \nu/Q) = Z_{\xi_n}^{-1/2}\, Z_n^{-1}\, C_n^{\text{bare}}$$
$$\mathcal{S}(m, \mu, \nu/\mu) = Z_S^{-1}\, \mathcal{S}^{\text{bare}} \qquad Z_{\xi_n} = 1 + \frac{\alpha_s C_F}{4\pi \epsilon}$$

$$Z_S = 1 - \frac{\alpha_s C_F}{\pi}\,\omega^2\left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\mu/m\right)^{2\epsilon}}{\eta} - \frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu}{\nu}\right]$$

$$Z_n = 1 + \frac{\alpha_s C_F}{\pi}\,\omega^2\left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\mu/m\right)^{2\epsilon}}{2\eta} + \frac{3}{8\epsilon} + \frac{1}{2\epsilon}\ln\frac{\nu}{p^-}\right]$$

μ- Anom. Dim

$$\gamma_\mu^S = -Z_S^{-1}\,\mu\frac{d}{d\mu} Z_S = \frac{\alpha_s(\mu)C_F}{\pi}\, 2\ln\frac{\mu}{\nu}$$

$$\mu\frac{d}{d\mu} S = \gamma_\mu^S S \quad \text{etc.}$$

$$\gamma_\mu^n = -Z_n^{-1}\,\mu\frac{d}{d\mu} Z_n = \frac{\alpha_s(\mu)C_F}{\pi}\left[\ln\frac{\nu}{p^-} + \frac{3}{4}\right]$$

$$\gamma_\mu^{\bar n} = \quad = \frac{\alpha_s(\mu)C_F}{\pi}\left[\ln \nu/p^+ + 3/4\right]$$

gives U_S, U_n Kernels

consistency $\gamma_\mu^S + \gamma_\mu^n + \gamma_\mu^{\bar n} = -\gamma_H = \frac{\alpha_s C_F}{\pi}\left(2\ln\frac{\mu}{Q} + \frac{3}{2}\right)$

ν Anom-Dim

$$\gamma_\nu^S = -Z_S^{-1}\,\nu\frac{\partial}{\partial\nu} Z_S = -\frac{\alpha_s(\mu)C_F}{\pi}\, 2\ln\frac{\mu}{m}$$

$$\gamma_\nu^n = -Z_n^{-1}\,\nu\frac{\partial}{\partial\nu} Z_n = \frac{\alpha_s(\mu)C_F}{\pi}\ln\frac{\mu}{m} = \gamma_\nu^{\bar n}$$

$\nu\frac{\partial}{\partial\nu} S = \gamma_\nu^S S$ etc. gives V_S, V_n Kernels

Path Independence: $Z^{-1}\left[\mu\frac{d}{d\mu}, \nu\frac{\partial}{\partial\nu}\right] Z = 0$

so $\mu\frac{d}{d\mu}\gamma_\nu^S = \nu\frac{\partial}{\partial\nu}\gamma_\mu^S$ ✓

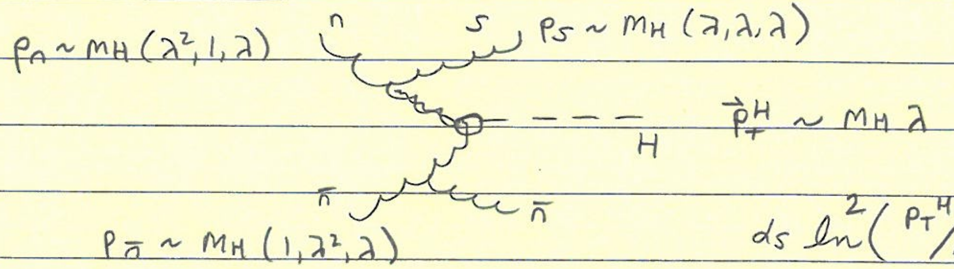
$\mu\frac{d}{d\mu}\gamma_\nu^n = \nu\frac{\partial}{\partial\nu}\gamma_\mu^n$ ✓

eg. $U_S^{LL}(\mu, \mu_S; \nu_S) = \exp\left[-\frac{8\pi C_F}{\beta_0^2}\left(\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu_S)} - \frac{1}{\alpha_s(\nu_S)}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_S)}\right)\right]$

$$V_S^{LL}(\nu, \nu_S; \mu) = \exp\left[\frac{2C_F}{\beta_0}\ln\left(\frac{\alpha_s(\mu)}{\alpha_s(m)}\right)\ln\left(\frac{\nu^2}{\nu_S^2}\right)\right]$$

