8.851 Course 2009

Lecture Notes
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Soft-Collinear Effective Theory (SCET)

Outline : Ch 6 SCET Formalism
Ch7 Applications of SCET

ch6 & 7
Topics & Refs [more as we go online]

(i) Intro, Degrees of Freedom, Scales,
Expansion of Spinors, Propagators,
Power counting see ②, ③

(ii) Construction of $\mathcal{L}_{SCET}$, Currents
Multipole Expansion, Labels
Zero-bin, $\overset{\text{I.R.}}{\text{Div.}}$ see ②, ③, ⑩

focus
(iii) SCET_{I}, Gauge Symmetry, ③, ④, ⑥
Reparameterization Invariance

(iv) Ultrasoft Collinear Factorization
Hard-Collinear Factorization,
Matching & Running for Hard Fn's ④, ⑧, ②, ③

(v) DIS, how SCET p.c. includes
twist expansion, renormalization with
convolutions (DGLAP from EFT)

(vi) SCET_{II}: Soft-Collinear Interaction
use of auxillary Lagrangians, Power
counting formula, ④, ⑦, ⑩, ⑤
$\gamma^*\gamma \to \pi^0$ ⑧ $B \to D\pi$ ⑨

(vii) Power Corrections, Deriving SCET_{II}
from SCET_{I}, $B \to \pi \ell \nu$ ⑪
$B \to X_s \gamma$ ⑫

Refs I used

① hep-ph/0005275 (d.o.f.)
② hep-ph/0011336 (d.o.f. $\mathcal{L}$, W, ...)
③ hep-ph/0107001 (hard-collin fact. $\bar{P}$,)
④ hep-ph/0109045 (Gauge Inv. soft-collin)
⑤ hep-ph/0205289 (power counting)
⑥ hep-ph/0204229 (RPI)
⑦ hep ph/0303156 (Gauge Inv. at λ^2,)
⑧ hep-ph/0202088 (Hard scattering)
⑨ hep-ph/0107002 ($B \to D\pi$)
⑩ hep-ph/0605001 (0-bin)
⑪ hep-ph/0211069 ($\text{SCET}_{\text{I}} \to \text{SCET}_{\text{II}}$)
⑫ hep-ph/0409045 ($B \to X_s \gamma$ subl. order)

(viii) $e^+e^- \Rightarrow$ dijets, resummation, power corrections & soft functions

Refs : see website

(ix) $e^+e^- \to$ massive particles

(x) Parton Shower from SCET

(xi) $pp \to X \, \ell^+ \ell^-$, $pp \to H \, X$
(Drell-Yan) (Higgs Production)

Inclusive vs. Threshold vs. Isolated Factorization. Beam Functions, Initial State Radiation

Section 1 Intro, Degrees of Freedom, Coordinates

SCET: an EFT for energetic hadrons $E_H \simeq Q \gg \Lambda_{QCD} \sim m_H$

an EFT for energetic jets $E_J \simeq Q \gg m_J = \sqrt{p_J^2}$

an EFT for massless hard ↔ soft ↔ collinear interactions

Why? • Our main probe of short distance physics is hard collisions ($e^+e^- \to$ stuff, $pp \to$ stuff). Disentangling the physics of QCD & other interactions requires a separation of scales → EFT → SCET ("Factorization")

- jets, energetic hadrons are very common

eg. Hard Scattering $e^- p \to e^- X$ (DIS), $p\bar{p} \to X \ell^+ \ell^-$ (Drell-Yan)

$Q \gg m_H$

$Q \gg m_J$

$\gamma^* \gamma \to \pi^0$, $\gamma^* p \to \gamma^{(*)} p'$ (Deeply Virt. Compton)

$e^+e^- \to$ jets, $e^+e^- \to J/\psi X$, $pp \to HX$,

...

eg. B-decays $B \to X_u e \bar{\nu}$, $B \to D\pi$, $B \to \pi e \bar{\nu}$, $B \to X_s \gamma$

$B \to \pi\pi$, ...

$m_B = 5.279$ GeV $\gg \Lambda_{QCD}$

- Need to separate perturbative $\alpha_s(Q)$ & non-perturbative effects in QCD (eg. hard scattering vs. parton distributions)

- Sum large Sudakov Logs $\sim (\alpha_s \ln^2)^k$

- SCET involves new EFT tools

Prelude (What makes SCET different from other EFT's?)

- we will have multiple fields for the same particle

  ξ_n = collinear quark field

  q_s = soft " "
- we will integrate out offshell modes but not entire d.o.f. (like HQET)
- SCET has convolutions $\sum_i C_i O_i \to \int d\omega \; C(\omega)\, O(\omega)$
- power counting parameter $\lambda \ll 1$ is not related to mass dimension of fields
- Wilson Lines $P \exp\left(ig \int ds \; n \cdot A(ns)\right)$ everywhere, subtle & interesting gauge symmetry structure
- $1/\epsilon^2$ divergences at 1-loop that require UV counterterm

Degrees of feedom for SCET:

eg 1 $B \to D\pi$ hadrons

in B rest frame $p_\pi^\mu = (2.310 \text{ GeV}, 0, 0, -2.306 \text{ GeV})$

$\simeq Q n^\mu$ to good approx.

$n^\mu = (1,0,0,-1)$, $n^2 = 0$ light like

↑ 0,1,2,3 basis

$Q \gg \Lambda_{QCD}$

<u>Light-cone coordinates</u> Basis vectors n^μ, $\bar{n}^\mu$

$$n^2 = 0 \,, \quad \bar{n}^2 = 0 \,, \quad n\cdot\bar{n} = 2$$

vectors $$p^\mu = \frac{n^\mu}{2}\,\bar{n}\cdot p + \frac{\bar{n}^\mu}{2}\, n\cdot p + p_\perp^\mu$$

metric $$g^{\mu\nu} = \frac{n^\mu \bar{n}^\nu}{2} + \frac{\bar{n}^\mu n^\nu}{2} + g_\perp^{\mu\nu}$$

epsilon $$\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta}\,\frac{\bar{n}_\alpha n_\beta}{2}$$

Notation

$$p^+ \equiv n\cdot p$$
$$p^- \equiv \bar{n}\cdot p$$
$$p^2 = p^+ p^- + p_\perp^2$$
$$= p^+ p^- - \vec{p}_\perp^{\,2}$$

- $n^2 = 0$ requires complementary vector $\bar{n}^\mu$ for decomposition (dual vector for orthogonality)
- choice $n^\mu = (1,0,0,-1)$, $\bar{n}^\mu = (1,\underbrace{0,0}_{\perp},1)$ works

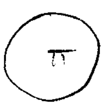
but other choices do too [eg. $n = (1,0,0,-1)$, $\bar{n} = (3,2,2,1)$]
(more later)

Constituent Quark & Gluons:

In $B\to D\pi$ the B, D are soft $E_H \sim M_H$ (use HQET for their constituents. quarks & gluons with $p^\mu \sim \Lambda$

But pion is "collinear" $E_\pi \gg M_\pi$

In rest frame



has quark & gluon constituents $p^\mu \sim (\overset{+}{\Lambda}, \overset{-}{\Lambda}, \overset{\perp}{\Lambda})$

boosting along $\hat{z}$

$p^- \to K p^-$, $p^+ \to \dfrac{p^+}{K}$
$(K \gg 1)$
$p_\perp \to p_\perp$

has constituents $p^\mu \sim \Big(\underbrace{\frac{\Lambda^2}{Q}, Q}, \Lambda\Big)$

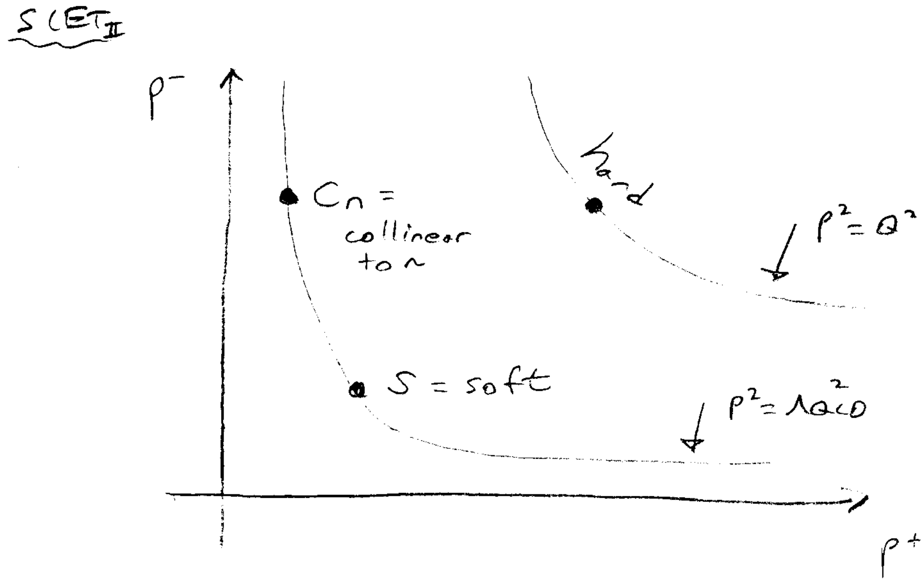
relative scaling <u>defines</u> collinear

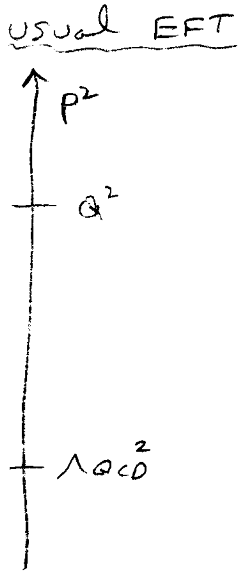
fluctuations about $(0, Q, 0) = p_\pi^\mu$

Generically $(p^+, p^-, p^\perp) \sim Q(\lambda^2, 1, \lambda)$ is collinear

where $\lambda \ll 1$ is small parameter (our eg. has $\lambda = \frac{\Lambda}{Q}$)

Degrees of freedom occupy momentum regions in SCET





$p^2 = p^+ p^- - \vec{p}_\perp^{\,2}$,

enough to look at $\vec{p}_\perp = 0$ plane

the theory with these d.o.f. is known as $SCET_{II}$, it applies for energetic hadron production

eg 2. inclusive decay to a jet $\quad B \to X_s \gamma \quad (b \to s\gamma)$

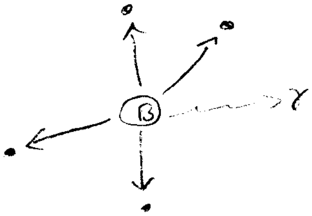
$\uparrow \geq 1$ hadron, summed over

two-body kinematics

$$E_\gamma = \frac{m_B^2 - m_X^2}{2m_B} \in \left[0, \frac{m_B^2 - m_{K^*}^2}{2m_B}\right]$$

for $m_X \in [m_B, m_{K^*}]$

3 regions for p.c. ⓘ $m_X^2 \sim m_B^2$



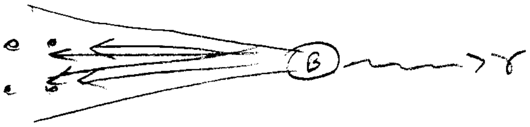
Standard OPE (HQET)

X has hadrons in all directions

(iii) $m_X^2 \sim \Lambda^2$

exclusive Decay ($SCET_{II}$)

(ii) $\Lambda^2 \ll m_X^2 \ll m_B^2$ (say $m_X^2 \sim m_B \Lambda$ to be definite)



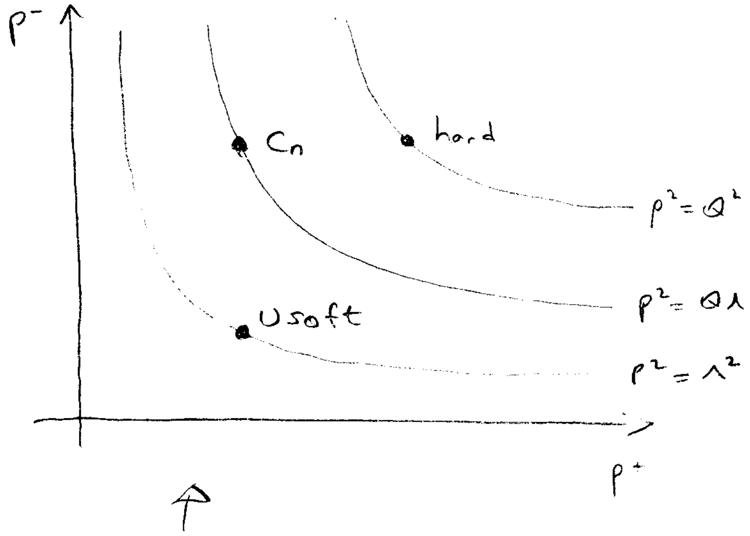
jet of hadrons in X

Jet constituents $(p^+, p^-, p_\perp) \sim (\Lambda, Q, \sqrt{\Lambda Q}) \sim Q(\lambda^2, 1, \lambda)$

collinear

here $\lambda = \sqrt{\Lambda/Q} \ll 1$

Modes

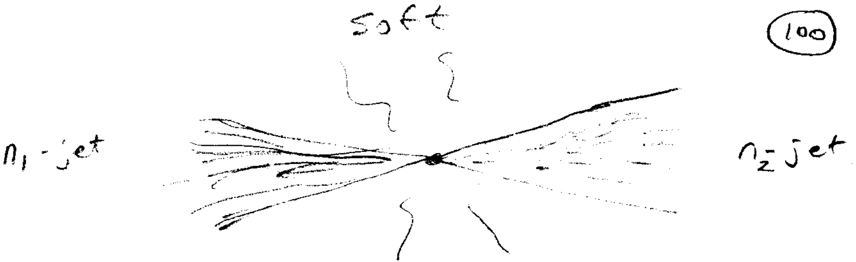


IR degrees of freedom with $p^2 \lesssim Q^2\lambda^2$

	$(+,-,\perp)$	p^2
collinear	$Q(\lambda^2, 1, \lambda)$	$Q^2\lambda^2$
ultrasoft	$Q(\lambda^2, \lambda^2, \lambda^2)$	$Q^2\lambda^4$
[soft	$Q(\lambda, \lambda, \lambda)$	$Q^2\lambda^2$]

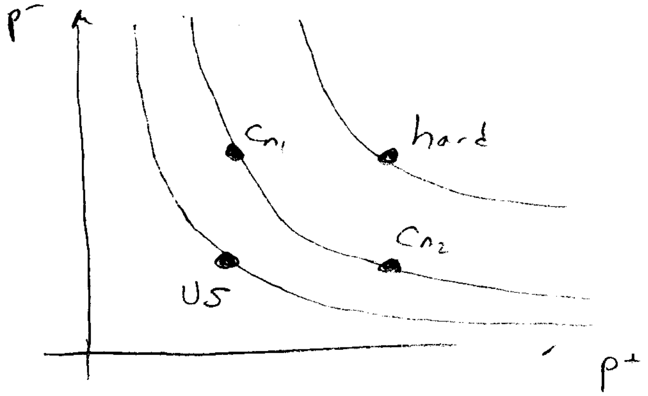
↑ recall

this is $SCET_I$, an EFT for energetic jets eg 3. $e^+e^- \to$ 2 jets



$\Lambda^2 \ll m_J^2 \ll Q^2$, $\lambda = \frac{m_J}{Q}$ (again)

$n_1 = n$
$n_2 = \bar{n}$ (say)



	$(+, -, \perp)$
n-collin	$(\lambda^2, 1, \lambda)Q$
$\bar{n}$-collin	$(1, \lambda^2, \lambda)Q$
usoft	$(\lambda^2, \lambda^2, \lambda^2)Q$

To Discuss:

i) multiple modes for IR ↔ p.c. ↔ multiple fields

ii) integrate out modes above given hyperbola (invariant mass)

iii) frame dependence

The theory "SCET$_{II}$" can be derived from "SCET$_I$", so we'll study I first.

Collinear Spinors $\qquad u_n$ labelled by direction n

(analog of HQET spinor u_v)

massless QCD spinors (Dirac Rep.) $\qquad u(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0}\, u \end{pmatrix}, \quad v(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{p^0}\, v \\ v \end{pmatrix}$

let $n^\mu = (1,0,0,1)$ $\qquad$ expand $\bar{n}\cdot p = p^0 + p^3 >> p_\perp >> n\cdot p = p^0 - p^3$

$\bar{n}^\mu = (1,0,0,-1)$ $\qquad \frac{\vec{\sigma}\cdot\vec{p}}{p^0} = \sigma^3$

$$u_n = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ \sigma^3 u \end{pmatrix} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \right\} \quad \text{particles}$$

$$v_n = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^3 v \\ v \end{pmatrix} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} \right\} \quad \text{antiparticles}$$

$$\not{n} = \begin{pmatrix} \mathbb{1} & -\sigma^3 \\ \sigma^3 & \underset{\uparrow}{-\mathbb{1}} \end{pmatrix} \qquad \text{so} \qquad \boxed{\not{n}\, u_n = \not{n}\, v_n = 0}$$

$$\frac{\not{n}\not{\bar{n}}}{4} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & \sigma^3 \\ \sigma^3 & \mathbb{1} \end{pmatrix} \qquad \text{so} \qquad \boxed{\frac{\not{n}\not{\bar{n}}}{4}\, u_n = u_n\,, \quad \frac{\not{n}\not{\bar{n}}}{4}\, v_n = v_n}$$

$\uparrow$

Projection Operator

Decompose $\quad \mathbb{1} = \frac{\not{n}\not{\bar{n}}}{4} + \frac{\not{\bar{n}}\not{n}}{4}$

$$\mathbb{1}\,\psi^{QCD} = \xi_n + \varphi_{\bar{n}}$$

$\uparrow$ we'll integrate out "small" component $\varphi_{\bar{n}}$

Collinear Propagators

$$p^2+i0 = \bar n\cdot p\, n\cdot p + p_\perp^2 + i0$$

$$\sim \lambda^0 * \lambda^2 + \lambda * \lambda \qquad \text{same size}$$

Fermions

$$\frac{i\not p}{p^2+i0} = \frac{i\not n}{2}\,\frac{\bar n\cdot p}{p^2+i0} + \overset{\lambda \text{ suppressed}}{\cdots}$$

$$\xrightarrow[p]{} \quad = \frac{i\not n}{2}\,\frac{1}{n\cdot p + \frac{p_\perp^2}{\bar n\cdot p} + i0\,\text{sign}(\bar n\cdot p)} + \cdots$$

↑ both particles $\bar n\cdot p>0$

& antiparticle $\bar n\cdot p<0$

from $T\{\xi_n(x), \bar\xi_n(0)\}$

Power counting of fields from free kinetic term

$$\mathcal{L} = \int d^4x \;\; \bar\xi_n \; \frac{\bar{\not n}}{2} \; [\, i n\cdot\partial + \cdots] \; \xi_n$$

$$\lambda^{-4} \quad \lambda^a \quad [\lambda^2 + \cdots] \quad \lambda^a \quad = \quad \lambda^{2a-2}$$

set $\mathcal{L}\sim\lambda^0$, normalize kinetic term so no λ's

then $\boxed{\xi_n\sim\lambda}$

Note: mass dimension $[\xi_n] = 3/2$

λ dimension $[\xi_n]^\lambda = 1$

Collinear Gluons

consider general covariant gauge

$$\int d^4x \; e^{ik\cdot x} \langle 0| T A_n^\mu(x) A_n^\nu(0) |0\rangle = \frac{-i}{k^2}\left(g^{\mu\nu} - \overset{\text{gauge param.}}{\tau} \frac{k^\mu k^\nu}{k^2} \right)$$

as above $k^2 = k^+k^- + k_\perp^2 \sim \lambda^2$, no expansion

Also $g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$ has two terms of same size

eg. $g_\perp^{\mu\nu} \sim \lambda^0 \sim \frac{k_\perp^\mu k_\perp^\nu}{k^2} \sim \frac{\lambda^2}{\lambda^2}$, $g^{+-} \sim \lambda^0 \sim \frac{k^+k^-}{k^2} \sim \frac{\lambda^2\lambda^0}{\lambda^2}$

dot $n_\mu n_\nu$: $g^{++} = 0$, $\frac{(n\cdot k)^2}{k^2} \sim \frac{\lambda^4}{\lambda^2} = \lambda^2$