$SCET_{II}$ examples with rapidity RGE

gg → Higgs p_T distribution



$\alpha_s \ln^2(p_T^H/m_H)$

Since we only measure $p_T^H \sim \lambda$ we can have soft radiation

Factorize cross-section $(|\mathrm{Ampl}|^2)$

$J_{full} = h\, G^{\mu\nu} G_{\mu\nu}$ $\qquad$ h = Higgs field

$$\langle J_{full}(x)\, J_{full}(0)\rangle = H(m_H)\ \langle p_n|\, g\mathcal{B}_{n\perp}\, g\mathcal{B}_{n\perp}\, |p_n\rangle$$ ← gluon pdfs

$$\langle p_{\bar{n}}|\, g\mathcal{B}_{\bar{n}\perp}\, g\mathcal{B}_{\bar{n}\perp}\, |p_{\bar{n}}\rangle$$

$$\langle 0|\, S_n\, S_{\bar{n}}\, S_n^T\, S_{\bar{n}}^T\, |0\rangle$$

↑ adjoint rep for soft wilson lines

$$\frac{d\sigma}{d p_T^{H2}\, dy} = N_0\, H(m_H,\mu) \int d^2p_{1\perp}\, d^2p_{2\perp}\, d^2p_{s\perp}\ \delta\left(p_T^{H2} - |\vec{p}_{1\perp}+\vec{p}_{2\perp}+\vec{p}_{s\perp}|^2\right)$$

$$* f_{g/p}^{\mu\nu}\left(\frac{m_H}{E_{cm}}e^{-Y}, \vec{p}_{1\perp}, \mu, \frac{\nu}{m_H e^{-Y}}\right)$$

$$* f_{g/p}^{\mu\nu}\left(\frac{m_H}{E_{cm}}e^{Y}, \vec{p}_{2\perp}, \nu/m_H e^{Y}\right)\ S\left(\vec{p}_{s\perp}^{\,2}, \mu, \nu/\mu\right)$$

↑ p_T dependent soft fn.

↑ transverse momentum dependent PDF which had rapidity divergences (prior to ν renormalization)

Jet Broadening $\quad e^+e^- \xrightarrow{Q^2}$ dijets

here only measure $\vec{p}_\perp$ (relative to thrust axis)

Broadening $= B = \sum_i \frac{|\vec{p}_{i\perp}|}{Q} = B_L + B_R = \sum_{i\in L}(\;) + \sum_{i\in R}(\;)$

Again we only measure $\perp$-momenta, $p_\perp \sim \lambda$, $B \sim \lambda$

so have $\mathrm{SCET}_{\mathrm{II}}$: $C_n, C_{\bar n}, S$

$$\begin{aligned}\frac{1}{\sigma_0}\frac{d\sigma}{dB_L\,dB_R} = H(Q^2,\mu) &\int de_n\, de_{\bar n}\, de_s^L\, de_s^R \int d^2\vec{k}_{1\perp}\, d^2\vec{k}_{2\perp} \\ &\delta(B_R - e_n - e_s^R)\,\delta(B_L - e_{\bar n} - e_s^L) \\ &J_n\left(Q, e_n, \vec{k}_{1\perp}, \mu, \frac{\nu}{Q}\right) J_{\bar n}\left(Q, e_{\bar n}, \vec{k}_{2\perp}, \mu, \frac{\nu}{Q}\right) \\ &* S\left(e_s^R, e_s^L, \vec{k}_{1\perp}, \vec{k}_{2\perp}, \mu, \nu/\mu\right)\end{aligned}$$

Another inclusive example: $B \to X_s \gamma$ [case where usoft modes matter]

Here we will need both usoft & collinear d.o.f. in SCET$_{I}$

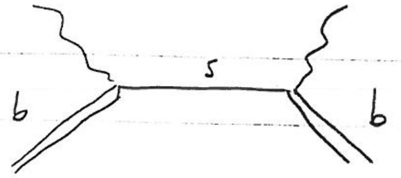
$$H_{eff} = \frac{-4G_F}{\sqrt{2}} V_{tb} V_{ts}^{*} C_7 \mathcal{O}_7 \quad , \quad \mathcal{O}_7 = \frac{e}{16\pi^2} m_b \bar{s} \sigma^{\mu\nu} F_{\mu\nu} P_R b$$

photon $q^\mu = E_\gamma \bar{n}^\mu$

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dE_\gamma} = \frac{4E_\gamma}{m_b^3} \left(\frac{-1}{\pi}\right) \operatorname{Im} T$$

$$T = \frac{i}{m_B} \int d^4x \, e^{-iq\cdot x} \langle \bar{B} | T \, J_\mu^\dagger(x) J^\mu(0) | \bar{B} \rangle$$

$$J^\mu = \bar{s} \, i\sigma^{\mu\nu} q_\nu P_R b$$

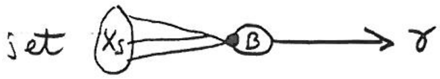


looks like DIS

Consider endpoint region

$$m_B/2 - E_\gamma \lesssim \Lambda_{QCD}$$

$$p_X^2 \simeq m_B \Lambda$$



B-rest frame $\quad p_B = \frac{m_B}{2}(n^\mu + \bar{n}^\mu) = p_X + q$

$$p_X = \frac{m_B}{2} n^\mu + \frac{\bar{n}^\mu}{2} \underbrace{(m_B - 2E_\gamma)}_{\Lambda}$$

collinear

so quarks and gluons in X are collinear with $p_c^2 \sim m_B \Lambda$

B has usoft light d.o.f.

match onto LO SCET operator

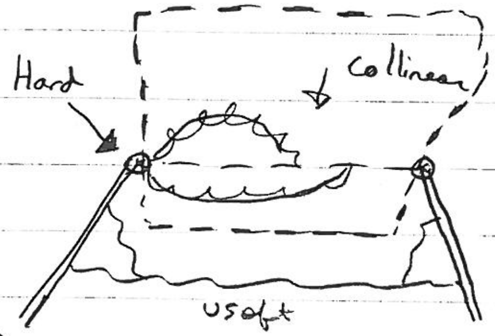
$$J_\mu = -E_\gamma \, e^{i(\bar{P}\frac{n}{2} - m_b v)\cdot x} \; \bar{\xi} W \gamma_\mu^\perp P_L h_v \; C(\bar{P}^+, \mu)$$

↑ our heavy-to-light current from earlier $\equiv J_{eff}^\mu$

The coefficient $C(\bar{P}^+)$ has $\bar{P}^+ = m_b$ since this is total momentum of s-quark jet in $\bar{n}\cdot P_X$

Factor with Field redefn

$$J_{eff}^\mu = \bar{\xi}_n^{(0)} W^{(0)} \gamma_\mu^\perp P_L Y^+ h_v$$



$$T_{eff} = i\int d^4x \; e^{i(m_b\frac{\bar{n}}{2} - q)\cdot x} \langle \bar{B} | T \, J_{eff}^{+\mu}(x) \, J_{eff\,\mu}(0) | \bar{B} \rangle$$

factored

$$= i\int d^4x \; e^{i()} \quad \langle \bar{B} | T (\bar{h}_v Y)(x) \; (Y h_v)(0) \; | \bar{B} \rangle$$
$$* \; \langle 0 | T \left(W^{+(0)} \xi^{(0)} \right)(x) \; \left(\bar{\xi}^{(0)} W \right)(0) \; | 0 \rangle$$

↖ spin & color indices & structures $\gamma_\mu^\perp P_L$ suppressed

$$= \frac{1}{2} \int d^4x \int đ^4k \; e^{i(m_b\frac{\bar{n}}{2} - q - k)\cdot x} \; \langle \bar{B} | T (\bar{h}_v Y)(x) \; (Y^+ h_v)(0) \; | \bar{B} \rangle$$
$$* \; J_P(k)$$

$$\langle 0 | T \, (W^+ \xi)_{P,\,0_\perp} \, (\bar{\xi} W) \, | 0 \rangle = \frac{i}{P^-} \int đ^4k \; e^{-ik\cdot x} \; J_P(k) \, \frac{\bar{n}\!\!\!/}{2}$$

↑ minus & ⊥ labels

↑ only depend on k^+! so do $k^-, k^\perp$ integrals

in T_{eff} we then get $\rightarrow \; \delta(x^+) \, \delta^2(x_\perp) \; \rightarrow$

$$S(\ell^+) = \frac{1}{2} \int \frac{dx^-}{4\pi} \; e^{-i/_2 \, \ell^+ x^-} \; \langle \bar{B} | T [\bar{h}_v Y) (\tfrac{n}{2} x^-) \; (Y^+ h_v)(0) | \bar{B} \rangle$$

↖ ↗ $Y(\frac{n}{2}x^-, 0)$

$$= \frac{1}{2} \langle \bar{B}_v | \; \bar{h}_v \, \delta(in\cdot D - k^+) \, h_v | \bar{B}_v \rangle$$