$d^4x \sim \lambda^{-4} \sim \frac{1}{(k^2)^2}$ so $\underline{A_n^\mu \sim k^\mu} \sim \overset{+,-,\perp}{(\lambda^2, 1, \lambda)}$

$A_n^\mu = (A_n^+, A_n^-, A_n^\perp) \sim (\lambda^2, 1, \lambda)$

ie $k^\mu + gA^\mu = iD^\mu$ homogeneous covariant derivative

Note: $A_n^- \sim \lambda^0$ no suppression to add A_n^- fields

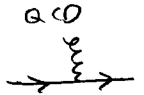
To see how this has an impact, consider an external weak $\boxed{\text{current}}$

eg. $b \to u e \bar{\nu}$ QCD $J = \bar{u}\,\Gamma\, b$ $\Gamma = \gamma^\mu(1-\gamma_5)$

consider heavy b (HQET), energetic u (SCET)

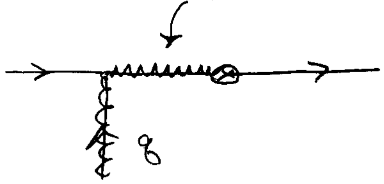
↑ dashed collinear quark

$J_{eff} = \bar{\xi}_n \Gamma h_v$



$= i g T^A \gamma^\mu$ sign convention

Consider



$\bar{n}\cdot A_n \sim \lambda^0$

$$k^\mu = m_b v^\mu + \frac{n^\mu}{2}\,\bar{n}\cdot q + \cdots$$
$$k^2 = m_b^2 + n\cdot v\, m_b\, \bar{n}\cdot q + \cdots$$
$$k^2 - m_b^2 \sim m_b^2 \quad \text{for} \quad \bar{n}\cdot q \sim \lambda^0 \sim m_b$$

no power suppression for these gluons

Find

$$A_{n\mu}\,\bar{\xi}_n \Gamma \frac{i(\not{k}+m_b)}{k^2-m_b^2}\, i g T^A \gamma^\mu h_v = -g A_n^{\mu A}\bar{\xi}_n\Gamma \frac{[\cancel{m_b}(1+\not{v}) + \cancel{\frac{\not{n}}{2}\bar{n}\cdot q}]}{n\cdot v\,\cancel{m_b}\,\bar{n}\cdot q}\frac{\not{n}}{2}\bar{n}_\mu T^A h_v$$

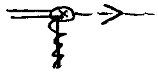
$0 = \not{n}^2$

$+ \cdots$

$$= \frac{-g\,\bar{n}\cdot A^A}{\bar{n}\cdot q}\,\bar{\xi}_n \Gamma T^A \left[\frac{+\cancel{\frac{\not{n}}{2}(1-\not{v})}^{\to 0} + \cancel{n\cdot v}}{\cancel{n\cdot v}}\right] h_v$$

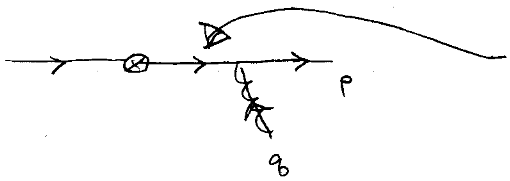
$\not{v} h_v = h_v$

$$= \frac{-g}{\bar{n}\cdot q}\,\bar{\xi}_n \Gamma\, \bar{n}\cdot A\, h_v =$$



same order in λ.

Consider

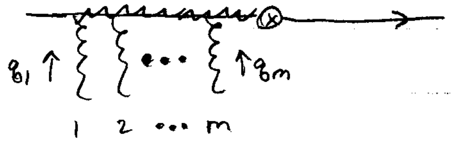


$p-q$ = collinear for p & q both collinear, so not offshell

$\leftrightarrow$ Lagrangian interaction

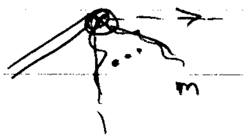
QCD graph

Consider



\+ crossed gluon graphs $\rightarrow$

SCET graph



$$= (-g)^m \;\Gamma \sum_{\substack{\text{perms}\\ \{1,\ldots,m\}}} \frac{\bar{n}^{\mu_m}T^{A_m}\cdots\bar{n}^{\mu_1}T^{A_1}}{[\bar{n}\cdot q_1][\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot\sum_{i=1}^m q_i]}$$

when we write fields for external lines we must be a bit careful

Since SCET vertex is localized with m identical fields

$$\to \frac{(\bar{n}\cdot A)^m}{m!}$$

Complete tree level matching is

$$\bar{u}\,\Gamma\, b \;\to\; \bar{\xi}_n\, W\, \Gamma\, h_v$$

where $$W = \sum_{k}\sum_{perms} \frac{(-g)^k}{k!}\left(\frac{\bar{n}\cdot A_{q_1}\cdots\bar{n}\cdot A_{q_k}}{[\bar{n}\cdot q_1][\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot\sum_{i=1}^{k} q_i]}\right)$$

is momentum space Wilson Line

position space Wilson line is

$$W(0,-\infty) = \underset{\uparrow}{P}\,\exp\left(ig\int_{-\infty}^{0} ds\;\bar{n}\cdot A_n(\bar{n}s)\right)$$

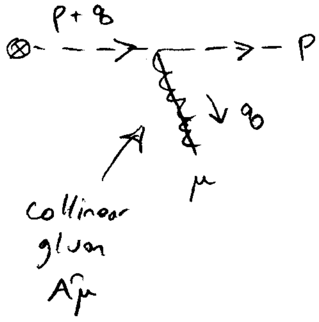
path ordering puts fields with larger arguement to the left $\quad \bar{n}\cdot A_n(\bar{n}s)\;\bar{n}\cdot A_n(\bar{n}s')$

for $s>s'$

Effectively: $\bar{n}\cdot A$ field gets traded for $W[\bar{n}\cdot A]$

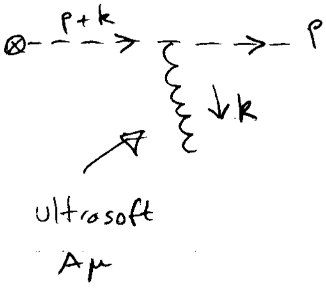
Consider SCET$_I$, collinear & usoft

$(\lambda^2,1,\lambda) \qquad (\lambda^2,\lambda^2,\lambda^2)$



$$\text{propagator} = \frac{\bar{n}\cdot(q+p)}{n\cdot(q+p)\,\bar{n}\cdot(q+p) + (q_\perp+p_\perp)^2 + i0}$$

$q^\mu \sim p^\mu$ so nothing dropped in denominator



here $k^\mu \sim \lambda^2$ $\qquad \bar{n}\cdot k \ll \bar{n}\cdot p \sim \lambda^0$

$k_\perp^\mu \ll p_\perp^\mu \sim \lambda$

$n\cdot k \sim n\cdot p$

$$\text{propogator} = \frac{\bar{n}\cdot p}{n\cdot(k+p)\,\bar{n}\cdot p + p_\perp^2 + i0} + \underset{\uparrow}{\cdots\cdot}$$

higher order terms

SCET Collinear Quark Lagrangian

Should: yield proper spin structure of propogator
have interactions with collinear gluons & usoft gluons
have both quarks & antiquarks
yield correct LO propogator for different situations
(without requiring an additional expansion)

(We'll meet (& resolve) some technical hurtles along the way.)

Step 1: Start with $\mathcal{L}_{QCD} = \bar\psi i \not{D} \psi$

Write $\psi = \xi_n + \varphi_{\bar n}$ where $\xi_n = \frac{\not{n}\not{\bar n}}{4}\psi$, $\varphi_{\bar n} = \frac{\not{\bar n}\not{n}}{4}\psi$

$\frac{\not{n}\not{\bar n}}{4}\xi_n = \xi_n$

$\frac{\not{\bar n}\not{n}}{4}\varphi_{\bar n} = \varphi_{\bar n}$

$$\mathcal{L} = (\bar\varphi_{\bar n} + \bar\xi_n)\left(i\frac{\not{\bar n}}{2}n\cdot D + \frac{i\not{n}}{2}\bar n\cdot D + i\not{D}_\perp\right)(\xi_n + \varphi_{\bar n})$$

$$= \bar\xi_n \frac{\not{\bar n}}{2} i n\cdot D\, \xi_n + \bar\varphi_{\bar n} i\not{D}_\perp \xi_n + \bar\xi_n i\not{D}_\perp \varphi_{\bar n} + \bar\varphi_{\bar n}\frac{\not{n}}{2} i\bar n\cdot D\, \varphi_{\bar n}$$

other terms are zero eg. $\bar\xi_n i\not{D}_\perp \xi_n = \bar\xi_n i\not{D}_\perp \frac{\not{n}\not{\bar n}}{4}\xi_n = \underbrace{\bar\xi_n \frac{\not{n}\not{\bar n}}{4}}_{0} i\not{D}_\perp \xi_n$

so for this is just QCD written in terms of $\varphi_{\bar n}$, ξ_n vars.

- $\varphi_{\bar n}$ corresponds to subleading spinor components. We will not consider a source for $\varphi_{\bar n}$ in path integral e.o.m. $\frac{\delta}{\delta \bar{\varphi}_{\bar{n}}}$: $\frac{\not{n}}{2}\, i\bar{n}\cdot D\, \varphi_{\bar{n}} + i\not{D}_\perp \xi_n = 0$

$$i\bar{n}\cdot D\, \varphi_{\bar{n}} + \frac{\not{\bar{n}}}{2}\, i\not{D}_\perp \xi_n = 0$$

$$\varphi_{\bar{n}} = \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \frac{\not{\bar{n}}}{2}\,\xi_n \quad , \quad \psi = \left(1 + \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \frac{\not{\bar{n}}}{2}\right)\xi_n$$

Plug back into ⊛: already used/satisfied 2nd & 4th terms, 1st & 3rd give

$$\mathcal{L} = \bar{\xi}_n \left(i n\cdot D + i\not{D}_\perp \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \right) \frac{\not{\bar{n}}}{2}\,\xi_n \qquad (**)$$

← insert (107.5) Aside

We're not yet done. We still need to:

② separate collinear & usoft gauge fields

③ " " " " momenta

④ expand and put pieces together

Step ② : $A_n^\mu \sim (\lambda^2, 1, \lambda) \sim p_n^\mu$, $A_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2) \sim k_{us}^\mu$

write $A^\mu = A_n^\mu + A_{us}^\mu + \dots$

like a classical background field to ξ_n, A_n^μ

$p_{us}^2 \sim Q^2\lambda^4 \ll p_c^2 \sim Q^2\lambda^2$

↑ long wavelength

there are some more terms that will matter for power corrections (& are fixed by gauge invariance). Ignore them for now.

Power counting

$$\left.\begin{aligned} \bar{n}\cdot A_n &\sim \lambda^0 \gg \bar{n}\cdot A_{us} \\ A_{\perp n}^\mu &\sim \lambda \gg A_{us}^\perp \\ n\cdot A_n &\sim \lambda^2 \sim n\cdot A_{us} \end{aligned}\right\}$$

so $A_{us}^\perp$ & $\bar{n}\cdot A_{us}$ can be dropped at leading order

What does $\frac{1}{i\bar{n}\cdot\partial}$ mean?

Its the analog of how you define $\frac{1}{\hat{r}}$ in quantum mechanics, you use the eigenbasis:

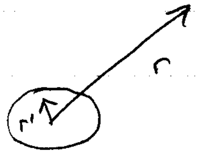
$$\frac{1}{i\bar{n}\cdot\partial}\,\phi(x) = \frac{1}{i\bar{n}\cdot\partial}\int d^4p\; e^{-ip\cdot x}\,\phi(p) = \int d^4p\; e^{-ip\cdot x}\,\frac{1}{\bar{n}\cdot p}\,\phi(p)$$

Step ③ We had a λ-expansion for a propagator carrying collinear & soft momenta

$$\frac{1}{(p_n+k_{us})^2} = \underset{\sim \lambda^{-2}}{\frac{1}{p_n^-(p_n^+ + k_{us}^+) + p_{n\perp}^2}} - \underset{\sim \lambda^{-1}}{\frac{2\, k_{us}^\perp \cdot p_n^\perp}{[p_n^-(p_n^+ + k_{us}^+) + p_{n\perp}^2]^2}} + \cdots$$

There must be Feyn. Rules in SCET to reproduce 2nd term too, so when we expand $k_{us}^\perp \ll p_n^\perp$, $k_{us}^- \ll p_n^-$ we can't just ignore $k_{us}^{\perp,-}$. We need a systematic (gauge invariant) multipole expansion.

[Recall E&M

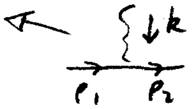


$r' \ll r$

$$V(r) = \frac{1}{r}\int d^3r'\, \rho + \frac{1}{r^2}\int r'\cos\theta\, \rho\, d^3r' + \cdots$$

Position Space (1-dim), consider

- $\int dx\, \bar{\psi}(x)\, A(0)\, \psi(x) = \int dx \int dp_1\, dp_2\, dk\; e^{ip_1\cdot x}\, e^{-i0\cdot x}\, e^{-ip_2\cdot x}\, \bar{\psi}(p_1) A(k) \psi(p_2)$
  $= \int dp_1 dp_2 dk\;\; \delta(p_1-p_2)\;\; \bar{\psi}(p_1)\, A(k)\, \psi(p_2)$



k gets dropped [momentum not conserved]

- $\int dx\; \bar{\psi}(x)\;\; x\cdot i\partial\, A(0)\;\; \psi(x)$
  $= \int dp_1 dp_2 dk\;\; \delta'(p_1-p_2)\; k\;\; \bar{\psi}(p_1) A(k)\, \psi(p_2)$

must int. by parts...

We will carry out the multipole expansion in momentum space

- more directly get mom. space Feyn. Rules
- simplifies formulation of gauge transformations
- mom. expansion sits in propagators rather vertices

eg. $\frac{k_{us}^\perp \cdot p_n^\perp}{[\cdots]^2} \sim \to\!\times\!\to$ propagator insertion

Call $q_n(x)$ field from

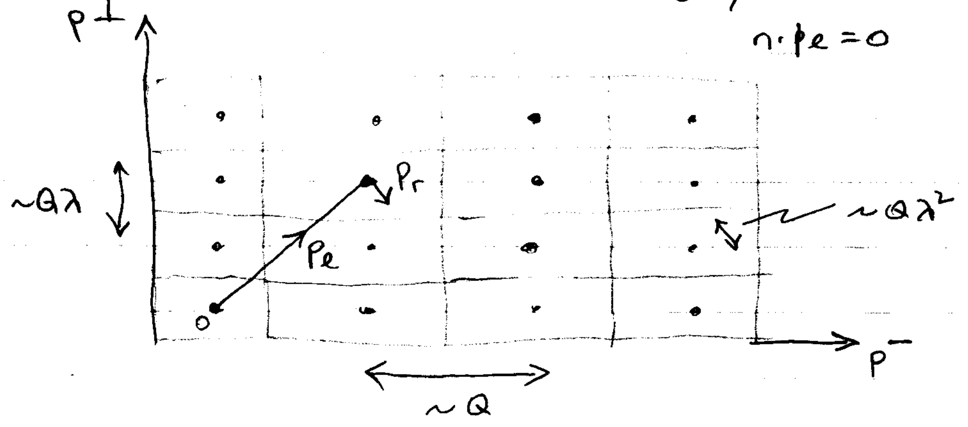
Eq. (**) pg. (107) $\to \hat{q}_n(x)$. [Consider only quark part, $a^s_{\vec{p}}$, to start.]

Let $\quad \tilde{q}_n(p) = \int d^4x \; e^{ip\cdot x} \; \hat{q}_n(x)$

$$\begin{array}{llcll} & & & \text{label} & \text{residual} \\ \text{Analogy} & \text{HQET:} & p^\mu = & m v^\mu & + k^\mu \\ & \text{SCET:} & p^\mu = & p_\ell^\mu & + p_r^\mu \end{array}$$

$$\underset{\substack{(p_\ell^-,\, p_\ell^\perp) \sim (1, \lambda) \\ \text{only}}}{\uparrow} \qquad \qquad \underset{p_r^\mu \sim (\lambda^2, \lambda^2, \lambda^2)}{\uparrow}$$

$n \cdot p_\ell = 0$



p_ℓ^μ discrete grid points

p_r^μ continuous

$p^\mu = p_\ell^\mu + p_r^\mu$ unique for given grid

$\int d^4p = \sum\limits_{p_\ell \neq 0} \int d^4 p_r \qquad$ for collinear p

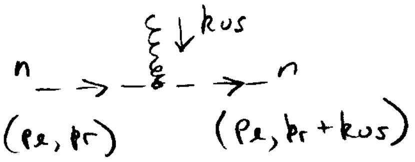
[$p_\ell = 0$ is not collinear]

$\int d^4p = \int d^4 p_r \qquad$ for usoft $p \qquad$ [usoft has $p_\ell = 0$]

Write: $\quad \tilde{q}_n(p) \;\to\; \tilde{q}_{n,p_\ell}(p_r)$

Note: We have separate conservation of label & residual momenta

$$\int d^4x \; e^{i(p_\ell - q_\ell)\cdot x} \; e^{i(p_r - q_r)\cdot x} = \delta_{p_\ell, q_\ell} \, \delta^4(p_r - q_r) \, (2\pi)^4$$



"non-conservation" of momenta is replaced by two separate conservations where some fields don't carry label momenta.