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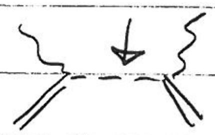
imaginary part is in jet function

let $J(k^+) \hat{=} -\frac{1}{\pi}\, \text{Im}\, J_p(k^+)$

(tree level $J(k^+) = \delta(k^+)$ from



)

All order's factorization

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_\gamma} = N \underset{\underset{p^2 \sim m_b^2}{\uparrow}}{C(m_b,\mu)} \int_{2E_\gamma - m_b}^{\bar{\Lambda}} dl^+ \underset{\underset{p^2 \sim \Lambda^2}{\uparrow}}{S(l^+)} \underset{\underset{p^2 \sim m_b\Lambda}{\uparrow}}{J(l^+ + m_b - 2E_\gamma)}$$

$\uparrow$

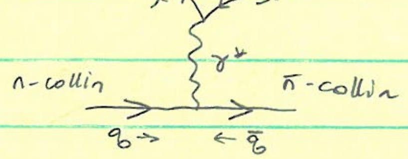
shape function
is <u>seen</u> in the
data

Final Example: Drell-Yan $pp \to X\, \ell^+ \ell^-$

- prototype LHC process (pp in, measure leptons, replace $\ell^+\ell^-$ by jets, ..., etc)

Kinematics $\quad p\ p \to X\ \underbrace{(\ell^+\ell^-)}$ $\qquad$ CM frame

$$p_A + p_B = p_X + q$$

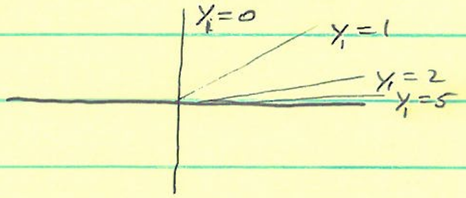


$E_{cm}^2 = (p_A+p_B)^2$ collision energy

q^2 hard scale of partonic collision

$\tau \equiv q^2/E_{cm}^2 \leq 1$

$Y = \frac{1}{2} \ln\left(\frac{p_b \cdot q}{p_a \cdot q}\right)$ total lepton rapidity (angular variable)



$$\left.\begin{aligned} x_a &\equiv \sqrt{\tau}\, e^{Y} \\ x_b &= \sqrt{\tau}\, e^{-Y} \end{aligned}\right\}$$ analogs of Bjorken Var in DIS

$$\tau \leq x_{a,b} \leq 1$$

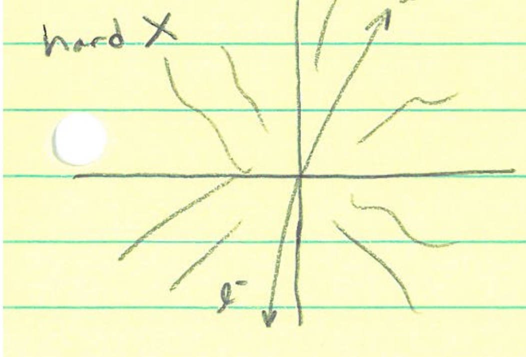
parton fractions $\qquad x_a \leq \xi_a \leq 1$

($\xi_i = x_i$ tree level) $\qquad x_b \leq \xi_b \leq 1$

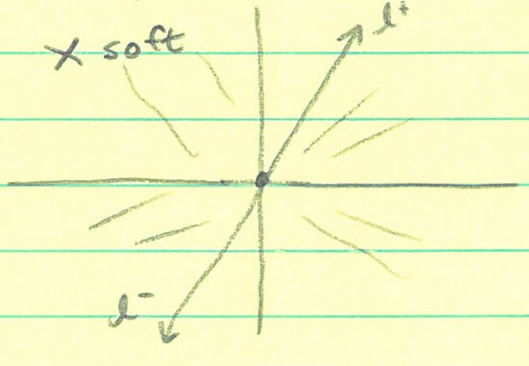
$$p_X^2 \leq E_{cm}^2 (1-\sqrt{\tau})^2$$

Cases:	Inclusive	$\tau \sim 1$	$p_X^2 \sim q^2 \sim E_{cm}^2$	$x_{a,b} \sim 1$	$\xi_{a,b} \sim 1$
	Endpoint	$\tau \to 1$	$p_X^2 \ll q^2 \to E_{cm}^2$ $\uparrow$ usoft	$x_{a,b} \to 1$	$\xi_{a,b} \to 1$
	(Small x	$\tau \to 0$	take $\xi_a, \xi_b \to 0$)		
	"Isolated"	$\tau \sim 1$	$p_X^2 \to$ two ISR jets	$x_{a,b} \sim \xi_{ab} \sim 1$	

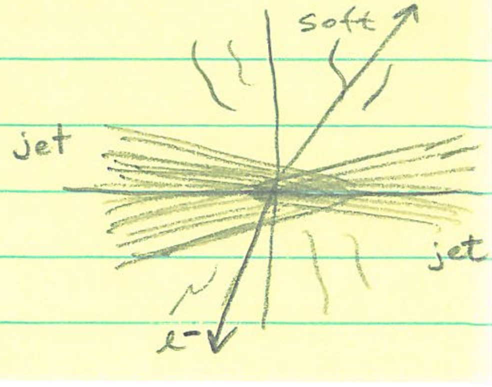
Incusive



Endpoint

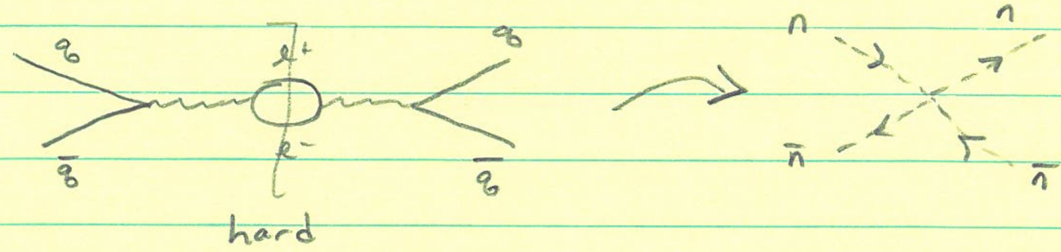


Isolated



Inclusive $\qquad p_n\, p_{\bar n} \to X_{hard}\,(\ell^+\ell^-)$

Factorization: $\quad SCET_I$ problem (hard-collinear Factorization)



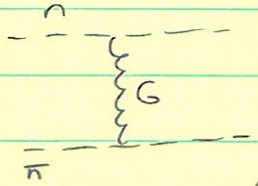
4-quark operator in SCET, which after Fierzing is

$$\Big[(\bar\xi_n W_n)\frac{\bar{n}\!\!\!/}{2}(W_n^\dagger \xi_n)\Big]\Big[(\bar\xi_{\bar n} W_{\bar n})\frac{n\!\!\!/}{2}(W_{\bar n}^\dagger \xi_{\bar n})\Big]$$

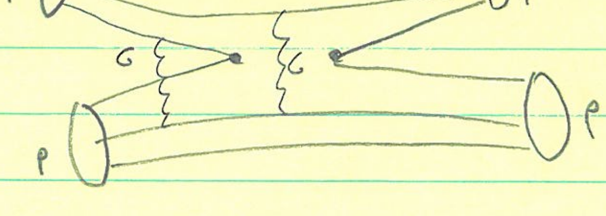
- $T^A \otimes T^A$ octet structure vanishes under $\langle p_n| \cdots |p_n\rangle$
- $\xi_n \to Y_n \xi_n$, $\xi_{\bar n} \to Y_{\bar n}\xi_{\bar n}$ etc, no coupling to soft gluons, they cancel out
- $\langle p_n|\, \bar\chi_{n,\omega} \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega'}\,|p_n\rangle$ gives PDF
  $\langle p_{\bar n}|\, \bar\chi_{\bar n,\bar\omega} \frac{n\!\!\!/}{2} \chi_{\bar n,\bar\omega'}\,|p_{\bar n}\rangle$ " "

$$\frac{1}{\sigma_0}\frac{d\sigma}{dq^2\,dY} = \sum_{i,j}\int_{x_a}^{1}\frac{d\xi_a}{\xi_a}\int_{x_b}^{1}\frac{d\xi_b}{\xi_b}\, H_{ij}^{incl}\Big(\frac{x_a}{\xi_a},\frac{x_b}{\xi_b},q^2,\mu\Big)\, f_i(\xi_a,\mu)\, f_j(\xi_b,\mu)$$
$$* \Big[1 + \mathcal{O}\Big(\frac{\Lambda_{QCD}}{\sqrt{q^2}}\Big)\Big]$$

- One more (important) caveat, "Glauber Gluons"



$$\begin{array}{c} \quad\ + \quad - \quad \perp \\ p_G^\mu \sim (\lambda^2, \lambda^2, \lambda) \end{array}$$