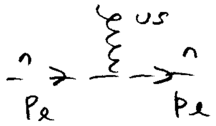
Final Step

Since all fields carry residual momenta the conservation law just corresponds to locality with respect to Fourier transform $p_r \to x$

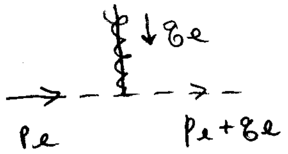
$$\xi_{n,p_\ell}(x) = \int \frac{d^4 p_r}{(2\pi)^4}\, e^{-ip_r\cdot x}\, \tilde{\xi}_{n,p_\ell}(p_r)$$

↑ build action from these fields

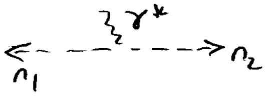
- usoft gluons leave labels conserved



- collinear gluons change labels



- label n for collinear direction always preserved by usoft & collinear gluons
  only a hard interaction can couple fields with different n's eg



back-to-back production by hard γ^*

All together

$$\hat{\xi}_n(x) = \int d^4 p\, e^{-ip\cdot x}\, \tilde{\xi}_n(p) = \sum_{p_\ell \neq 0} \int d^4 p_r\, e^{-ip_\ell\cdot x}\, e^{-ip_r\cdot x}\, \tilde{\xi}_{n,p_\ell}(p_r)$$

$$= \sum_{p_\ell \neq 0} e^{-ip_\ell\cdot x}\, \xi_{n,p_\ell}(x)$$

Define <u>two</u> derivative operators:

$$i\partial^x_\mu \, \xi_{n,p_\ell}(x) \sim \lambda^2 \, \xi_{n,p_\ell}(x)$$ residual

$$\mathcal{P}_\mu \, \xi_{n,p_\ell}(x) \equiv p_\ell^\mu \, \xi_{n,p_\ell}(x) \sim (\overset{+}{0}, \overset{-}{1}, \overset{\perp}{\lambda}) \, \xi_{n,p_\ell}(x)$$

$$\Rightarrow \quad i\bar{n}\cdot\partial \ll \bar{\mathcal{P}} = \bar{n}\cdot\mathcal{P} \, , \qquad i\partial_\perp^\mu \ll \mathcal{P}_\perp^\mu$$

implements multipole expansion
similar structure to expansion for
gauge fields → gauge symmetry easier

Notation is friendly:

$$\hat{\xi}_n(x) = \sum_{p_\ell \neq 0} e^{-ip_\ell\cdot x} \, \xi_{n,p_\ell}(x) = e^{-i\mathcal{P}\cdot x} \sum_{p_\ell\neq 0} \xi_{n,p_\ell}(x)$$

$$\equiv e^{-i\mathcal{P}\cdot x} \underbrace{\xi_n(x)}_{\sum_{p_\ell\neq 0} \xi_{n,p_\ell}(x)}$$

← suppress labels if we don't need them explicitly

Field products

$$\hat{\xi}_n(x)\,\hat{\xi}_n(x) = e^{-i\mathcal{P}\cdot x} \, \xi_n(x)\,\xi_n(x)$$

↑ acts on both fields & just gives <u>label</u> conservation

Last Step is to consider anti-quarks & gluons

Mode Expn

$$\psi(x) = \int d^4p \;\; \delta(p^2)\,\theta(p^0)\,\left[\,u(p)\,a(p)\,e^{-ip\cdot x} + v(p)\,b^\dagger(p)\,e^{ip\cdot x}\right]$$

$$= \psi^+ + \psi^-$$

QCD

Write
$$\left.\begin{aligned}\psi^+(x) &= \sum_{p_\ell\neq 0} e^{-ip_\ell\cdot x}\,\xi^+_{n,p_\ell}(x)\\ \psi^-(x) &= \sum_{p_\ell\neq 0} e^{ip_\ell\cdot x}\,\xi^-_{n,p_\ell}(x)\end{aligned}\right\}$$

both have $\theta(p^0) = \theta(\bar n\cdot p)$

$\not{n}\,\xi^\pm_{n,p_\ell} = 0$

Define $\xi_{n,p_\ell}(x) \equiv \xi^+_{n,p_\ell}(x) + \xi^-_{n,-p_\ell}(x)$ any p_ℓ signs

$\bar n\cdot p_\ell > 0$ particles destroy

$\bar n\cdot p_\ell < 0$ antiparticles create

$\bar\xi_{n,p_\ell}$ $\bar n\cdot p_\ell > 0$ part. create

$\bar n\cdot p_\ell < 0$ anti. destroy

then $\hat\xi_n(x) = e^{-i\mathcal{P}\cdot x}\,\xi_{n,p_\ell}(x)$ as before

Collinear Gluons $A^\mu_{n,q_\ell}(x)$, $\left[A^\mu_{n,q_\ell}(x)\right]^* = A^\mu_{n,-q_\ell}(x)$

$q_\ell > 0$ destroy

$q_\ell < 0$ create

$\hat A_n(x) = e^{-i\mathcal{P}\cdot x}\,A_n(x)$

$\uparrow \sum_{q_\ell} A_{n,q_\ell}(x)$

General Results

$$\mathcal{P}^\mu\left(\phi^\dagger_{q_1}\,\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots\right) = \left(p_1^\mu + p_2^\mu + \cdots - q_1^\mu - q_2^\mu - \cdots\right)\left(\phi^\dagger_{q_1}\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots\right)$$

eigenvalue eqtn

$$i\partial^\mu \sum_p e^{-ip\cdot x}\,\phi_{n,p}(x) = \sum_p e^{-ip\cdot x}\left(\mathcal{P}^\mu + i\partial^\mu\right)\phi_{n,p}(x)$$

$$= \underbrace{e^{-i\mathcal{P}\cdot x}}\left(\mathcal{P}^\mu + i\partial^\mu\right)\phi_n(x)$$

later we'll suppress this & recall that labels conserved

Step ④ Expand $\mathcal{L} = \bar{\hat{\xi}}_n(x)\left[in\cdot D + i\not{D}_\perp \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \right]\frac{\not{\bar{n}}}{2}\,\hat{\xi}_n(x)$

$$iD^\mu = \mathcal{P}^\mu + gA_n^\mu + i\partial^\mu + gA_{us}^\mu + \cdots$$

$$in\cdot D = in\cdot\partial + g\,n\cdot A_n + g\,n\cdot A_{us} \qquad (\text{exact, all} \sim \lambda^2)$$

$$iD_\perp = \underbrace{(\mathcal{P}_\perp + gA_n^\perp)}_{\lambda} + \underbrace{(i\partial_\perp + gA_\perp^{us})}_{\lambda^2} + \cdots$$

$$i\bar{n}\cdot D = \underbrace{(\bar{\mathcal{P}} + g\bar{n}\cdot A_n)}_{\lambda^0} + \underbrace{(i\bar{n}\cdot\partial + g\bar{n}\cdot A_{us})}_{\lambda^2} + \cdots$$

From before $\hat{\xi}_n(x) \sim \lambda \overset{\text{so}}{\sim} \xi_n(x)$

$$d^4x\, \underbrace{e^{-ix\cdot\mathcal{P}}}_{\mathcal{O}(1)} \sim \lambda^{-4}$$

phases implies $x^- \sim 1/p^+$, $x^+ \sim \frac{1}{p^-}$, $x^\perp \sim 1/p_\perp$

Leading Order $\mathcal{L}$ is $\mathcal{O}(\lambda^4)$

$$\mathcal{L}_{\xi\xi}^{(0)} = e^{-ix\cdot\mathcal{P}}\,\bar{\xi}_n\left[in\cdot D + i\not{D}_\perp^n \frac{1}{i\bar{n}\cdot D_n}\, i\not{D}_\perp^n\right]\frac{\not{\bar{n}}}{2}\,\xi_n$$

where $\left.\begin{aligned} iD_\perp^{n\,\mu} &= \mathcal{P}_\perp^\mu + gA_n^{\perp\mu} \\ i\bar{n}\cdot D_n &= \bar{\mathcal{P}} + g\bar{n}\cdot A_n \end{aligned}\right\}$ collinear cov. derivatives

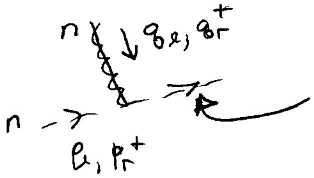
Note:

- both terms $\sim \lambda\cdot\lambda^2\cdot\lambda \sim \lambda^4$
- all fields at x, derivatives $i\partial \sim \lambda^2$, action is explicitly local at $Q\lambda^2$ scale

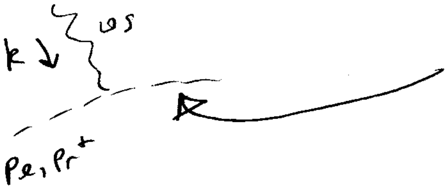
also local at $Q\lambda$ too ($D_\perp^n$ in numerator, momentum space version of locality)

only nonlocal at $\sim Q$ from $\frac{1}{\bar{n}\cdot p}$ factors

- Collinear propogators



$$\frac{\bar n\cdot(q_\ell+p_\ell)}{\bar n\cdot(q_\ell+p_\ell)\, n\cdot(q_r+p_r)+(q_\ell^\perp+p_\ell^\perp)^2+i0}$$

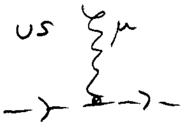


$$\frac{\bar n\cdot p_\ell}{\bar n\cdot p_\ell\;\; n\cdot(p_r+k)\;+(p_\ell^\perp)^2+i0}$$

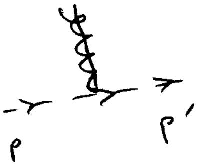
because <u>no</u> $i\bar n\cdot\partial$ or $i\partial^\perp$ in $\mathcal{L}_{\xi\xi}^{(0)}$

$\mathcal{L}_{\xi\xi}^{(0)}$ knows how to give LO propogator in both situations <u>without</u> further expansions

<u>Feyn. Rules</u>

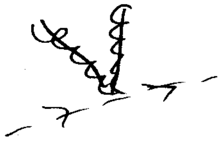


$= ig\frac{\not{\bar n}}{2}n^\mu T^A$ only $n\cdot A_{us}$ gluons



$$= ig\,T^A\frac{\not{\bar n}}{2}\left[n^\mu+\frac{\gamma_\perp^\mu \not{p}_\perp}{\bar n\cdot p}+\frac{\not{p}_\perp'\gamma_\perp^\mu}{\bar n\cdot p'}-\frac{\not{p}_\perp'\not{p}_\perp}{\bar n\cdot p'\,\bar n\cdot p}\,\bar n^\mu\right]$$

all 4 components couple



$= \cdots$ terms with ≥ 2 gluons also exist but have at most $2\perp$ gluons & rest $\bar n\cdot A$

trade $\bar{n}\cdot A_n \Leftrightarrow W$

Wilson Line Eqtns

$$i\bar{n}\cdot D_x\, W(x,-\infty) = 0$$

equivalent def'n of position space W-line

$$i\bar{n}\cdot D_n\, W_n = 0$$

momentum space W_n

$$(\bar{\mathcal{P}} + g\bar{n}\cdot A_n)\, W_n = 0$$

$$i\bar{n}\cdot D_n\, W_n\, \underset{\uparrow \text{ some operator}}{\mathcal{O}} = W_n\, \bar{\mathcal{P}}\, \mathcal{O}$$

so $\boxed{i\bar{n}\cdot D_n\, W_n = W_n\, \bar{\mathcal{P}}}$ as operator eqtn.

and since $(W(x,-\infty))^\dagger\, W(x,-\infty) = 1$

$$W_n^\dagger\, W_n = 1$$

we have $\boxed{i\bar{n}\cdot D_n = W_n\, \bar{\mathcal{P}}\, W_n^\dagger}$ $\bar{\mathcal{P}} = W_n^\dagger\, i\bar{n}\cdot D_n\, W_n$

$$\frac{1}{\bar{\mathcal{P}}} = W_n \frac{1}{i\bar{n}\cdot D} W_n^\dagger \,, \qquad \frac{1}{i\bar{n}\cdot D_n} = W_n^\dagger \frac{1}{\bar{\mathcal{P}}} W_n$$

(easy to check these are inverses)

$$\mathcal{L}^{(0)}_{\xi\xi} = e^{-ix\cdot\wp}\, \bar{\xi}_n \frac{\bar{n}\!\!\!/}{2}\left[in\cdot D + i\not{D}_{n\perp} W_n^\dagger \frac{1}{\bar{\mathcal{P}}} W_n\, i\not{D}_{n\perp} \right] \xi_n$$

Collinear Gluon Lagrangian

QCD $$\mathcal{L} = \underbrace{-\frac{1}{2}\,\text{tr}\{G^{\mu\nu}G_{\mu\nu}\}}_{\text{standard } -\frac{1}{4}G^{\mu\nu}_A G^{\mu\nu A}} + \underbrace{\tau\,\text{tr}\{(i\partial_\mu A^\mu)^2\}}_{\text{gen. cov. gauge fixing}} + \underbrace{2\,\text{tr}\{\bar c\, i\partial_\mu\, iD^\mu c\}}_{\text{gen. cov. ghost}}$$

ghost < adjoint, scalar, fermi statistics

$$G^{\mu\nu} = G^{\mu\nu}_A T^A = \frac{i}{g}[D^\mu, D^\nu]$$

SCET: same steps as for quark action

Let $$i\mathcal{D}^\mu = \frac{n^\mu}{2}(\bar{\mathcal{P}} + g\,\bar n\cdot A_n) + (\mathcal{P}_\perp^\mu + g A^\mu_{n\perp}) + \frac{\bar n^\mu}{2}(in\cdot\partial + g\,n\cdot A_n + g\,n\cdot A_{us})$$

$\boxed{iD^\mu \to i\mathcal{D}^\mu}$ at LO

$$i\mathcal{D}^\mu_{us} = \frac{n^\mu}{2}\bar{\mathcal{P}} + \mathcal{P}_\perp^\mu + \frac{\bar n^\mu}{2}(in\cdot\partial + g\,n\cdot A_{us})$$

recall A^μ_{us} behaves like background to A^μ_n. Maintaining gauge inv. for the background even in the A^μ_n gauge fixing terms requires

$\boxed{i\partial^\mu \to i\mathcal{D}^\mu_{us}}$ at LO

$$\mathcal{L}^{(0)}_{cg} = \frac{1}{2g^2}\,\text{tr}\{([i\mathcal{D}^\mu, i\mathcal{D}^\nu])^2\} + \tau\,\text{tr}\{([i\mathcal{D}^\mu_{us}, A_{n\mu}])^2\} + 2\,\text{tr}\{\bar c_n\,[i\mathcal{D}^{us}_\mu, [i\mathcal{D}^\mu, c_n]]\}$$

$$\boxed{\mathcal{L}^{(0)}_{SCET_I} = \mathcal{L}^{(0)}_{\xi\xi} + \mathcal{L}^{(0)}_{cg} + \underbrace{\mathcal{L}^{(0)}_q}_{} + \mathcal{L}^{(0)}_A}$$

full QCD actions for usoft quark q_{us} and for US gluon A^μ_{us}. These have no collinear fields

Arguement so far was <u>tree level</u>. To go further we need symmetries & power counting

① Gauge Symmetry

② Reparameterization Invariance

$\Big]$ very useful

③ Spin Symmety (?)

Consider ③ :

first lets revisit spinors $\psi(x) = e^{-ix\cdot p}\left(1+\frac{1}{i\bar{n}\cdot D_n}\, i\not{D}_{\perp n}\frac{\bar{\not{n}}}{2}\right)\xi_n(x)$

$$\text{so} \quad u = \left(1+\frac{1}{\bar{n}\cdot p}\not{p}_\perp\frac{\bar{\not{n}}}{2}\right)u_n$$

$$u_n = \frac{\not{n}\bar{\not{n}}}{4}u$$

$$\left[\ \not{n}\, u_n = 0, \quad \frac{\not{n}\bar{\not{n}}}{4}u_n = u_n\ \right]$$

Note:

- $\sum_s u_n^s \bar{u}_n^s = \frac{\not{n}\bar{\not{n}}}{4}\sum_s u^s\bar{u}^s\frac{\bar{\not{n}}\not{n}}{4} = \frac{\not{n}\bar{\not{n}}}{4}\not{p}\frac{\bar{\not{n}}\not{n}}{4} = \frac{\not{n}}{2}\bar{n}\cdot p$

  [Quantized ξ_n field does give proper collinear propogator, including numerator.]

- u_n is <u>NOT</u> equal to our earlier result of $\frac{(2p^0)^{1/2}}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}$ even though it obeys same relations

Dirac $u_n = \frac{1}{2}\begin{pmatrix}\mathbb{1} & \sigma^3\\ \sigma^3 & \mathbb{1}\end{pmatrix}\sqrt{p^0}\begin{pmatrix}u\\ \vec{\sigma}\cdot\vec{p}/_{p^0}\,u\end{pmatrix} = \frac{\sqrt{p^0}}{2}\begin{pmatrix}\left(1+\frac{p_3}{p_0}+\frac{i\vec{\sigma}\times\vec{p}_\perp}{p^0}\right)u\\ \sigma_3\left(1+\frac{p_3}{p_0}+\frac{i\vec{\sigma}\times\vec{p}_\perp}{p^0}\right)u\end{pmatrix}$

as before with extra factor of $u = \begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$

let two componet $\tilde{u} = \frac{1}{\sqrt{2}}\left(1+\frac{p_3}{p_0}+\frac{i\vec{\sigma}\times\vec{p}_\perp}{p^0}\right)\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\sqrt{\frac{p^0}{p^-}}$

$$u_n = \sqrt{\frac{p^-}{2}}\begin{pmatrix}\tilde{u}\\ \sigma_3\tilde{u}\end{pmatrix}, \quad u_n\bar{u}_n = \frac{p^-}{2}\begin{pmatrix}\tilde{u}\tilde{u}^+ & -\tilde{u}\tilde{u}^+\sigma_3\\ \sigma_3\tilde{u}\tilde{u}^+ & -\sigma_3\tilde{u}\tilde{u}^+\sigma_3\end{pmatrix}$$

$$\sum_s \tilde{u}^s\tilde{u}^{+s} = \mathbb{1}_{2\times2}$$

Extra terms ensure proper structure under ②, RPI

Projector's $P_n' = \frac{\not{n}\bar{\not{n}}}{4}+\frac{\not{n}}{2}$, $P_{\bar{n}}' = \frac{\bar{\not{n}}\not{n}}{4}-\frac{\not{n}}{2}$ give $\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$ but are not RPI invariant.

Spin Symmetry easiest to analyze in two-component form

$\xi_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_n \\ \sigma^3 \varphi_n \end{pmatrix}$ where $\dim \xi_n = \dim \varphi_n$

$$\mathcal{L} = \varphi_n^{\dagger} \left\{ i n\cdot D + i D_{n\perp}^{\mu} \frac{1}{i\bar{n}\cdot D} i D_{n\perp}^{\nu} \left(g^{\perp}_{\mu\nu} + i \epsilon^{\perp}_{\mu\nu} \sigma_3 \right) \right\} \varphi_n$$

not SU(2)

just U(1) helicity $h = \frac{i \epsilon_\perp^{\mu\nu}}{4} [\gamma_\mu, \gamma_\nu] \sim \sigma_3$ generator, spin along the direction of collinear motion n

- broken by masses
- broken by non-perturbative effects
- useful in perturbation theory
- related to chiral rotation $\gamma_5 \xi_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_n \\ \sigma^3 \varphi_n \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma^3 \varphi_n \\ \varphi_n \end{pmatrix}$
  ie $\varphi_n \to \sigma_3 \varphi_n$

---

① Gauge Symmetry $U(x) = \exp\left[i\, \alpha^A(x)\, T^A \right]$

Need to consider U's which leave us within EFT

eg. $i\partial^\mu \alpha^A \sim Q\, \alpha^A$ then $\xi_n' = U(x)\,\xi_n$ would no longer have $p^2 \lesssim Q^2 \lambda^2$.

global $U = e^{i\alpha^A T^A}$

collinear $U_c(x)$ $i\partial^\mu U_c(x) \sim Q(\lambda^2, 1, \lambda)\, U_c(x) \leftrightarrow A_n^\mu$

usoft $U_u(x)$ $i\partial^\mu U_u(x) \sim Q(\lambda^2, \lambda^2, \lambda^2)\, U_u(x) \leftrightarrow A_{us}^\mu$

- two classes of gauge trnsfm for two gauge fields
- in label momentum space we have $\xi_{n,p_\ell} \to \sum_{q} (U_c)_{p_\ell - q_\ell}\, \xi_{n,q_\ell}$
  (analog of $\psi(x) \to U(x)\psi(x)$
  $\tilde{\psi}(p) \to \int dq\, \tilde{U}(p-q)\, \tilde{\psi}(q)$)

Let $(U_c)_{p_\ell - q_\ell} = \overset{\text{matrix}}{(U_c)_{p_\ell , q_\ell}}$ ie $\{p_\ell, q_\ell\}$'th entry is number $(U_c)_{p_\ell - q_\ell}$

For A_n^μ we let its U_c transformation be that of quantum gauge trnsfm of a quantum field in a A_{us}^μ background (in manner homogeneous in p.c.)

<u>$U_c(x)$</u>

- $\xi_n(x) \to U_c(x) \xi_n(x)$ matrix notation
- $A_n^\mu \to U_c \left(A_n^\mu + \frac{i}{g} \mathcal{D}_{us}^\mu \right) U_c^\dagger$
- Also $q_{us} \xrightarrow{U_c} q_{us}$
  $A_{us}^\mu \xrightarrow{U_c} A_{us}^\mu$ since otherwise we give large momentum to usoft field

For $U_{us}(x)$ the fields ξ_n, A_n^μ transform like quantum fields under background gauge trnsfm. That is, they transform like matter fields of appropriate rep.

<u>$U_{us}(x)$</u>

- $\xi_n(x) \to U_{us}(x) \xi_n(x)$, $A_n^\mu \to U_{us} A_n^\mu U_{us}^\dagger$

  ↑ one number for all $\xi_{n,p}$ "vector" components

- $q_{us} \to U_{us} q_{us}$, $A_{us}^\mu \to U_{us} \left(A_{us}^\mu + \frac{i \partial^\mu}{g} \right) U_{us}^\dagger$

  ↑ ↗ usual gauge transformations

These transformations are fundamental, they are not corrected by power corrections.

U_c, U_{us}

Gauge transformations are homogeneous in λ

no mixing of term of different orders

eg. recall our heavy-to-light current

$$\bar{\xi}_n \Gamma h_v^{us} \xrightarrow{U_c} \bar{\xi}_n U_c^\dagger \Gamma h_v^{us}$$ is not gauge inv!

BUT recall offshell propagators generated Wilson line

$\bar{W}(x, -\infty)$

In general $\bar{W}(x,y) \rightarrow U(x)\, \bar{W}(x,y)\, U^\dagger(y)$. To avoid double counting with U_{global}, we are free to take $U^\dagger(-\infty) = 1$

$\bar{W}(x,-\infty) \rightarrow U_c(x)\, \bar{W}(x,-\infty)$

Momentum Space $$W = \sum_{m=0}^{\infty} \sum_{perms} \sum_{q_i} \frac{(-g)^m}{m!} \frac{\bar{n}\cdot A_{n,q_1}^{a_1}(x) \cdots \bar{n}\cdot A_{n,q_m}^{a_m}(x)\, T^{a_m} \cdots T^{a_1}}{\bar{n}\cdot q_1\ \bar{n}\cdot(q_1+q_2) \cdots \bar{n}\cdot(\Sigma q_i)}$$

$$W(x) = \left[\sum_{perms} \exp\left(\frac{-g}{\bar{P}}\, \bar{n}\cdot A_n(x) \right) \right]$$

the dependence on x encodes residual momenta in Wilson line. For $x=0$ the Fourier transform w.r.t P_e^- gives the line $\bar{W}(x,-\infty)$ where x is conjugate P_e^-.

* $W(x) \overset{U_c}{\rightarrow} U_c(x)\, W(x)$ in label matrix space

* $W(x) \overset{U_u}{\rightarrow} U_{us}(x)\, W(x)\, U_{us}^\dagger(x)$ from transformation of A_n directly

$$\bar{\xi}_n W \Gamma h_v \xrightarrow{U_c} \bar{\xi}_n \cancel{U_c^\dagger} \cancel{U_c} W \Gamma h_v = \bar{\xi}_n W \Gamma h_v$$ invariant

$$\bar{\xi}_n W \Gamma h_v \xrightarrow{U_{us}} \bar{\xi}_n \cancel{U_{us}^\dagger} \cancel{U_{us}} W \cancel{U_{us}^\dagger} \Gamma \cancel{U_{us}} h_v = \bar{\xi}_n W \Gamma h_v$$ "

- the Wilson line carries n-collinear gluons, which in full QCD combine with attachments to ξ_n -→- to give gauge invariant answers.
- usoft can be taken to include global, ~~and~~ connects all fields.

Gauge Symmetry ties together

$$i n\cdot D = i n\cdot\partial + g n\cdot A_n + g n\cdot A_{us}$$

$$i D_{n\perp}^{\mu} = \mathcal{P}_\perp^{\mu} + g A_{n\perp}^{\mu}$$

$$i \bar{n}\cdot D = \bar{\mathcal{P}} + g \bar{n}\cdot A_n$$

$$i D_{us}^{\mu} = i\partial^{\mu} + g A_{us}^{\mu} \quad \text{acting on usoft fields}$$

Is Gauge & Power Countin Symmetry Enough for $\mathcal{L}_{qq}^{(0)}$?

$i n\cdot D \sim \lambda^2$, $\frac{1}{\bar{\mathcal{P}}}(iD_\perp)^2 \sim \lambda^2$ no other λ^2 operators with right mass dimension

So far nothing rules out $\bar{\xi}_n \, i D_{n\perp}^{\mu} \frac{1}{i\bar{n}\cdot D} i D_{\perp n\mu} \frac{\bar{n}\!\!\!/}{2} \xi_n$.

Final Symmetry

② Reparameterization Invariance (RPI)

$n, \bar{n}$ break Lorentz Invariance , generators $\underbrace{n^{\mu} M_{\mu\nu} , \bar{n}^{\mu} M_{\mu\nu}}_{5 \text{ total}}$ ← usual 6 antisymm. gen of Lorentz symm.

only $\epsilon_\perp^{\mu\nu} M_{\mu\nu}$, rotations about $\vec{n}$-axis are preserved

3 types of RPI that keep $n^2 = 0$, $\bar{n}^2 = 0$, $n\cdot\bar{n} = 2$

	inf		inf		finite
I.	$n \to n + \Delta_\perp$	II.	$n \to n$	III.	$n \to e^{\alpha} n$
	$\bar{n} \to \bar{n}$		$\bar{n} \to \bar{n} + \epsilon_\perp$		$\bar{n} \to e^{-\alpha}\bar{n}$

Power Counting: $\Delta_\perp \sim \lambda$ eg. $n\cdot p \to n\cdot p + \Delta_\perp\cdot p_\perp \sim \lambda^2$

$\epsilon_\perp \sim \alpha \sim \lambda^0$ ie unconstrained

[N : So far we've talked mostly about 1 collinear sector. With multiple collinear sectors n_i^{μ} we have freedom to define $\bar{n}_i^{\mu}$ & have RPI trnsfm's for each pair $\{n_i^{\mu}, \bar{n}_i^{\mu}\}$]

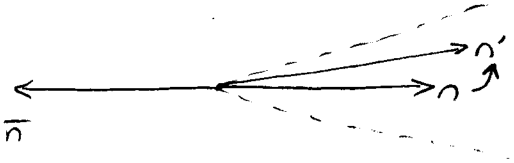
↑ later

Type III simple, just implies for any operator with $\bar{n}^\mu$ in numerator there must be another n^μ in numerator, or $\bar{n}$ in denominator

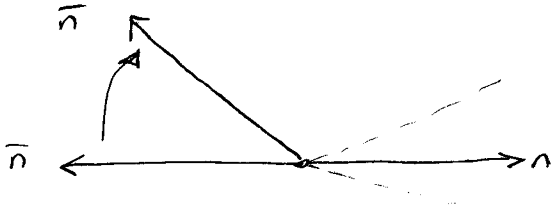
eg in $\mathcal{L}_{qq}^{(0)}$: had $\bar{\not{n}}\,\frac{1}{i\bar{n}\cdot D}$, $\bar{\not{n}}\,n\cdot D$ ✓

[no $\bar{\not{n}}\,\bar{n}\cdot D$]

Type I & II



type-I



type-II

We only can care about restoring Lorentz Inv. for the set of fluctuations described by SCET

ector $p^\mu = \frac{n^\mu}{2}\bar{n}\cdot p + \frac{\bar{n}^\mu}{2} n\cdot p + p_\perp^\mu$ is invariant to choice for decomposition

→ implies transformations for $p_\perp^\mu$ to compensate $n,\bar{n}$'s.

Find

type-I	type-II
$n \to n+\Delta_\perp$	$\bar{n} \to \bar{n} + \epsilon_\perp$
$n\cdot D \to n\cdot D + \Delta_\perp\cdot D_\perp$	$n\cdot D \to n\cdot D$
$D_\perp^\mu \to D_\perp^\mu - \frac{\Delta_\perp^\mu}{2}\bar{n}\cdot D - \frac{\bar{n}^\mu}{2}\Delta^\perp\cdot D^\perp$	$D_\perp^\mu \to D_\perp^\mu - \frac{\epsilon_\perp^\mu}{2} n\cdot D - \frac{n^\mu}{2}\epsilon_\perp\cdot D^\perp$
$\bar{n}\cdot D \to \bar{n}\cdot D$	$\bar{n}\cdot D \to \bar{n}\cdot D + \epsilon_\perp\cdot D_\perp$
$\xi_n \to \left[1+\frac{\not{\Delta}^\perp\bar{\not{n}}}{4}\right]\xi_n$	$\xi_n \to \left[1+\frac{\not{\epsilon}^\perp}{2}\,\frac{1}{i\bar{n}\cdot D}\,\not{D}_\perp\right]\xi_n$
$W \to W$	$W \to \left[\left(1-\frac{1}{i\bar{n}\cdot D}\,i\epsilon^\perp\cdot D^\perp\right)W\right]$

[I write D^μ everywhere, but your free to think of it as $\mathcal{P}^\mu$ or $i\partial^\mu$ with appropriate gauging from symmetry ①] e.g.

$$\delta^{(I)}\left(\bar\xi_n\, i\not{D}_{n\perp}\frac{1}{i\bar n\cdot D}\, i\not{D}_{n\perp}\frac{\bar{\not n}}{2}\xi_n\right) = -\bar\xi_n\, i\Delta^\perp\cdot D^\perp\frac{\bar{\not n}}{2}\xi_n$$

$$\delta^{(I)}\left(\bar\xi_n\, in\cdot D\,\frac{\bar{\not n}}{2}\xi_n\right) = +\underbrace{\bar\xi_n\, i\Delta^\perp\cdot D^\perp\frac{\bar{\not n}}{2}\xi_n}_{\text{connected, no non-trivial Wilson coefficient btwn them}}$$

type-II rules out $\bar\xi_n\, iD^\mu_{n\perp}\frac{1}{i\bar n\cdot D_n}\, iD_{n\perp\mu}\frac{\bar{\not n}}{2}\xi_n$ operator in $\mathcal{L}^{(0)}_{\xi\xi}$.

So

$$\mathcal{L}^{(0)}_{\xi\xi} = \bar\xi_n\left[in\cdot D + i\not{D}_{n\perp}\frac{1}{i\bar n\cdot D_n}\, i\not{D}_{n\perp}\right]\frac{\bar{\not n}}{2}\xi_n$$

is unique by p.c, gauge inv, & RPI.

More: Freedom in the label + residual decomposition

$$\bar n\cdot(p_\ell + p_r)\ , \quad p^\mu_{\ell\perp} + p^\mu_{r\perp}$$

$$\mathcal{P}_\mu \to \mathcal{P}_\mu + \beta_\mu\ , \quad i\partial_\mu \to i\partial_\mu - \beta_\mu \qquad n\cdot\beta = 0$$

$$\xi_{n,p}(x) \to e^{i\beta\cdot x}\,\xi_{n,p+\beta}(x)$$

Connects: $\mathcal{P}^\mu + i\partial^\mu$ connects leading & subleading Wilson coefficients in $\mathcal{L}^{(i)}$ and operators $\mathcal{O}^{(i)}$

<u>Gauge It</u> recall

$iD^\mu_{n\perp} \to \mathcal{U}_c\, iD^\mu_{n\perp}\,\mathcal{U}_c^\dagger$ or $\mathcal{U}_u\, iD^\mu_{n\perp}\,\mathcal{U}_u^\dagger$

$i\bar n\cdot D_n \to \mathcal{U}_c\, i\bar n\cdot D_n\,\mathcal{U}_c^\dagger$ $\quad\mathcal{U}_u\, i\bar n\cdot D_n\,\mathcal{U}_u^\dagger$

$in\cdot D \to \mathcal{U}_c\, in\cdot D\,\mathcal{U}_c^\dagger$ $\quad\mathcal{U}_u\, in\cdot D\,\mathcal{U}_u^\dagger$

$iD^\mu_{us} \to iD^\mu_{us}$ or $\mathcal{U}_u\, iD^\mu_{us}\,\mathcal{U}_u^\dagger$

Simplest idea:

$$\left.\begin{array}{l} iD^\mu_{n\perp} + iD^\mu_{us} \\ i\bar n\cdot D_n + i\bar n\cdot D_{us}\end{array}\right\}\ \text{doesn't work due to lack of trnsfm of } iD^\mu_{us} \text{ under } \mathcal{U}_c$$

The object that can compensate is $W \to U_c W$.
The (unique) result that preserves are choice for gauge symmetries [choice: strictly LO, homogeneous in λ] is

$$i D_{n\perp}^{\mu} \;+\; W\, i D_{\perp}^{us\,\mu}\, W^{\dagger} \;\equiv\; i D_{\perp}^{\mu}$$

$$i \bar{n}\cdot D_n \;+\; W\, i \bar{n}\cdot D_{us}\, W^{\dagger} \;\equiv\; i \bar{n}\cdot D$$

$\uparrow$ extra terms from $W, W^{\dagger}$ induce the $+\cdots$ in our earlier $(A^{\mu} = A_n^{\mu} + A_{us}^{\mu} + \cdots)$ expression

Coments

- Just like HQET, RPI can connect Wilson coefficients of leading order & subleading order external currents
- More collinear fields for >1 energetic hadron or >1 energetic jet

Generalize to $\sum_n \mathcal{L}_n^{(0)} = \sum_n \left[\mathcal{L}^{(0)}_{\xi_n \xi_n} + \mathcal{L}^{(0)}_{cg_n} \right]$

For $n_1, n_2, n_3, \ldots$ the collinear modes are distinct only if $n_i \cdot n_j \gg \lambda^2$ for $i \neq j$

eg. $p_2 = Q n_2$, $n_1 \cdot p_2 = Q\, n_1 \cdot n_2 \sim \lambda^2$ if $n_1 \cdot n_2 \sim \lambda^2$
but then p_2 is collinear to n_1, ie n_1-collinear.
So n_2 is within RPI equivalence class defined by $[n_1]$.

- Discrete Symmetries $n = (1,0,0,1)$, $\bar{n} = (1,0,0,-1)$

$$C^{-1}\, \xi_{n,p}\, C = -\left[\bar{\xi}_{n,-p}\, \mathcal{C} \right]^{T} \qquad p = (p^+, p^-, p^\perp)$$

$$P^{-1}\, \xi_{n,p}\, P = \gamma_0\, \xi_{\bar{n},\tilde{p}}\, (x_P) \qquad \tilde{p} = (p^-, p^+, -p^\perp)$$

$$T^{-1}\, \xi_{n,p}\, T = \mathcal{T}\, \xi_{\bar{n},\tilde{p}}\, (x_T) \qquad x_P = (x^-, x^+, -x_\perp)$$

$$x_T = (-x^-, -x^+, x^\perp)$$

JL20

Study $\mathcal{L}_{\xi\xi}^{(0)}$

① Propagator

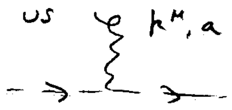
$$\frac{i\not{\bar n}}{2}\,\frac{\theta(\bar n\cdot p)}{n\cdot p+\frac{p_\perp^2}{\bar n\cdot p}+i\epsilon}+\frac{i\not{\bar n}}{2}\,\frac{\theta(-\bar n\cdot p)}{+n\cdot p+\frac{p_\perp^2}{\bar n\cdot p}-i\epsilon}=\frac{i\not{\bar n}}{2}\,\frac{\bar n\cdot p}{n\cdot p\,\bar n\cdot p+p_\perp^2+i\epsilon}$$

particles $\bar n\cdot p>0$ anti $\bar n\cdot p<0$

✓ expn. of QCD

② Interactions

- only $n\cdot A_{us}$ gluons at LO



$$= i g T^a n^\mu \frac{\not{\bar n}}{2}$$

& only sees $n\cdot k$ usoft momentum (multipole expn.)

$$\frac{\bar n\cdot p}{\bar n\cdot p\; n\cdot(p+k)+p_\perp^2+i\epsilon}=\frac{\bar n\cdot p}{\bar n\cdot p\; n\cdot k+p^2+i\epsilon}$$

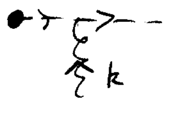
$$\underset{\text{on-shell}}{=}\frac{\bar n\cdot p}{\bar n\cdot p\; n\cdot k+i\epsilon}$$

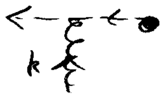
(Compare Collinear Gluon

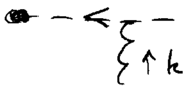
$$\frac{\bar n\cdot(p+q)}{(p+q)^2+i\epsilon}$$)

Propagator reduces to eikonal approx when appropriate

$\bar n\cdot p>0$ $\bar n\cdot p<0$



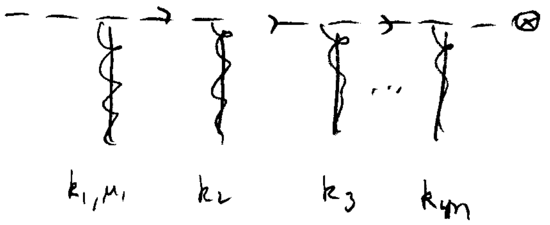




$$\frac{n^\mu}{n\cdot k+i\epsilon}\qquad\frac{n^\mu}{-n\cdot k+i\epsilon}\qquad\frac{n^\mu}{-n\cdot k-i\epsilon}\qquad\frac{n^\mu}{n\cdot k-i\epsilon}$$

Usoft - Collinear Factorization

Consider



$$= \Gamma \sum_m \sum_{\text{perms}} \frac{(-g)^m\, n\cdot A^{a_1}\cdots n\cdot A^{a_m}\, T^{a_m}\cdots T^{a_1}}{n\cdot k_1\; n\cdot(k_1+k_2)\cdots n\cdot(\Sigma k_i)} * u_n$$

on-shell so $\frac{1}{n\cdot k + \frac{p^2}{\bar n\cdot p}} \to \frac{1}{n\cdot k}$

Motivates us to consider a field redefinition

$$\xi_{n,p}(x) = Y(x)\,\xi^{(0)}_{n,p}(x) \qquad A_{n,p} = Y A^{(0)}_{n,p} Y^\dagger$$

↑ adjoint version

$$Y(x) = P\exp\left(ig\int_{-\infty}^{0} ds\; n\cdot A^a_{us}(x+ns)\,T^a\right)$$

$$n\cdot D\;Y = 0\;, \quad Y^\dagger Y = 1 \qquad \text{find} \quad W = Y\,W^{(0)}\,Y^\dagger$$

$$\mathcal{L}^{(0)}_{\xi\xi} = \bar\xi_{n,p'}\,\frac{\bar{n}\!\!\!/}{2}\left[in\cdot D + \cdots\right]\xi_{n,p}$$

$$= \bar\xi^{(0)}_{n,p'}\,\frac{\bar{n}\!\!\!/}{2}\left[Y^\dagger in\cdot D_{us} Y + Y^\dagger(Y g\bar n\cdot A_n Y^\dagger)Y + \cdots\right]\xi_{n,p}$$

$$= \bar\xi^{(0)}_{n,p'}\,\frac{\bar{n}\!\!\!/}{2}\left[\underbrace{in\cdot\partial + g\bar n\cdot A_n}_{in\cdot D_c} + \cdots\right]\xi_{n,p}$$

↑ all $n\cdot A_{us}$'s disappear!