These gluons cancel out at Leading Order (Proving this would take us too far afield)

Threshold Limit only certain terms in H_{ij}^{incl} contribute (most singular in $1-\tau$)

$$H_{ij}^{incl} \to S_{q\bar{q}}^{thr}\left[\sqrt{q^2}\left(1-\frac{\tau}{\xi_a \xi_b}\right), \mu\right] H_{ij}(q^2,\mu) \left[1 + \mathcal{O}(1-\tau)^0\right]$$

$\uparrow$ $ij = u\bar{u}, d\bar{d}, \ldots$ quarks
no glue

$\xi_{a,b} \to 1$ so one parton in each proton carries all the momentum (not the dominant LHC region
but pdf's may enhance the importance of these terms)

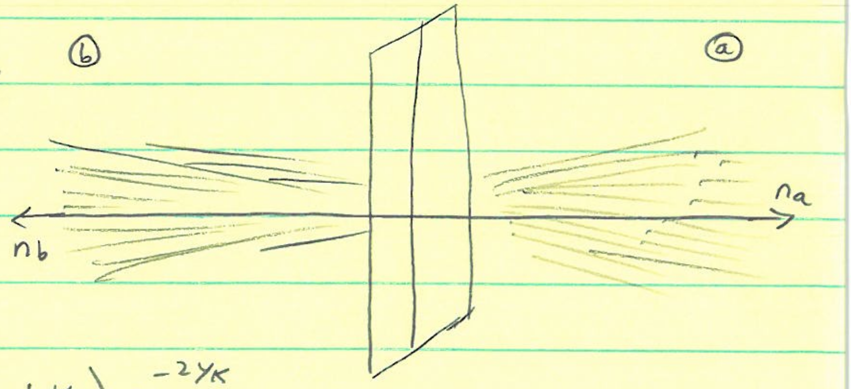
Isolated DY - allow forward jets to carry away part of Ecm, so $\xi_{a,b} \not\to 1$

- restrict central region to still only have soft radiation (signal region is bkgnd free, no jets, ie jet veto)

need to __observe__ something to guarantee this.

__Observable__ $p_X = B_a + B_b$

- two hemispheres, $\perp$ to the beam axis



$$B_a^+ = n_a \cdot B_a = \sum_{k \in a} n_a \cdot p_k$$
$$= \sum_{k \in a} E_k \left(1 + \tanh y_k\right) e^{-2y_k}$$

plus momenta for n-collinear radiation should be small

Take $B_a^+ \le Q\, e^{-2y_{cut}} \ll Q$ $\qquad Q = \sqrt{q^2}$

$B_b^+ \equiv n_b \cdot B_b \le$ " $\ll Q$

does the trick
(inclusive variable for jet veto)

n-collinear : proton ⓐ <u>and</u> jet ⓐ

we do not simply get a PDF from the hard-collinear-soft factorization

[Glauber's again cancel]

$$\frac{1}{\sigma_0} \frac{d\sigma}{dq^2\, dY\, dB_a^+ dB_b^+} = \sum_{ij} H_{ij}(q^2,\mu) \int dk_a^+ dk_b^+ \; Q^2 B_i[\omega_a(B_a^+ - k_a^+), x_a, \mu]$$
$$* \; B_j[\omega_b(B_b^+ - k_b^+), x_b, \mu]$$
$$* \; S_{ihemi}(k_a^+, k_b^+, \mu)$$
$$* \left[1 + \mathcal{O}\left(\frac{\Lambda_{QCD}}{Q}, \frac{\sqrt{B_{a,b}^+ \omega_{a,b}}}{Q} \right) \right]$$

where $\omega_{a,b} = x_{a,b} E_{cm}$

B_i = "beam function"

$$B_q(\omega b^+, \omega/P^-, \mu) = \frac{\theta(\omega)}{\omega} \int \frac{dy^-}{4\pi} e^{ib^+ y^-/2} \langle p_n(P^-) | \bar{\chi}_n(y^- \tfrac{n}{2}) \, \delta(\omega - \bar{P}) \frac{\bar{n}\!\!\!/}{2} \chi_n(0) | p_n(P^-) \rangle$$

recall jet fn $\langle 0 | \bar{\chi}_{n,\omega}(y^- \tfrac{n}{2}) \frac{\bar{n}\!\!\!/}{2} \chi_n(0) | 0 \rangle$