True for gluon action too

$$\mathcal{L}(\xi_{n,p}, A^\mu_{n,q}, n\cdot A_{us}) = \mathcal{L}(\xi^{(0)}_{n,p}, A^{(0)}_{n,q}, 0)$$

Interactions don't disappear, but are moved out of L.O. $\mathcal{L}$ and into currents

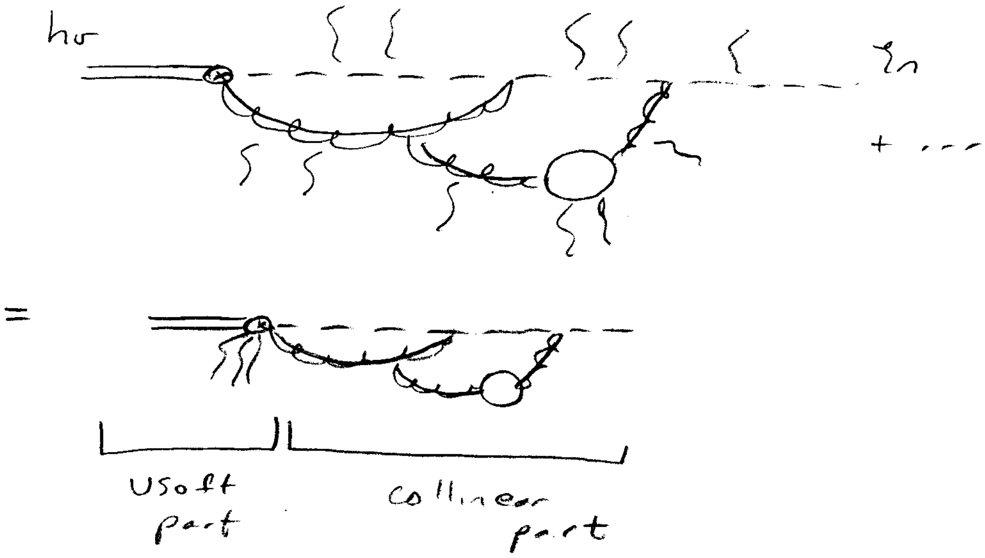
eg 1 $$J = \bar\xi W \Gamma h_v = \bar\xi_n^{(0)} Y^\dagger Y W^{(0)} Y^\dagger \Gamma h_v$$
$$= (\bar\xi_n^{(0)} W^{(0)}) \Gamma (Y^\dagger h_v)$$

If our current was a collinear color singlet

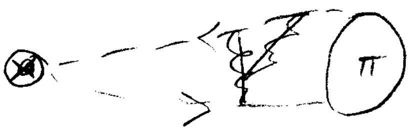
eg 2 $$J = (\bar\xi_n W) \Gamma (W^\dagger \xi_n) = \bar\xi_n^{(0)} W^{(0)} \cancel{Y^\dagger}\cancel{Y} \Gamma (W^{\dagger(0)} \xi_n^{(0)})$$

Quite powerful, sums an ∞ class of diagrams

in eg 1



in eg 2 usoft gluons decouple at L.O. from any graph
This is color transparency



- usoft gluons decouple from energetic partons in color singlet state
- they just "see" overall color singlet due to multipole expansion

What about Wilson Coefficients ?

have $C(\bar{\mathcal{P}},\mu)$ ie depend on large momenta picked out by label operator $\bar{\mathcal{P}} \sim \lambda^0$

eg. $C(-\bar{\mathcal{P}},\mu)\,(\bar{\xi}_n W)\,\Gamma h_v = (\bar{\xi}_n W)\,\Gamma h_v\, C(\bar{\mathcal{P}}^\dagger)$

must act on product $(\bar{\xi} W)$ since only momentum of this combination is gauge invariant

Write $(\bar{\xi} W)\,\Gamma h_v\, C(\bar{\mathcal{P}}^\dagger) = \int d\omega\, C(\omega,\mu)\,\left[(\bar{\xi} W)\,\delta(\omega - \bar{\mathcal{P}}^\dagger)\,\Gamma h_v\right]$

$$= \int d\omega\, \underset{\uparrow}{C(\omega,\mu)}\;\; \underset{\uparrow}{O(\omega,\mu)}$$

convolution (as promised) of "C" and collinear "O"

Hard-Collinear Factorization

Recall defn of W, $\quad i\bar{n}\cdot D_c\, W = 0$, $\quad W^\dagger W = 1$

as operator

$$i\bar{n}\cdot D_c\, W = W\,\bar{\mathcal{P}}$$
$$i\bar{n}\cdot D_c = W\,\bar{\mathcal{P}}\,W^\dagger$$
$$(i\bar{n}\cdot D_c)^k = W\,\bar{\mathcal{P}}^k\,W^\dagger$$

$$f(i\bar{n}\cdot D_c) = W\, f(\bar{\mathcal{P}})\, W^\dagger$$

trades $\bar{n}\cdot A \to W$

Part of collin op. $p^2 \sim \lambda^2 Q^2$

hard coefficient

$$= \int d\omega\, f(\omega)\;\; W\,\delta(\omega - \bar{\mathcal{P}})\,W^\dagger$$

In general we can define a convenient set of (collinear gauge invariant) <u>building blocks</u> for operators:

- $\chi_n \equiv (W_n^\dagger \xi_n)$ "quark jet-field"
  $\chi_{n,\omega} \equiv \delta(\omega - \bar{\mathcal{P}})(W_n^\dagger \xi_n)$
  operators $\int d\omega_1 \, d\omega_2 \; C(\omega_1, \omega_2) \; \bar{\chi}_{n,\omega_1} \Gamma \chi_{n,\omega_2}$ etc.

- $$\mathcal{B}_{n\perp}^\mu = \Big[\frac{1}{\bar{\mathcal{P}}} W_n^\dagger \, [i\bar{n}\cdot D_n , iD_{n\perp}^\mu] \, W_n \Big] = g A_{n\perp}^\mu + \ldots$$
  "gluon jet-field" for two physical gluon-pol.
  $$\mathcal{B}_{n\perp,\omega}^\mu = [\mathcal{B}_{n\perp}^\mu \, \delta(\omega - \bar{\mathcal{P}}^\dagger)]$$
  $\uparrow$ convention/choice, acts left inside $[\cdots]$

<u>Comments</u>

All operators can be constructed solely from $\{ \chi_n , \mathcal{B}_{n\perp}^\mu , \mathcal{P}_\perp^\nu \}$ + usoft fields & D_{us}^μ.

① Let $i\mathcal{D}_n^\mu = W_n^\dagger \, iD_n^\mu \, W_n$ where iD_n^μ has $\begin{Bmatrix} \bar{\mathcal{P}} \\ \mathcal{P}_\perp \\ in\cdot\partial \end{Bmatrix} + g \begin{Bmatrix} n\cdot A_n \\ A_n^\perp \\ \bar{n}\cdot A_n \end{Bmatrix}$

$\bar{n}\cdot i\mathcal{D}_n = \bar{\mathcal{P}}$

$i\mathcal{D}_n^{\perp\mu} = \mathcal{P}_\perp^\mu + g\,\mathcal{B}_{n\perp}^\mu$, $i\,n\cdot\mathcal{D}_n = in\cdot\partial + \underbrace{g\,n\cdot\mathcal{B}_n}_{}$

analogous to defn $\mathcal{B}_{n\perp}^\mu$

derivatives $\bar{\mathcal{P}}\,\chi_{n,\omega} = \omega\,\chi_{n,\omega}$ can be absorbed in coefficients

$in\cdot\partial\,\chi_n = -(g\,n\cdot\mathcal{B}_n)\,\chi_n - i\not{\mathcal{D}}_{n\perp} \frac{1}{\bar{\mathcal{P}}}\, i\not{\mathcal{D}}_{n\perp}\,\chi_n$ equation of motion for χ_n

$\uparrow$ remove $in\cdot\partial$'s

$in\cdot\partial\,\mathcal{B}_{n\perp}^\mu = \ldots$ eqtn of motion

2) $\omega\,(g\,\bar{n}\cdot\mathcal{B}_n)_\omega = 2\,\mathcal{P}^\perp_\nu\, g\,\mathcal{B}^\nu_{\perp,\omega} + \ldots$

also part of gluon e.o.m.

All other collinear operators , $W_n^\dagger\,[i\mathcal{D}^\mu_{n\perp}, i\mathcal{D}^\nu_{n\perp}]\,W_n$, ...
are reducible to $\{\chi_n, \mathcal{B}_\perp^\mu, \mathcal{P}_\perp^\nu\}$

③ Do need usoft derivatives, Field strengths, q_{us}, etc
Statement of RPI becomes
$i\mathcal{D}^\mu_{n\perp} + iD^\mu_{us\perp}$, $\bar{\mathcal{P}}_n + i\bar{n}\cdot D_{us}$

(equiv. to earlier, but W's around collinear D^μ here, rather than usoft)

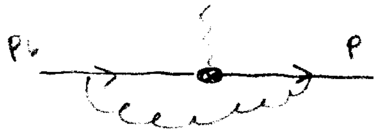
Loops, IR divergences, Matching & Running

Consider heavy-to-light current for $b \to s\gamma$

$J^{QCD} = \bar{s}\,\Gamma\, b$ $\Gamma = \sigma^{\mu\nu} P_R F_{\mu\nu}$

$J^{SCET}_{LO} = (\bar{\xi}W)\,\Gamma\, h_v\, C(\bar{\mathcal{P}}^\dagger)$ [pre Y-field redefn]

QCD graphs at one-loop , take $p^2 \neq 0$ to regulate IR of coll'r quark , Feyn Gauge



$$= -\,\bar{u}_s\,\Gamma\, u_b\,\frac{\alpha_s C_F}{4\pi}\left[\ln^2\!\left(\frac{-p^2}{m_b^2}\right) + 2\ln\!\left(\frac{-p^2}{m_b^2}\right) + \ldots\right]$$

$f(p\cdot p_b/m_b^2)$, IR finite

$$Z_{\psi_b} = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} + 3\ln\frac{\mu^2}{m_b^2} + \ldots\right]$$

full Z's (not $\overline{MS}$) match for S-matrix

$$Z_{\psi_s} = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} - \ln\frac{p^2}{\mu^2}\right]$$

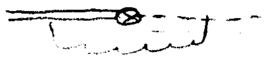
$$Z_{tensor} = 1 + \frac{\alpha_s C_F}{4\pi}\,\frac{1}{\epsilon}$$ ← Tensor current in QCD not conserved

$$\text{um} = \bar{u}_s \Gamma u_b \left[1 - \frac{\alpha_s C_F}{4\pi} \left\{ \ln^2\left(\frac{-p^2}{M_b^2}\right) + \frac{3}{2} \ln\left(\frac{-p^2}{M_b^2}\right) + \frac{1}{\epsilon_{IR}} + \dots \right\} \right]$$

$\boxed{\text{SCET}_I}$

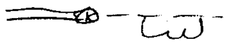
Feyn Gauge

usoft-loops

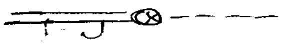


$$\int \frac{d^dk \quad n\cdot v}{(v\cdot k+i0)(k^2+i0)(n\cdot k + p^2/\bar{n}\cdot p + i0)}$$

$$= -\bar{u}_n \Gamma u_v \frac{\alpha_s C_F}{4\pi} \left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} \ln\left(\frac{\mu \bar{n}\cdot p}{-p^2-i0}\right) + 2 \ln^2\left(\frac{\mu \bar{n}\cdot p}{-p^2}\right) + \frac{3\pi^2}{4} \right]$$

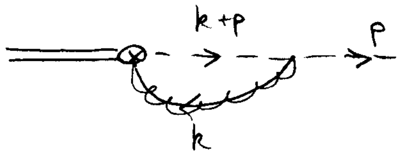


$\propto \; n^\mu n_\mu = 0$ in Feyn. Gauge



$$Z_{HQET} = 1 + \frac{\alpha_s C_F}{4\pi} \left[\frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}} \right]$$

Collinear graphs



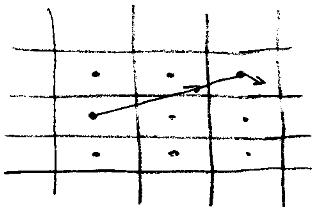
$$= \bar{u}_n \Gamma h_v \sum_{\substack{k_\ell \neq 0 \\ k_\ell \neq -p_\ell}} \int \frac{d^dk_r \quad n\cdot\bar{n} \quad \bar{n}\cdot(p+k)}{(\bar{n}\cdot k)(k^2)(k+p)^2}$$

⇑ each momentum has (k_ℓ, k_r), label & residual

label & residual ensure we can pick out LO piece (in particular for mixed collinear & usoft graphs). But now we want to turn $\sum_{k_\ell} \int dk_r$ back into $\int dk_\ell$ to ~~do~~ do loop integration

k_r^+ is only +-momentum. So worry about $k_\ell^\perp, k_r^\perp$ & k_ℓ^-, k_r^-

Recall grid



was like Wilsonian EFT (with finite edges)

Continuum EFT : each grid point specifies an ∞-space of residual momenta ($k_r^\mu \in \mathbb{R}$) , subject to rules

Ignore $k_\ell \neq 0$, $k_\ell \neq -p_\ell$

i) $\displaystyle\sum_{k_\ell} \int dk_r = \int dk_\ell$ for $-$ & $\perp$ momenta

(use 1-dim notation for simplicity)

ii) $\displaystyle\sum_{k_\ell} \int dk_r \, \underset{\uparrow \text{ constant throughout each box}}{F(k_\ell)} = \sum_{k_\ell} \int dk_r \, F(k_\ell + k_r) = \int dk_\ell \, \underset{\uparrow \text{ continuous dummy var.}}{F(k_\ell)}$

- This is the (simplified version of) main rule for obtaining $\int dk_\ell$. For each label loop momentum k_ℓ, there will always be a corresponding k_r that we can absorb in this fashion.
- Recall that grid facilitated multipole expansion. For a purely collinear loop there is often no physical p_r^+, p_r^- flowing through it. In this case answer must reduce to what we get from considering $\int d^dk_n$

iii) $\displaystyle\sum_{k_\ell} \int dk_r \, (k_r)^j \, F(k_\ell) = 0$ for $j > 0$ integer

dim-reg type rule which maintains Lorentz-Inv in residual Space

iv) Ultrasoft external particles or loops give non-trivial ℓ_r^μ & hence residual momenta that we can not absorb

eg. $\displaystyle\sum_{k_\ell} \int dk_r \int d\ell_r \, F(k_\ell, \underset{\uparrow \text{ usoft propagator (say)}}{\ell_r}) = \int dk_\ell \int d\ell_r \, F(k_\ell, \ell_r)$

(133)

Our example 

$$\sum_{\substack{k_\ell\neq 0\\ k_\ell\neq -p_\ell}} \int \frac{đ^d k_r \;\; n\cdot\bar n \;\; \bar n\cdot(p_\ell+k_\ell)}{\bar n\cdot k_\ell\,(k_\ell^- k_r^+ + k_\ell^{\perp 2})\left[(k_\ell^-+p_\ell^-)(k_r^+ + p_r^+) + (k_\ell^\perp + p_\ell^\perp)^2\right]}$$

{ ignore restrictions on the sum for now

$$= \int \frac{đ^d k \;\; n\cdot\bar n \;\; \bar n\cdot(p+k)}{\bar n\cdot k \;\; k^2 \;\; (k+p)^2}$$

do as standard loop integral

$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} + \frac{2}{\epsilon}\ln\left(\frac{\mu^2}{-p^2}\right) + \ln^2\left(\frac{\mu^2}{-p^2}\right) + 2\ln\left(\frac{\mu^2}{-p^2}\right) + 4 - \frac{\pi^2}{6}\right]$$

minimized ↑ $\mu^2 \sim p^2$ consistent with power counting

 collinear w. fn. renormalization (same as massless QCD)

$$Z_\xi = 1 + \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \ln\frac{\mu^2}{-p^2}\right]$$

 ,  scaleless power-divergent* (Feyn. Gauge)

$\propto \bar n^2 = 0$ (Feyn.) ↘ 0

* cancels unphysical singularity for $\bar n\cdot(p+k)\to 0$, $k_\perp$ fixed in 

Matching Compare QCD & SCET (Kinematics $b\to s\gamma$ sets $p^- = m_b$)

$$(\text{sum QCD})^{\text{ren}} = -\frac{\alpha_s C_F}{4\pi}\Big[\underline{\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}}} + 2\ln\left(\frac{\mu^2}{m_b^2}\right) + \cdots\Big]$$

$$(\text{sum SCET})^{\text{bare}} = -\frac{\alpha_s C_F}{4\pi}\Big[\underline{\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}}}$$

$$\underbrace{-\frac{1}{\epsilon^2} - \frac{5}{2\epsilon} - \frac{2}{\epsilon}\ln(\mu/m_b)}_{\text{UV renormalization}} \underbrace{- 2\ln^2\left(\frac{\mu}{m_b}\right) - \frac{3}{2}\ln\frac{\mu^2}{m_b^2} + \cdots}_{\text{difference gives one-loop } C(m_b,\mu)}$$

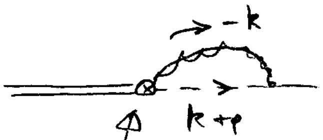
——— = Same IR divergences

discuss later

Note ① $\sum_{\omega} C(\omega,\mu)\, \underbrace{\bar\chi_{n,\omega}}_{(\bar\xi_n W)\,\delta_{\omega,\bar{\mathcal{P}}^{+}}} \Gamma\, h_v$ total momentum of $\bar\xi_n$ & W fixed as ω

so its always $\omega = p^-$ above

- non-trivial example



sum $= \bar n\cdot(k+p) + \bar n\cdot(-k) = \bar n\cdot p$

② Should be careful with $k_\ell \neq 0$, $k_\ell \neq -p_\ell$ (Zero-Bin's)
Collinear momenta have non-zero labels
When $k_\ell = 0$ gluon is usoft ($k_\ell = -p_\ell$ quark is usoft)
These restrictions avoid double counting in SCET fields and hence also in results for loop integrations

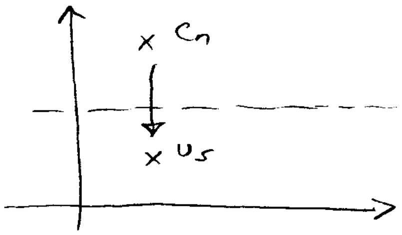
Rule ii) above with restrictions (encoded via propagators) is

$$\begin{aligned}\sum_{k_\ell\neq 0}\int dk_r\, F(k_\ell) &= \sum_{k_\ell}\int dk_r\, F(k_\ell) - \int dk_r\, F^{k_\ell\to 0}(0)\\ &= \sum_{k_\ell}\int dk_r\, F(k_\ell+k_r) - \int dk_r\, F^{k_\ell\to 0}(k_r)\\ &= \int dk\,\Big[\, F(k) - \underset{\uparrow}{F^{k_\ell\to 0}(k)}\,\Big]\end{aligned}$$

zero-bin subtraction term

$F^{k_\ell\to 0}(k)$ is defined by taking scaling limit $k_n^\mu \to k_{us}^\mu$
ie $k_n^\mu \sim \lambda^2$

and expanding to keep all subtractions that are same order in λ (dropping power suppressed terms, a "minimal subtractio



subtraction ensures "C_n" has no non-trivial support in ultrasoft "us" region our eg. 

$$\int d^dk \left[\frac{n\cdot\bar n \quad \bar n\cdot(k+p)}{\bar n\cdot k\;(k+p)^2\;k^2} - \underset{\uparrow\text{ scaling limit}}{\frac{n\cdot\bar n \quad \bar n\cdot p}{\bar n\cdot k\;(\bar n\cdot p\, n\cdot k + p^2)\;k^2}} \right]$$

$$= \frac{i}{16\pi^2}\Bigg[\left(\frac{2}{\epsilon_{IR}\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} \ln\frac{\mu^2}{-p^2} + \ln^2\frac{\mu^2}{-p^2} + \left(\frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}}\right)\ln\frac{\mu}{\bar n\cdot p} + \cdots \right)$$
$$- \left(\underbrace{\left(\frac{2}{\epsilon_{IR}} - \frac{2}{\epsilon_{UV}}\right)}_{\text{zero in pure-dim reg}} \left(\frac{1}{\epsilon_{UV}} + \ln\frac{\mu^2}{-p^2} - \ln\frac{\mu}{\bar n\cdot p} \right) \right) \Bigg]$$

subtraction :
- cancels $\bar n\cdot q \to 0$ IR singularity of first term,
- UV. divergences come from $\bar n\cdot q \to \infty$ & are indep. of IR regulator
- here $\epsilon_{IR} = \epsilon_{UV}$ and ignoring subtraction gives correct answer

- for other less inclusive calculations (eg. jet algorithms) or other regulators (eg. $\Omega_\perp^2 \le \vec k_\perp^2 \le \Lambda_\perp^2$, $\Omega_-^2 \le (k^-)^2 \le \Lambda_-^2$) the subtraction is crucial to avoid double counting (get correct IR structure) & have UV div. indep of IR regulator.