PDF $\langle p | \bar{\chi}_{n,\omega}(0) \frac{\bar{n}\!\!\!/}{2} \chi_n(0) | p \rangle$

beam function is mix of both

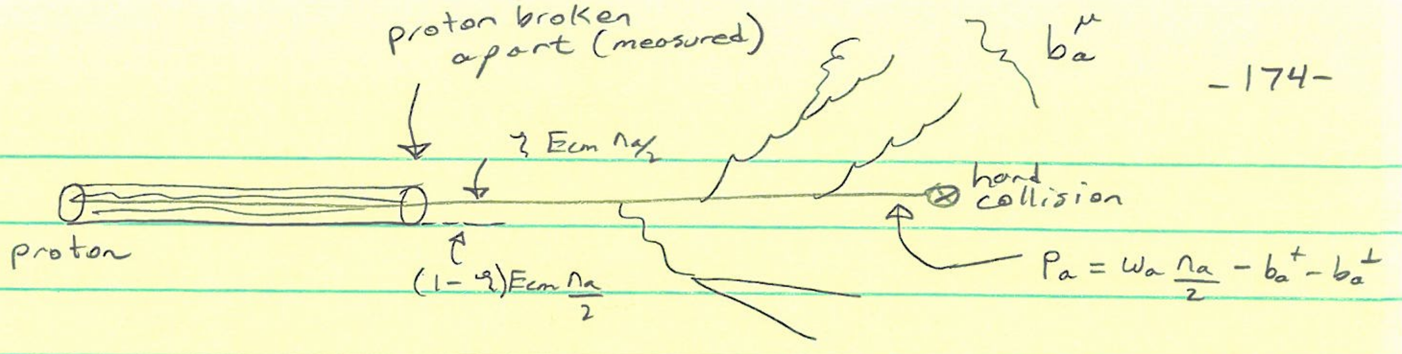
proton = $SCET_{II}$ collinear

jet = $SCET_{I}$ collinear (B_q is in $SCET_I$)

Match $SCET_I \to SCET_{II}$:

$$B_i(t, x, \mu) = \sum_j \int_x^1 \frac{d\xi}{\xi} \, \mathcal{I}_{ij}\left(t, \frac{x}{\xi}, \mu\right) f_j(\xi, \mu) \quad \left[1 + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{t} \right) \right]$$

$\uparrow$ f_g & f_q contribute to B_q (B_g)

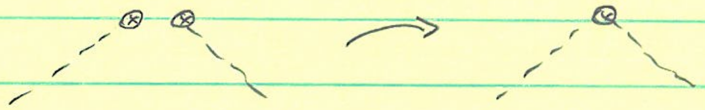


$$b_a^\mu = (\xi - x) E_{cm} \frac{n_a}{2} + b_a^+ \frac{\bar{n}_a}{2} + b_{a\perp}$$

$$p_a^2 = \underbrace{-\omega_a b_a^+}_{t_a \gg \Lambda_{QCD}} - \vec{b}_{\perp a}^{\,2} \leq 0$$

spacelike active parton participates in hard collision

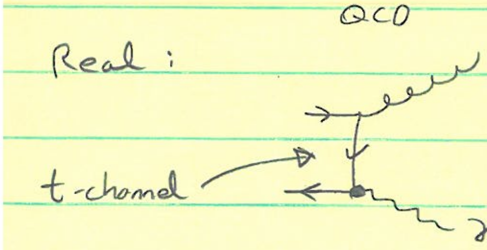
<u>Tree-Level</u>



$$B_i(t, x, \mu) = \delta(t)\, f_i(x, \mu)$$

<u>Order α_s</u> Real & Virtual Contractions

Real:



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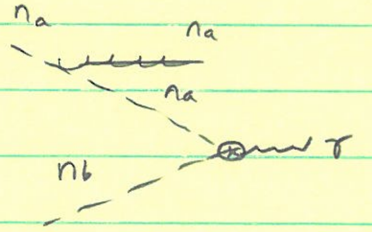
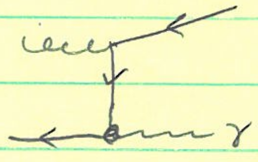


$\mathcal{I}_{qq}^{(\alpha_s)}$



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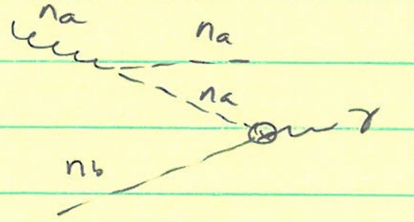
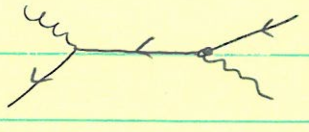