Renormalization in SCET & Summing Sudakov Logs

our eg.

$$C^{bare} = C + (Z_C - 1)\,C$$

need counter term $\quad Z_C = 1 - \frac{\alpha_s(\mu)\,C_F}{4\pi}\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu}{\omega} + \frac{5}{2\epsilon} \right)$

$$\mu\, d/d\mu\; C^{bare} = 0 \quad \to \quad \mu\, d/d\mu\; C(\omega,\mu) = \gamma_c(\omega,\mu)\; C(\omega,\mu)$$

$$\gamma_c = -Z_c^{-1}\,\mu\, d/d\mu\; Z_c = \mu\, d/d\mu\; \frac{C_F\,\alpha_s(\mu)}{4\pi}\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu}{\omega} + \frac{5}{2\epsilon}\right)$$

$$= \frac{C_F\,\alpha_s(\mu)}{4\pi}\left(-\cancel{\frac{2}{\epsilon}} - 4\ln\frac{\mu}{\omega} - 5 + \cancel{\frac{2}{\epsilon}}\right) \quad ; \quad \mu\, d/d\mu\; \alpha_s = -2\epsilon\,\alpha_s + \cdots$$

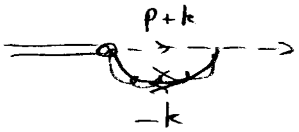
$$= -\frac{\alpha_s(\mu)}{4\pi}\left(\underset{\substack{\uparrow \\ \text{LL, cusp} \\ \text{anomalous} \\ \text{dimension}}}{4C_F\ln\frac{\mu}{\omega}} + \underset{\substack{\uparrow \\ \text{part of} \\ \text{NLL}}}{5C_F}\right)$$

Running

In general we must be careful with coeffs since they act like operators $C(\mu, \bar{P})$

In our eg. $\bar{P} \to \bar{n}\cdot p$ of external field always

non-trivial case



$$C(\mu, \bar{n}\cdot(p+k) + \bar{n}\cdot(-k)) = C(\mu, \bar{n}\cdot p)$$

$$\mu \, {}^{d}/_{d\mu} \, C(\mu) = -\frac{\alpha_s(\mu)\, C_F}{\pi} \ln\left(\frac{\mu}{\bar{P}}\right) C(\mu)$$ LO anom dim

Soln. QED α_s = fixed, $C_F = 1$

$$C(\mu) = \exp\left[-\frac{\alpha}{2\pi} \ln^2\left(\frac{\mu}{\bar{P}}\right)\right]$$ Sudakov Suppression

QCD $$C(\mu) = \exp\left[\frac{-4\pi C_F}{\beta_0^2\, \alpha_s(m_b)}\left(\frac{1}{z} - 1 + \ln z\right)\right]$$

$z = \frac{\alpha_s(\mu)}{\alpha_s(m_b)}$

here m_b = matching scale

In more complicated cases $C(\bar{P}, \bar{P}^+)$ will be sensitive to $\bar{n}\cdot k$ loop momentum and we'll get

$$\mu \frac{\partial}{\partial\mu} C(\mu,\omega) = \int d\omega' \, \gamma(\omega,\omega')\, C(\mu,\omega')$$

examples	DIS	Alterelli - Parisi evolution
	$\gamma^* \pi^0 \to \pi^0$	Brodsky - Lepage "
	$\gamma^* p \to \gamma p'$	Deeply - Virtual Compton Scattering

these are actually all the evolution of a single SCET operator

$$(\bar{\xi}_n W)\; C(\bar{P}, \bar{P}^+)\; (W^+ \xi_n)$$

Note: series in $\ln C(\mu)$

		one-loop	two-loop	3-loop
LL	$\alpha_s^n \ln^{n+1}$	$1/\epsilon^2$	—	—
NLL	$\alpha_s^n \ln^n$	$1/\epsilon$	$1/\epsilon^2$	—
NNLL	$\alpha_s^n \ln^{n-1}$	matching	$1/\epsilon$	$1/\epsilon^2$

$1/\epsilon^2 \to 1/\epsilon \ln(\mu)$ term

Differs from single log case somewhat

At LHC, Sudakov effects are important in

Parton showers [Prob. to evolve without branching]

Jets

Recall

$\boxed{SCET_I}$	hard	$p^\mu \sim (Q, Q, Q)$	$C = H$
	collin	$(Q\lambda^2, Q, Q\lambda)$	
	usoft	$(Q\lambda^2, Q\lambda^2, Q\lambda^2)$	

$\uparrow$ non-trivial communication between sectors

$\boxed{SCET_{II}}$ (still to come)

hard	(Q, Q, Q)
hard-collin	$(Q\lambda, Q, \sqrt{Q\lambda})$
collin	$(Q\lambda^2, Q, Q\lambda)$
soft	$(Q\lambda, Q\lambda, Q\lambda)$

Results for observables which tie d.o.f. together are "Factorization Theorems"

Processes

- $\gamma^* \gamma \to \pi^0$ $\pi - \gamma$ form factor at $Q^2 \gg \Lambda^2$ for γ^*

  Breit frame $q^\mu = \frac{Q}{2}(n^\mu - \bar{n}^\mu)$, $p_\gamma^\mu = E\,\bar{n}^\mu$

  $$p_\pi^\mu = \frac{Q}{2} n^\mu + \underbrace{\left(E - \frac{Q}{2}\right)}_{m_\pi^2/2Q} \bar{n}^\mu$$

  pion = collinear in n-direction (SCET$_{\text{II}}$)

- $\gamma^* M \to M'$ $M - M'$ (meson) form factor $Q^2 \gg \Lambda^2$ for γ^*

  $M =$ collinear in n

  $M' =$ " " $\bar{n}$ (say) (SCET$_{\text{I}}$)

- $B \to D\pi$ Matrix Elt. of 4-quark operators

  $Q = \{m_b, m_c, E_\pi\} \gg \Lambda$

  B, D are soft $p^2 \sim \Lambda^2$, π-collinear (SCET$_{\text{II}}$)

- DIS Structure Functions at $Q^2 \gg \Lambda^2$

  $e^- p \to e^- X$ and $1 - x \gg \Lambda/Q$ (ie not near endpts in Bjorken x)

  Breit frame: proton n-collinear , X - hard (SCET$_{\text{II}}$ or SCET$_{\text{I}}$)

- Drell-Yan $\frac{d\sigma}{dQ^2}$ Q^2 = inv. mass of $l^+ l^- \gg \Lambda^2$

  $p\bar{p} \to l^+ l^- X$

  p - n-collin, $\bar{p}$ - $\bar{n}$-collin, X - hard

- $e^+ e^- \to$ jets

  $\bar{p} \to$ jets

  $pp \to$ jets

  - depends on observable we formulate

    eg two jets n-collin jet

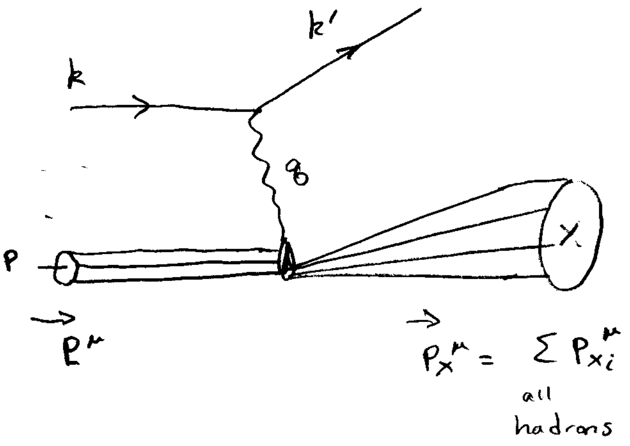
    $\bar{n}$-collin jet

etc.

DIS A rich subject, only aspects related to QCD factorization are covered here using SCET

Refs: § 1.8 of text
Aneesh M.'s review: hep-ph/9204208
Bob J.'s review: hep-ph/9602236
paper: hep-ph/0202088 (for material below)

$e^- p \to e^- X$



$Q^2 \gg \Lambda^2$

$q^2 = -Q^2 \,, \qquad x = \dfrac{Q^2}{2P\cdot q}$

$p_X^\mu = P^\mu + q^\mu$

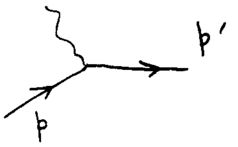
$p_X^2 = \dfrac{Q^2}{x}(1-x) + M_p^2$

regions

p_X^2	$(\frac{1}{x}-1)$	
$\sim Q^2$	~ 1	inclusive OPE
$\sim Q\Lambda$	$\sim \Lambda/Q$	endpt. region
$\sim \Lambda^2$	$\sim \Lambda^2/Q^2$	resonance region

$e^- p \to e^- p'$
$\uparrow$
eg. excited state

Parton Variables



struck quark carries some fraction ξ of proton momentum

$\bar{n}\cdot p = \xi\, \bar{n}\cdot P$ ← we'll see how to formulate ξ in QCD

$p'^2 \simeq Q^2\left(\dfrac{\xi}{x} - 1\right)$

Frames

Breit Frame

$$q^\mu = \frac{Q}{2}\left(\bar{n}^\mu - n^\mu\right)$$

$$P^\mu = \frac{n^\mu}{2}\,\bar{n}\cdot P + \frac{\bar{n}^\mu M_p^2}{2\,\bar{n}\cdot P} = \frac{n^\mu}{2}\,\frac{Q}{x} + \cdots \qquad \text{collinear}$$

$$P_X^\mu = \frac{n^\mu}{2}\,Q + \frac{\bar{n}^\mu}{2}\,\frac{Q(1-x)}{x} + \cdots \qquad \text{hard}$$

Proton is made of collinear quarks and gluons

Rest Frame

$$P^\mu = \frac{m_p}{2}\left(n^\mu + \bar{n}^\mu\right) \qquad \text{soft}$$

$$q^\mu = \frac{\bar{n}^\mu}{2}\,\frac{Q^2}{m_p x} - \frac{n^\mu}{2}\,m_p x + \cdots$$

$$P_X^\mu = \text{sum} \qquad \text{"collinear" } P_X^2 \sim Q^2$$

Like $B \to X_c e \nu$ we can write cross-section in terms of leptonic & hadronic tensors

$$d\sigma = \frac{d^3k'}{2|\vec{k}'|}\,\frac{e^4}{s Q^4}\,L^{\mu\nu}(k,k')\,W_{\mu\nu}(P,q)$$

we'll look at spin-avg. case

$$W_{\mu\nu} = \frac{1}{\pi}\,\mathrm{Im}\,T_{\mu\nu}$$

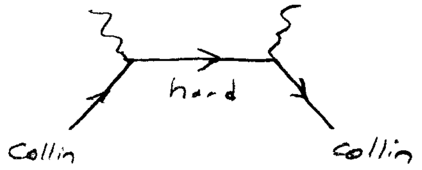
$$T_{\mu\nu} = \frac{1}{2}\sum_{\text{spin}} \langle p|\,\hat{T}_{\mu\nu}(q)\,|p\rangle$$

$$\hat{T}_{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\, T\big[\underset{\substack{\uparrow\\ \text{e.m. currents}}}{J_\mu(x)}\, J_\nu(0)\big]$$

$$T_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(x,Q^2) + \left(P_\mu + \frac{q_\mu}{2x}\right)\left(P_\nu + \frac{q_\nu}{2x}\right) T_2(x,Q^2)$$

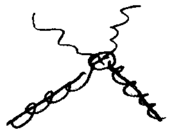
Satifies current conservation, P, C, T, etc.

Want imaginary part of forward scattering



First Match onto SCET Ops.

at L.O. :



$$\hat T^{\mu\nu} = \frac{g_\perp^{\mu\nu}}{Q}\left(O_1^{(i)} + \frac{O_1^{g}}{Q}\right) + \frac{(n^\mu+\bar n^\mu)(n^\nu+\bar n^\nu)}{Q}\left(O_2^{(i)} + \frac{O_2^{g}}{Q}\right)$$

$O(\lambda^2)$ operators

flavor = u, d, ...

$$O_j^{(i)} = \bar\xi^{(i)}_{n,p} W \frac{\bar{n}\!\!\!/}{2} C_j^{(i)}(\bar P_+, \bar P_-) W^{+} \xi^{(i)}_{n,p}$$

$$O_j^{(g)} = \mathrm{tr}\left[W^{+} B_\perp^{\lambda} W C_j^{g}(\bar P_+, \bar P_-) W^{+} B_{\perp\lambda} W\right]$$

where $ig B_\perp^{\lambda} \equiv [i\bar n\cdot D_c , iD_{\perp c}^{\lambda}] \sim \lambda \sim \xi_n$

$\bar P_\pm = \bar P^{+} \pm \bar P$

$O_j^{(i)}$ will lead to quark, anti-quark p.d.f.'s

O_j^{g} " " gluon p.d.f.'s

Quark contribution in detail :

$$O_j^{(i)} = \int d\omega_1 d\omega_2\, C_j^{(i)}(\omega_+,\omega_-) \left[\underset{\substack{\uparrow \\ \delta(\omega_1-\bar P^{+})}}{(\bar\xi_n W)_{\omega_1}} \frac{\bar{n}\!\!\!/}{2} \underset{\substack{\uparrow \\ \delta(\omega_2-\bar P)}}{(W^{+} \xi_n)_{\omega_2}} \right]$$

$\omega_\pm = \omega_1 \pm \omega_2$

coord space $f_{i/p}(z) = \int dy\; e^{-i2z\bar{n}\cdot P y} \langle p|\, \bar{\xi}(y)\, W(y,-y)\, \bar{n}\!\!\!/\, \xi(y)\, |p\rangle$

parton distn for quark i in proton p

$\bar{f}_{i/p}(z) = -f_{i/p}(-z)$ for anti-quark

mom. space $\langle P_n|\, (\bar{\xi}_n W)_{\omega_1}\, \bar{n}\!\!\!/\, (W^{\dagger}\xi_n)_{\omega_2}\, |P_n\rangle = 4\bar{n}\cdot P \int_0^1 d\xi\; \delta(\omega_-)$

$$* \Big[\, \delta(\omega_+ - 2\xi\,\bar{n}\cdot P)\, f_{i/p}(\xi) - \delta(\omega_+ + 2\xi\,\bar{n}\cdot p)\, \bar{f}_{i/p}(\xi) \Big]$$

recall

positive $\omega_1 = \omega_2$ gives particles

negative $\omega_1 = \omega_2$ gives anti-particles

$\underbrace{(\bar{\xi}_n W)_\omega}_{\text{a parton}}\, \bar{n}\!\!\!/\, (W^{\dagger}\xi_n)_\omega$ is a number operator for collinear quarks with momentum ω

[If we tried to couple usoft or soft gluons to this op. its a singlet so they decouple, more later]

Charge Conjugation

$$C_j^{(i)}(-\omega_+, \omega_-) = -C_j^{(i)}(\omega_+, \omega_-)$$

- relates Wilson-Coeff for quarks & anti-quarks at operator level
- Only need matching for quarks
- δ-functions set $\omega_- = 0$, $\omega_+ = 2\xi\,\bar{n}\cdot P = 2Q\frac{\xi}{x}$

Relate basis

$$\frac{1}{\pi}\,\mathrm{Im}\,T_1 = \int [d\omega]\ \frac{-1}{Q}\left(\frac{1}{\pi}\,\mathrm{Im}\,C_1(\omega)\right) \langle O^{(i)}(\omega)\rangle$$

$$\frac{1}{\pi}\,\mathrm{Im}\,T_2 = \int [d\omega]\ \left(\frac{4x}{Q}\right)^2 \frac{1}{Q}\,\frac{1}{\pi}\,\mathrm{Im}\left(C_2(\omega) - \frac{C_1(\omega)}{4}\right) \quad \langle O^{(i)}(\omega)\rangle$$

Define $\quad H_j(z) = \frac{\mathrm{Im}}{\pi}\ C_j\big(\underset{\omega_+}{2Qz},\ \underset{\omega_-}{0},\ Q^2, \mu^2\big)$ $\qquad$ do $\omega_\pm$ with δ-functions

$$T_1(x,Q^2) = \frac{-1}{x}\int_0^1 d\xi\ H_1^{(i)}\left(\frac{\xi}{x}\right)\left[f_{i/p}(\xi) + \bar{f}_{i/p}(\xi)\right]$$

$$T_2(x,Q^2) = \frac{4x}{Q^2}\int_0^1 d\xi\ \left(4H_2^{(i)}\left(\frac{\xi}{x}\right) - H_1^{(i)}\left(\frac{\xi}{x}\right)\right)\left[f_{i/p}(\xi) + \bar{f}_{i/p}(\xi)\right]$$

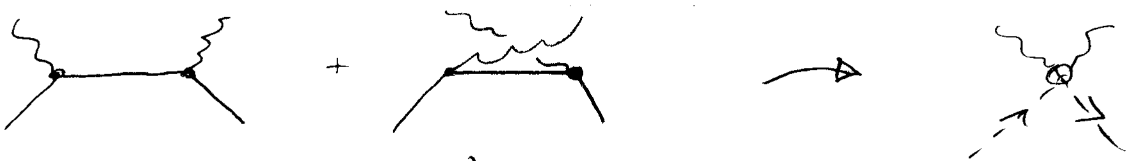
- this is factorization for DIS (to all order in α_s) into
  computable coefficients H_i
  universal non-pert. functions $f_{i/p}$, $\bar{f}_{i/p}$
  (show up in many processes)
- Coefficients C_j were dimensionless and can only have $\alpha_s(\mu)$ $\ln(\mu/Q)$ dependence on Q
  $\rightarrow$ Bjorken scaling

[Analysis valid to LO in $\frac{\Lambda^2}{Q^2}$]

- $H_i(\mu)$ $f_{i/p}(\mu)$ traditionally this μ-dependence is called the "factorization-scale" $\mu = \mu_F$
  & one also has "renorm. scale" $\alpha_s(\mu = \mu_R)$

In SCET the μ is just the ren. scale in SCET. We have new UV divergences associated with running of p.d.f., along with running for $\alpha_s(\mu)$.

- Tree Level Matching
  (upon which a lot of intuition is based)



find just $g_\perp^{\mu\nu}$ ie $C_2 = 0$

$\hookrightarrow$ Callan-Gross relation that $W_1 / W_2 = Q^2/4x^2$

$$C_1(\omega_+) = \underset{\overset{\uparrow}{\text{charges}}}{2e^2 Q_i^2} \left[\frac{Q}{(\omega_+ - 2Q) + i\epsilon} - \frac{Q}{-(\omega_+ + 2Q) + i\epsilon} \right]$$

$$H_1 = -e^2 Q_i^2 \; \delta\left(\frac{\xi}{x} - 1 \right)$$ gives parton-model interpretation $\xi = x$

One-Loop Renormalization of PDF

$$f_q(\xi) = \langle p_n |\, \bar\chi_n(0) \frac{\bar n\!\!\!/}{2}\, \chi_{n,\omega}(0)\, | p_n \rangle \qquad \text{with} \quad \omega = x\, p_n^- > 0$$

Renormalize operator in EFT with dim. reg., $1/\epsilon_{UV}$'s

Loops can change ω (or ξ), and parton type

$$f_i^{bare}(\xi) = \int d\xi'\ Z_{ij}(\xi,\xi')\, f_j(\xi',\mu)$$

$\nwarrow$ $1/\epsilon$'s, $\alpha_s(\mu)$ $\quad\nwarrow$ finite, but IR div. (encodes Λ_{QCD} effects)

$$\mu\, d/d\mu\, f_i(\xi,\mu) = \int d\xi'\ \gamma_{ij}(\xi,\xi')\, f_j(\xi',\mu)$$

$$\gamma_{ij} = -\int d\xi''\ \underbrace{Z_{ii'}^{-1}(\xi,\xi'')}_{\delta_{ii'}\,\delta(\xi-\xi'')+\cdots}\ \mu\, d/d\mu\ Z_{i'j}(\xi'',\xi')$$

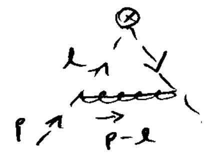
$$\gamma_{ij}^{1\text{-loop}} = -\mu\, d/d\mu \left[Z_{ij}(\xi,\xi') \right]^{1\text{-loop}}$$

tree level

$$= \underbrace{\bar u_n \frac{\bar n\!\!\!/}{2} u_n}_{p^-}\, \delta(\omega - p^-) = \delta(1 - \omega/p^-)$$

one-loop, use offshellness $p^2 = p^+ p^- \neq 0$ to regulate IR

ⓐ



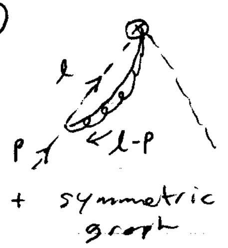
$$= -i g^2 C_F \int \bar{d}^d\ell\ \frac{p^-(d-2)\,\ell_\perp^2}{[\ell^2+i0]^2\,[(\ell-p)^2+i0]}\ \delta(\ell^- - \omega)\ \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}$$

$$= \frac{2 g^2 C_F}{(4\pi)^2} (1-\epsilon)^2\, \Gamma(\epsilon)\, e^{\epsilon\gamma_E}\ (1-z)\, \theta(z)\, \theta(1-z) \left(\frac{A}{\mu^2}\right)^{-\epsilon}$$

$$= \frac{\alpha_s C_F}{\pi} (1-z)\, \theta(z)\, \theta(1-z) \left[\frac{1}{2\epsilon} - 1 - \frac{1}{2}\ln(A/\mu^2)\right]$$

$$A = -p^+ p^- z(1-z) \quad , \quad z = \omega/p^-$$

ⓑ



two contractions $\bar{\xi}_n \underbrace{W \overset{\text{cut}}{\downarrow \delta} \overbrace{W^\dagger}^{\text{virtual}}}_{\text{real}} \xi_n$

$$= 2 i g^2 C_F \int \frac{d^d l \;\; \bar{u}_n \frac{\bar{n}\!\!\!/}{2} \frac{\alpha}{2}\, \bar{n}\cdot l\, \bar{n}\!\!\!/ u_n}{(l^- - p^-)(l^2)(l-p)^2} \Big[\overset{\text{real}}{\delta(l^- - \omega)} - \overset{\text{virtual}}{\delta(p^- - \omega)} \Big]$$

$$= \frac{C_F \alpha_s(\mu)}{\pi} e^{\epsilon \gamma_E} \Gamma(\epsilon) \Bigg[\frac{z\, \theta(z)\, \theta(1-z)}{(1-z)^{1+\epsilon}} \left(\frac{-p^- p^+ z - i0}{\mu^2} \right)^{-\epsilon}$$
$$- \delta(1-z) \left(\frac{-p^+ p^- - i0}{\mu^2} \right)^{-\epsilon} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)} \Bigg]$$

Distribution Identity

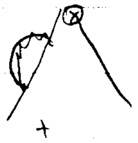
$$\frac{\theta(1-z)}{(1-z)^{1+\epsilon}} = -\frac{\delta(1-z)}{\epsilon} + \mathcal{L}_0(1-z) - \epsilon\, \mathcal{L}_1(1-z) + \cdots$$

plus-functions $\mathcal{L}_n(x) = \left[\frac{\theta(x) \ln^n x}{x} \right]_+$

$$\int_0^1 dx\, \mathcal{L}_n(x) = 0 \;, \quad \int_0^1 dx\, \mathcal{L}_n(x)\, g(x) = \int_0^1 dx \frac{\ln^n x}{x} [g(x) - g(0)]$$

- $1/\epsilon^2$ terms in real & virtual terms cancel
- remaining $1/\epsilon$ is UV

$$= \frac{C_F \alpha_s(\mu)}{\pi} \Bigg[\{ \delta(1-z) + z\,\theta(z)\, \mathcal{L}_0(1-z) \} \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^+ p^- z - i0} \right) - z\, \mathcal{L}_1(1-z)\, \theta(z)$$
$$+ \delta(1-z) \left(2 - \frac{\pi^2}{6} \right) \Bigg]$$



$$= \delta(1-z)\,(Z_\xi - 1) = \frac{\alpha_s C_F}{\pi} \left[-\frac{1}{4\epsilon} - \frac{1}{4} - \frac{1}{4} \ln \left(\frac{\mu^2}{-p^+ p^- - i0} \right) \right] \delta(1-z)$$

[We'll ignore

which mixes $\mathcal{O}^f_{\text{glue}}$ & $\mathcal{O}^f_{\text{quark}}$

this is $\overset{\text{only}}{\wedge}$ strictly correct if quark operator is not a flavor singlet

eg. $\bar{u}(\cdots)d$

$$\text{Sum} = \frac{C_F\,\alpha_s(\mu)}{\pi}\Big[\Big\{\frac{3}{4}\,\delta(1-z) + z\,\theta(z)\,\mathcal{L}_0(1-z) + \frac{(1-z)}{2}\,\theta(z)\,\theta(1-z)\Big\}\Big(\frac{1}{\epsilon} + \ln()$$

$$+ \text{ finite fn. of } z \Big]$$

$$= \frac{C_F\,\alpha_s(\mu)}{\pi}\Big[\underbrace{\frac{1}{2}\Big(\frac{1+z^2}{1-z}\Big)_+\Big[\frac{1}{\epsilon}}_{\text{determines } Z_{qq}^{\text{1-loop}}} + \ln\Big(\frac{--}{--}\Big)\Big] + \text{fn. } z\Big]$$

$$, \quad \mu\, d/d\mu\ \alpha_s = -2\epsilon\,\alpha_s + \cdots$$

Let total momentum of state be P^- , $p^-/P^- = \xi'$

$z = \omega/p^- = \frac{\xi P^-}{\xi' P^-} = \xi/\xi'$, & extra multiplicative

$P^-/p^- = 1/\xi'$ to swap

to proper norm for spinors

$$Z_{qq}^{\text{1-loop}} \rightarrow \gamma_{qq}(\xi, \xi') = \frac{C_F\,\alpha_s(\mu)}{\pi}\,\frac{\theta(\xi'-\xi)\,\theta(1-\xi')}{\xi'}\Big(\frac{1+z^2}{1-z}\Big)_+$$

Alterelli-Parisi (DGLAP) anomalous dimension

<u>Soft-Collinear Interactions</u> ($\mathrm{SCET_{II}}$)

Recall $q = q_s + q_c \sim Q(\overset{+}{\lambda}, \overset{-}{1}, \overset{\perp}{\lambda})$

$q^2 = Q^2\lambda \gg (Q\lambda)^2$

offshell w.r.t s, c

On-shell modes $q^\mu \sim Q(\lambda, 1, \sqrt{\lambda})$ are hard-collinear

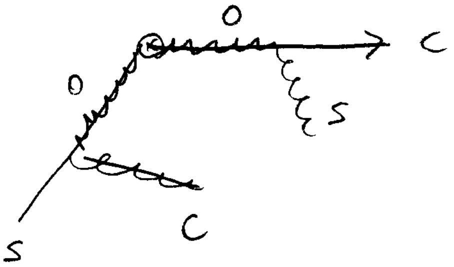
compared to collinear $q^\mu \sim Q(\lambda^2, 1, \lambda)$

Integrating out these fluctuations builds up a soft Wilson line S_n (analogous to $Y(n\cdot A_{us})$ but with soft fields)

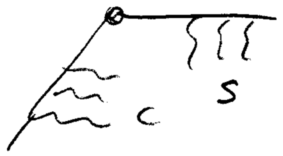
Toy eg. heavy-to-light soft-collin current $\bar{\xi}_n \Gamma h_v$

s = soft, c = collinear

o = offshell



adding more gives

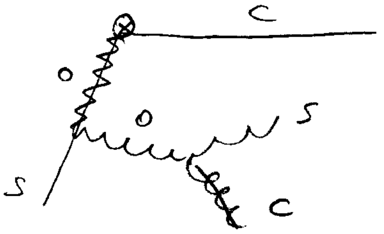


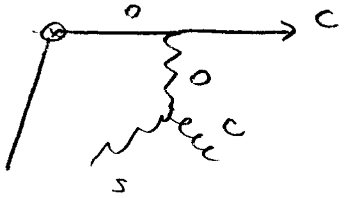
$\bar{\xi}_n S_n^+ \Gamma W h_v$

$S_n^+[n\cdot A_{us}]$

$W[\bar{n}\cdot A_c]$

In QCD need 3-gluon, 4-gluon vertices too: these flip order of S^+ & W





$$\underbrace{(\bar{\xi}_n W)}_{\substack{\text{Collinear}\\ \text{gauge}\\ \text{invariant}}} \Gamma \underbrace{(S_n^+ h_v)}_{\substack{\text{soft}\\ \text{gauge}\\ \text{invariant}}}$$

[can be extended to all orders]

this is <u>soft</u> - <u>collinear</u> <u>factorization</u>

Another Method

- construct SCET_{II} operators using SCET_I

i) Match QCD onto SCET_I usoft $p_u^2 \sim \Lambda^2$

collinear $p_c^2 \sim Q\Lambda$

ii) Factorize usoft with field redefinition

iii) Match SCET_I onto SCET_{II} soft $p_s^2 \sim \Lambda^2$

collin $p_c^2 \sim \Lambda^2$

<u>Notes</u>

- this gives us a simple procedure to construct SCET_{II} ops. (<u>even</u> <u>though</u> <u>they're</u> <u>non-local</u>)
- usoft fields in I are renamed soft for II

eg. i) $J^{I} = (\bar{\xi}_n W)\,\Gamma\, h_v$

ii) $J^{I} = (\bar{\xi}_n^{(0)} W^{(0)})\,\Gamma\,(Y^{\dagger} h_v)$

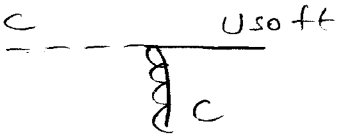
iii) $J^{II} = (\underset{\uparrow}{\bar{\xi}_n} W)\,\Gamma\,(S^{\dagger} h_v)$ as before

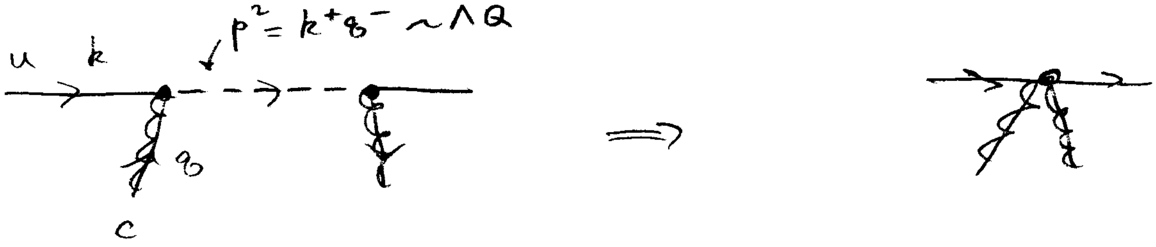
here all T-products in SCET_I & SCET_{II} match up, so matching was trivial

<u>"Thm"</u> • In cases where we have T-products in SCET_I with ≥ 2 operators involving both collin & usoft fields, we can generate a non-trivial coefficient in SCET_{II} (jet-function J)

T eg. $$\int dp_-\, dk_+\, \underset{\substack{\uparrow\\ \mathrm{SCET}_I \text{ loops}\\ p^2\sim Q\Lambda}}{J}(p_-, \underset{\substack{\uparrow\\ \text{ind's allow}\\ k^+ \text{ dependence}}}{k_+})\ \overbrace{(\bar{\xi} W)_{p_-}\,\Gamma\,(S^{\dagger} q_s)_{k^+}}^{p^2\sim\Lambda^2}$$

eg. two operators





when we lower offshellness of ext. collin fields the intermediate line still has $p^2 \sim Q\Lambda$ and must really be integrated out

P.C. $\quad T^{I} \sim \lambda^{2K} \quad \Rightarrow \quad O^{II} \sim \eta^{K+E}$

where $\lambda^2 = \eta = \dfrac{\Lambda}{Q}$,

factor $E > 0$ from changing the scaling of ext. fields

eg. $\xi_I \sim \lambda$

$\xi_{II} \sim \eta = \lambda^2$

$\Rightarrow$ No mixed soft-collin $\mathcal{L}$ at leading order

- after field redefn no mixed $\mathcal{L}_I$ ops at LO

- mixed $\mathcal{L}_I^{(1)}$ gives $T\{\mathcal{L}_I^{(1)}, \mathcal{L}_I^{(1)}\} \sim \lambda^2$
  matches onto $O_{II} \sim \eta$ or higher

SCET$_I$ $\quad \lambda^\delta$

$$\delta = 4 + \underset{\uparrow\ u=1 \text{ no c., else } u=0}{4u} + \overset{\text{rest}}{\sum_k (k-4)\, V_k^c} + \overset{\text{pure usoft}}{(k-8)\, V_k^u}$$

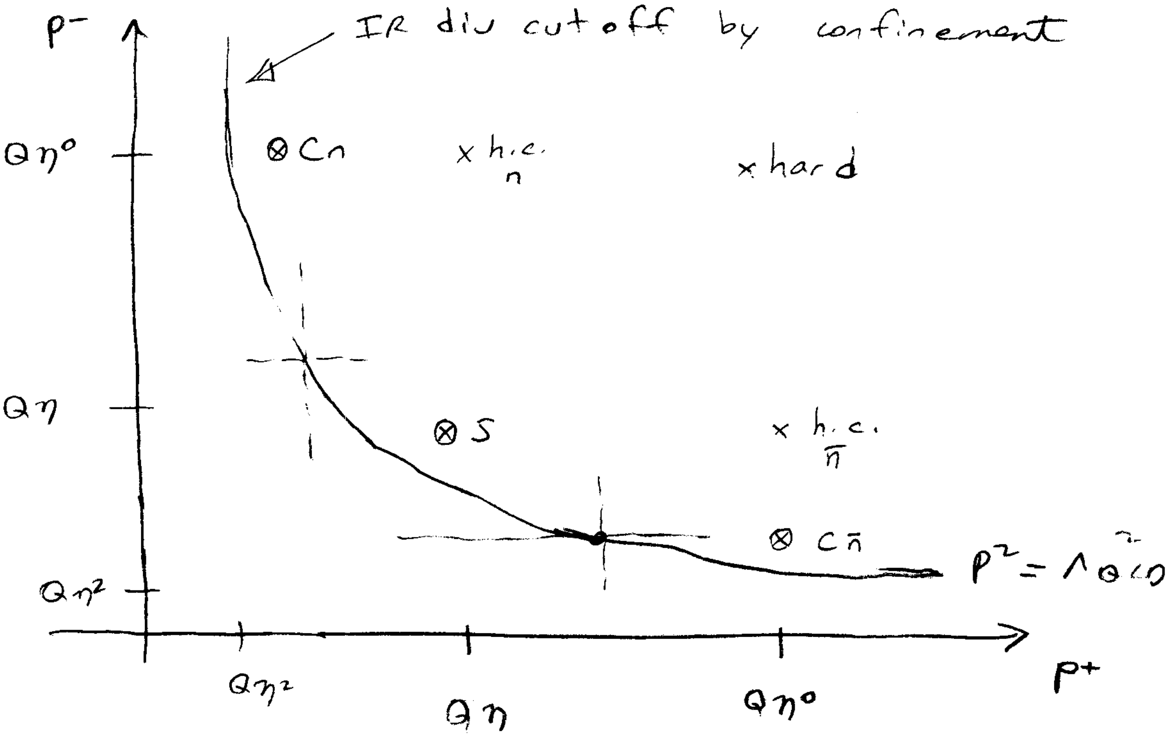
$\underline{SCET_{II}}$ $$\delta = 4 + \sum_k (k-4)\big(\underset{\substack{\uparrow \\ \text{pure} \\ c}}{V_k^c} + \underset{\substack{\uparrow \\ \text{pure} \\ s}}{V_k^s} + \underset{\substack{\uparrow \\ \text{mixed}}}{V_k^{sc}}\big) + \underset{\substack{\uparrow \\ p\sim(\eta^2,\eta,\eta) \\ \text{loops}}}{L^{sc}}$$

$$\delta = 5 - \underset{\uparrow}{N_c} - \underset{\uparrow}{N_s} + \sum_k (k-4)\left(V_k^S + V_k^c\right) + (k-3)\, V_k^{sc}$$

\# connected
soft, collin components

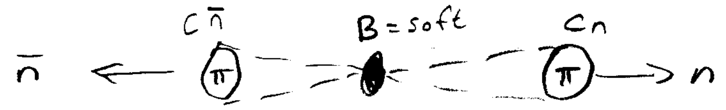
[in eg. $SCET_I$ $\lambda^3\lambda \frac{1}{\lambda^2}\lambda^3\lambda \sim \lambda^{6-4} \sim \lambda^2$ $\nRightarrow$ $\left(\eta^{3/2}\eta\right)^2\frac{1}{\eta} = \eta^{4-3} = \eta$]
or $\lambda * \lambda \sim \lambda^2$

$$\mathcal{L}_{SCET^{II}} = \mathcal{L}^{(0)}_{soft}[q_s, A_s] + \mathcal{L}^{(0)}_{collin\text{-}n}[\xi_n, A_n] + \mathcal{L}^{(0)}_{collin\text{-}\bar{n}}[\xi_{\bar{n}}, A_{\bar{n}}]$$



Non-pert d.o.f. in different sectors $B \to \pi\pi$

eg.



Exclusive

e.g. $\gamma^* \gamma \to \pi^0$ hard-collin factorization

[Breit frame: soft modes have no active role so this does not really probe difference between $SCET_I$ & $SCET_{II}$]

QCD has

$$\langle \pi^0(p_\pi) | J_\mu(0) | \gamma(p_\gamma,\epsilon)\rangle = ie\,\epsilon^\nu \int d^4z\, e^{-ip_\gamma\cdot z} \langle\pi^0(p_\pi)| T\, J_\mu(0)\, J_\nu(z)|0\rangle$$

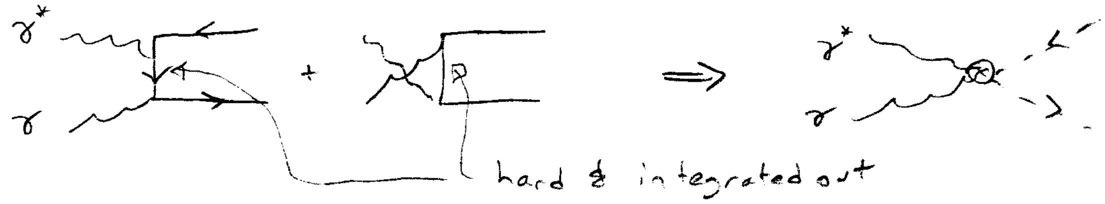
$$= -ie\, F_{\pi\gamma}(Q^2)\, \epsilon_{\mu\nu\rho\sigma}\, p_\pi^\nu \epsilon^\rho q^\sigma$$

e.m. current $J^\mu = \bar\Psi \hat Q \gamma^\mu \Psi$, $\hat Q = \frac{\tau_3}{2} + \frac{\mathbb{1}}{6} = \begin{pmatrix} 2/3 & \\ & -1/3 \end{pmatrix}$

For $Q^2 \gg \Lambda^2$ $F_{\pi\gamma}$ simplifies (a la Brodsky-Lepage)

frame $q^\mu = \frac{Q}{2}(n^\mu - \bar n^\mu)$, $p_\gamma^\mu = E\,\bar n^\mu$

$p_\pi^\mu = p + p_\gamma = \frac{Q}{2} n^\mu + (E - \frac{Q}{2})\bar n^\mu$



SCET Operator at Leading-order (for T-product) is

$$O = \frac{i\,\epsilon^\perp_{\mu\nu}}{Q}\, [\bar\xi_{n,p} W]\, \Gamma\, C(\bar P, \bar P^+, \mu)\, [W^\dagger \xi_{n,p'}]$$

order λ^2 ("twist-2")

- obeys current conservation
- dim analysis fixes $\frac{1}{Q}$ pre-factor for C dimless

- Charge Conj: $T\{J,J\}$ even so O even so $C(\mu,\bar P,\bar P^+) = C(\mu,-\bar P^+,-\bar P)$

- flavor & spin structure $\quad \Gamma = \underbrace{\bar{n}\!\!\!/\, \gamma_5}_{\text{for pion}} \; 3\sqrt{2}\; \underbrace{\hat{Q}^2}_{\text{2nd order e.m.}}$

- color singlet, purely collinear (again) so soft gluons decouple

$$\text{equate} \quad \frac{Q^2}{2} F_{\pi\gamma} = \frac{i}{Q} \overset{\text{SCET}_{II}}{\langle} \pi^0 | (\bar\xi W)\, \Gamma\, C\, (W^\dagger \xi) | 0 \rangle$$

write $\quad \bar P_\pm = \bar P^\dagger \pm \bar P$

now $\bar P_-$ gives total mom of $(\bar\xi W)\Gamma(W^\dagger\xi)$ operator

ie momentum of pion

$$\bar P_- = \bar n \cdot p_\pi = Q$$

$\longrightarrow$ total mom

$$F_{\pi\gamma}(Q^2) = \frac{2i}{Q^2} \int d\omega\, C(\omega,\mu) \langle \pi^0 | (\bar\xi W)\, \Gamma\, \delta(\omega - \bar P_+)(W^\dagger \xi) | 0 \rangle$$

Non-perturbative Matrix Elt

finite Wilson line $\left(P\exp ig \int_x^y ds \ldots \right)$

Fourier Transform of $\bar n \cdot p$ label

position space

$$\langle \pi^0(p) | \bar\xi_n(y) \frac{\bar n\!\!\!/\, \gamma_5 \tau^3}{\sqrt2} W(y,x)\, \xi_n(x) | 0 \rangle = -i f_\pi\, \bar n\cdot p \int_0^1 dz\, e^{i\bar n\cdot p\,(zy + (1-z)x)}\, \phi_\pi(\mu,z)$$

$$\int_0^1 dz\, \phi_\pi(z) = 1$$

momentum space

$$\langle \pi^0(p) | (\bar\xi_{n,p} W) \frac{\bar n\!\!\!/\, \gamma_5 \tau^3}{\sqrt2} \delta(\omega - \bar P_+)(W^\dagger \xi_{n,p'}) | 0 \rangle$$

$$= -i f_\pi\, \bar n\cdot p \int_0^1 dz\, \delta(\omega - (2z-1)\bar n\cdot p)\, \phi_\pi(\mu,z)$$

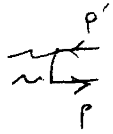
Plug it into $F_{\pi\gamma}(Q^2)$ and do integral over ω

$$F_{\pi\gamma}(Q^2) = \frac{2 f_\pi}{Q^2} \int_0^1 dz \; C\big((2z-1)Q, Q, \mu \big) \; \phi_\pi(z,\mu)$$

- ϕ_π is universal light-cone dist'n for pions
- C is process dependent (all order factorization in α_s)
- one-dim convolution again

Tree Level Matching

expand

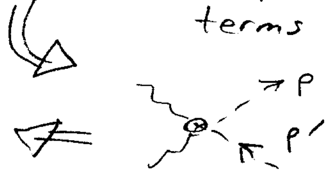
$i\Big($  +  $\Big) = \frac{ie}{2} \, \epsilon_{\mu\nu\rho\beta} \, \epsilon^\nu \bar n^\rho n^\beta \left(\frac{\bar{n}\!\!\!/}{2} \gamma_5 \right) \hat{Q}^2$

$$* \left(\frac{1}{\bar n \cdot p} - \frac{1}{\bar n \cdot p'} \right) + \ldots$$

non-pion terms

So $$C = \frac{1}{6\sqrt{2}} \left(\frac{Q}{\bar p^+} - \frac{Q}{\bar p} \right)$$

$$C(\omega = (2x-1)Q) = \frac{1}{6\sqrt{2}} \left(\frac{1}{x} + \frac{1}{1-x} \right)$$



match with $W=1$

Charge Conj $+1$ for $|\pi^0\rangle$ gives $\phi_\pi(x) = \phi_\pi(1-x)$

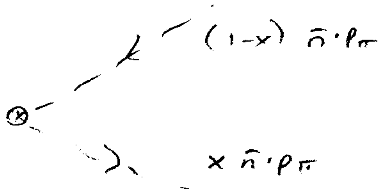
(Hmwk.)