$\mathcal{I}_{qg}^{(\alpha_s)}$



power correction $\sim \frac{t}{s} \sim \frac{\omega B_a^+}{Q^2}$

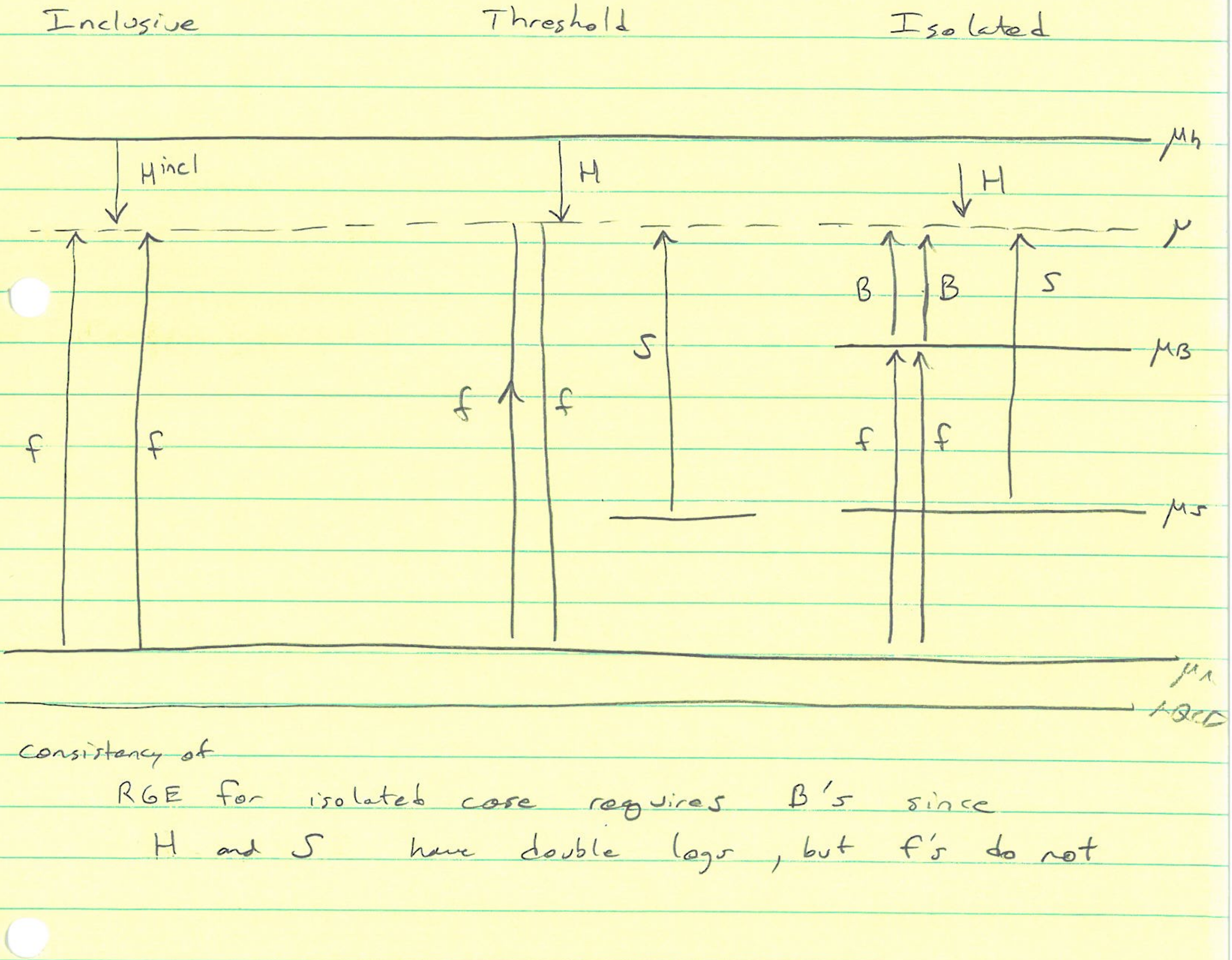
(would be ~1 for inclusive)

$\boxed{\text{RGE}}$ $$\mu \frac{d}{d\mu} B_i(t,x,\mu) = \int dt' \, \gamma_i(t-t',\mu) \, B_i(t',x,\mu)$$

like the jet function

(invariant mass evolution)

- sums $\ln^2(t/\mu)$
- indep of x & no mixing



consistency of

RGE for isolated case requires B's since H and S have double logs, but f's do not