So only $\displaystyle\int_0^1 dx \, \frac{\phi_\pi(x,\mu)}{x}$ appears in our prediction

integrate over all x, much different than DIS $\delta(1 - \xi/x) \Rightarrow f_{i/p}(x,\mu)$

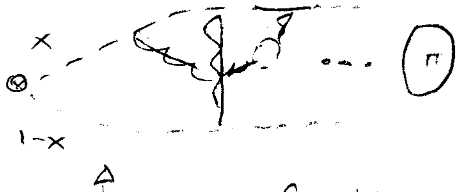
Interpretation:

Naively



mom fraction of quarks in pion

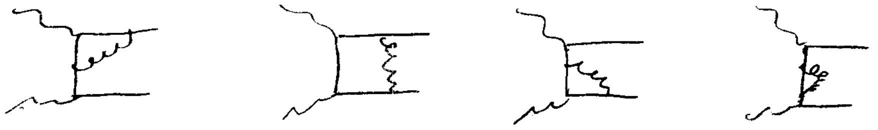
Really



↖ mom. fractions at point where quarks are produced. Hadronization process changes "x" carried by valence quarks which is encoded in $\phi_\pi(x)$

Higher Order Matching

full



+ w.fn graphs

SCET

+ w.fn

Differeence will be IR finite, and gives C at one-loop

Another Exclusive Example

(hep-ph/0107002)

<u>$B \to D\pi$</u> $\qquad \underbrace{m_b, m_c, E_\pi}_{Q} \gg \Lambda_{QCD}$

QCD Operators at $\mu \simeq m_b$

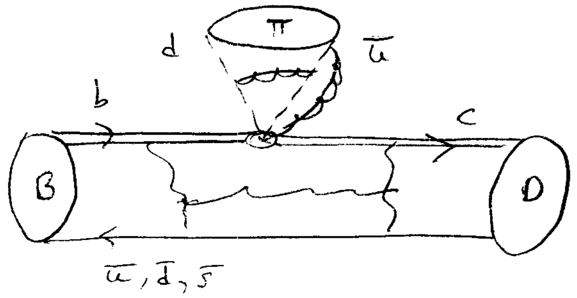
$$H_W = \frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cb} \left[C_0^F O_0 + C_8^F O_8 \right]$$

$$P_L = \frac{1-\gamma_5}{2}$$

where
$$O_0 = [\bar c \gamma^\mu P_L b][\bar d \gamma_\mu P_L u]$$
$$O_8 = [\bar c \gamma^\mu P_L T^a b][\bar d \gamma_\mu P_L T^a u]$$

Want to Factorize $\langle D\pi | O_{0,8} | B \rangle$

ie Show at LO



no gluons btwn B, D and quarks in pion

↖ expect $B \to D$ form factor ξ Isgur-Wise

$\phi_\pi(x)$ distn for pion

$$\left.\begin{array}{lll} B, D & \text{soft} & p^2 \sim \Lambda^2 \\ \pi & \text{collinear} & p^2 \sim \Lambda^2 \end{array}\right\} \text{SCET}_{II}$$

Use SCET_I as intermediate step

① Match at $\mu^2 \simeq Q^2$

$$\left.\begin{array}{l} O_0 \\ O_8 \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} Q_0^{1,5} = [\bar h_{v'}^{(c)} \Gamma_h^{1,5} h_v^{(b)}][(\bar\xi_n^{(d)} W) \Gamma_\ell C_0(\bar P_+) W^\dagger \xi_n^{(u)}] \\ Q_8^{1,5} = [\;\text{"}\; T^A \;\text{"}\;][\;\text{"}\;\;\text{"}\; C_8(\bar P_+) T^a \;\text{"}\;] \end{array}\right.$$

↑ usoft SCET_I

↑ collinear $p^2 \sim Q\Lambda$

$$\Gamma_h^{1,5} = \frac{\not{n}}{2} \{1, \gamma_5\}$$
$$\Gamma_\ell = \frac{\not{\bar n}}{4} (1-\gamma_5)$$

② Field redefinitions $\xi_{n,p} = Y \xi_{n,p}^{(0)}, \ldots$

in $Q_0^{1,5}$ get $\bar\xi_n^{(0)} W^{(0)} \cancel{Y^\dagger} \cancel{Y} W^{\dagger(0)} \xi_n^{(0)}$

$Q_8^{1,5}$ get $\bar\xi_n^{(0)} W^{(0)} Y^\dagger T^A Y W^{\dagger(0)} \xi_n^{(0)}$

$Y T^a Y^\dagger = \mathcal{Y}^{ba} T^b$ $\qquad\qquad Y^\dagger T^a Y = \mathcal{Y}^{ab} T^b$

↑ adjoint Wilson line

$T^a \otimes Y^\dagger T^a Y = Y T^a Y^\dagger \otimes T^a$

↑ moves usoft Wilson lines next to h_v fields

③ Match SCET$_{\text{I}}$ onto SCET$_{\text{II}}$ (trivial here again)

$Y \to S$

$\xi_n^{(0)} \to \xi_n$ in II etc.

$$Q_0^{1,5} = [\bar h_{v'}^{(c)} \Gamma_h h_v^{(b)}][\bar\xi_n^{(d)} W \Gamma_\ell C_0(\bar P_+) W^\dagger \xi_{n,p}^{(a)}]$$
$$Q_8^{1,5} = [\bar h_{v'}^{(c)} \Gamma_h S T^a S^\dagger h_v^{(b)}][\bar\xi_n^{(d)} W \Gamma_\ell C_8(\bar P_+) T^a W^\dagger \xi_{n,p}^{(a)}]$$

④ Take Matrix Elements

$$\langle \pi_{\bar n} | \bar\xi_n W \Gamma C_0(\bar P_+) W^\dagger \xi_n | 0\rangle = \frac{i}{2} f_\pi E_\pi \int_0^1 dx\, C(2E_\pi(2x-1))\phi_\pi(x)$$
$$\langle D_{v'} | \bar h_{v'} \Gamma h_v | B\rangle = N' \xi(w_0, \mu)$$

↑ $w_0 = v\cdot v'$

B, D purely soft → no contractions with collinear fields

π " collinear → no " " soft fields

which is why it factors into two matrix element

O8:

$$\langle D_{v'} | \underbrace{\bar h_{v'} Y T^a Y^\dagger h_v}_{\text{color octet operator between color singlet states}} | B_v\rangle = 0$$

Find Factorization Formula

$$\langle \pi D | H_W | B \rangle = \underset{\substack{\uparrow \\ \text{prefactors}}}{i N} \; \xi(w_0, \mu) \int_0^1 dx \; C(2E_\pi(2x-1), \mu) \, \phi_\pi(x, \mu) + O(\Lambda/Q)$$

- $\xi(w_0, \mu)$ is Isgur-Wise function at max. recoil
  $w_0 = \frac{m_B^2 - m_D^2}{2 m_B}$ (measured in $B \to De\nu$ recall)

- This applies to type-I (& III) decays
  $\bar{B}^0 \to D^+ \pi^-$ $\bar{B}^0 \to D^{*+} \pi^-$, $\bar{B}^0 \to D^+ \rho^-$, ...
  $B^- \to D^0 \pi^-$ $B^- \to D^{*0} \pi^-$ $B^- \to D^0 \rho^-$, ...

  predicts type-II decays are suppressed by Λ/Q
  $\bar{B}^0 \to D^0 \pi^0$, ... (we could derive fact. thm. for these too)

Another inclusive example . $B \to X_s \gamma$ [Case where usoft modes matter]

Here we will need both usoft & collinear d.o.f. in SCET$_I$

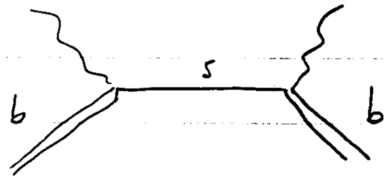
$$H_{eff} = \frac{-4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \; C_7 \, O_7 \quad , \quad O_7 = \frac{e}{16\pi^2} m_b \, \bar{s} \, \sigma^{\mu\nu} F_{\mu\nu} P_R \, b$$

photon $q^\mu = E_\gamma \, \bar{n}^\mu$

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dE_\gamma} = \frac{4E_\gamma}{m_b^3} \left(\frac{-1}{\pi}\right) \text{Im } T$$

$$T = \frac{i}{m_B} \int d^4x \; e^{-iq\cdot x} < \bar{B} \,|\, T \; J_\mu^+(x) \, J^\mu(0) \,|\, \bar{B} >$$

$$J^\mu = \bar{s} \; i\sigma^{\mu\nu} q_\nu \, P_R \, b$$

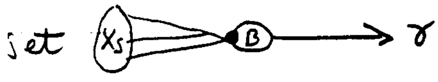


looks like DIS

Consider endpoint region

$$m_B/2 - E_\gamma \lesssim \Lambda_{QCD}$$

$$p_X^2 \simeq m_B \Lambda$$



B-rest frame $\quad p_B = \frac{m_B}{2}(n^\mu + \bar{n}^\mu) \quad = \quad p_X + q$

$$p_X = \frac{m_B}{2} n^\mu + \frac{\bar{n}^\mu}{2} \underbrace{(m_B - 2E_\gamma)}_{\Lambda}$$

collinear

so quarks and gluons in X are <u>collinear</u> with $p_c^2 \sim m_B \Lambda$

B has <u>usoft</u> light d.o.f.

match onto LO SCET operator

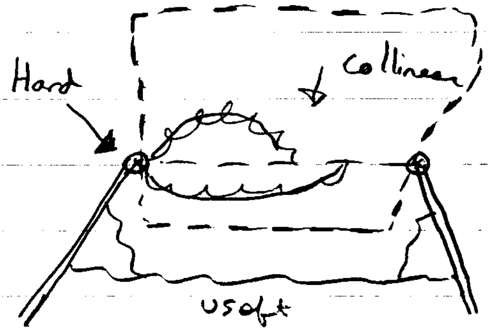
$$J_\mu = -E_\gamma \; e^{i(\bar{P}\frac{n}{2} - m_b v)\cdot x} \; \bar{\xi} W \gamma_\mu^\perp P_L h_v \; C(\bar{P}^+, \mu)$$

↑ our heavy-to-light current from earlier $\equiv J^\mu_{eff}$

The coefficient $C(\bar{P}^+)$ has $\bar{P}^+ = m_b$ since this is total momentum of s-quark jet in $\bar{n}\cdot P_X$

Factor with Field redefn

$$J_{eff}^\mu = \bar{\xi}_n^{(0)} W^{(0)} \gamma_\mu^\perp P_L Y^\dagger h_v$$



factored

$$T_{eff} = i\int d^4x \; e^{i(m_b\frac{\bar{n}}{2} - q)\cdot x} \langle \bar{B} | T J_{eff}^{\dagger\mu}(x) J_{eff\,\mu}(0) | \bar{B}\rangle$$

$$= i\int d^4x \; e^{i()} \langle \bar{B} | T(\bar{h}_v Y)(x) \; (Y h_v)(0) \; | \bar{B}\rangle$$
$$* \langle 0 | T(W^{\dagger(0)} \xi^{(0)})(x) \; (\bar{\xi}^{(0)} W)(0) \; | 0\rangle$$

↖ spin & color indices & structures $\gamma_\mu^\perp P_L$ suppressed

$$= \frac{1}{2}\int d^4x \int đ^4k \; e^{i(m_b\frac{\bar{n}}{2} - q - k)\cdot x} \langle \bar{B} | T(\bar{h}_v Y)(x) \; (Y^\dagger h_v)(0) \; | \bar{B}\rangle$$
$$* \; J_P(k)$$

$$\langle 0 | T(W^\dagger \xi)_{P,O_\perp} (\bar{\xi} W) \; | 0\rangle = i \int đ^4k \; e^{-ik\cdot x} J_P(k) \frac{\bar{n}\!\!\!/}{2}$$

↑ minus & ⊥ labels

↑ only depends on k^+! so do $k^-, k^\perp$ integrals

in Teff we then get

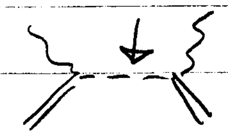
$$S(\ell^+) = \frac{1}{2}\int \frac{dx^-}{4\pi} \; e^{-i/2\, \ell^+ x^-} \langle \bar{B} | T[\bar{h}_v Y)(\tfrac{n}{2}x^-) \; (Y^\dagger h_v)(0) | \bar{B}\rangle$$

↖ ↗ $Y(\frac{n}{2}x^-, 0)$

$$= \frac{1}{2} \langle \bar{B}_v | \; \bar{h}_v \; \delta(in\cdot D - k^+) h_v | \bar{B}_v\rangle$$

Imaginary part is in jet function

let $J(k^+) \doteq -\frac{1}{\pi}\,\mathrm{Im}\, J_p(k^+)$

(tree level $J(k^+) = \delta(k^+)$ from )

All order's factorization

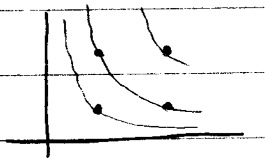
$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_\gamma} = \underset{\substack{\uparrow \\ p^2 \sim m_b^2}}{N\, C(m_b,\mu)} \int_{2E_\gamma - m_b}^{\bar{\Lambda}} dl^+ \underset{\substack{\uparrow \\ p^2\sim\Lambda^2}}{S(l^+)}\; \underset{\substack{\uparrow \\ p^2 \sim m_b\Lambda}}{J(l^+ + m_b - 2E_\gamma)}$$

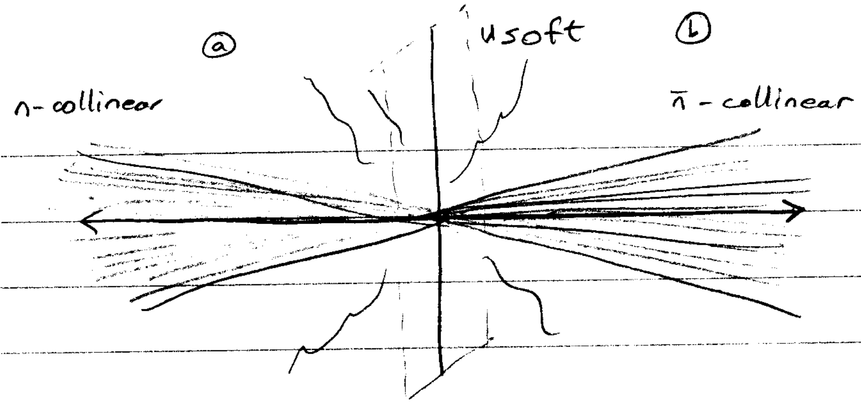
$\uparrow$

Shape function
is <u>seen</u> in the
data

$\boxed{e^+e^- \to \text{dijets}}$

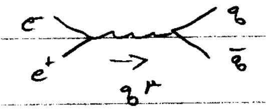
SCET$_I$





$e^+e^- \to \gamma^*$ or $Z^* \to X_n \, X_{\bar n} \, X_{usoft}$ $\qquad (e^+e^-)$ cm frame

Scales



$q^2 = Q^2$ $\qquad$ hard $\qquad \mu_h \sim Q$

- Hemisphere invariant mass $\quad$ divide $\quad p_X^\mu = p_{Xa}^\mu + p_{Xb}^\mu$

$$M^2 \equiv (p_{Xa}^\mu)^2 = \Big(\sum_{i\varepsilon a} (p_i^\mu)\Big)^2 , \qquad \bar{M}^2 = \Big(\sum_{i\varepsilon b} p_i^\mu\Big)^2$$

jet $\to$ $M^2 \ll Q^2$

n- collinear $\quad Q(\lambda^2, 1, \lambda)$ $\qquad \mu_J \sim M$

n̄- collinear $\quad Q(1, \lambda^2, \lambda)$ $\qquad \lambda = M/Q$

- Usoft Radiation
  - uniform in space
  - communication btwn jets
  - eikonal

energy $\sim Q\lambda^2 = M^2/Q$

$\mu_S \sim M^2/Q$

$\nearrow$ $M^2/Q \gg \Lambda_{QCD}$ perturbative

"tail region"

$\mu_h \gg \mu_J \gg \mu_S \gg \Lambda_{QCD}$

$\searrow$ $M^2/Q \sim \Lambda_{QCD}$ non-perturbative

"peak region"

$\mu_h \gg \mu_J \gg \mu_S \sim \Lambda_{QCD}$

In tail region we have power corrections $\left(\frac{\Lambda_{QCD}}{\mu_S}\right)^K \ll 1$. Leading order cross-section <u>perturbative</u>.

In peak region $\quad (\Lambda_{QCD}/\mu_S)^K \sim 1$ (any K) $\to$ non-pert. soft function

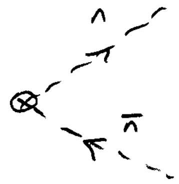
Other Power Corrections

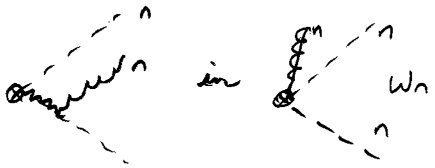
- μ_S/μ_J "Kinematic" expansion of kinematic variables
- Λ_{QCD}/μ_h hard power corr. (HwK)
- $\Lambda_{QCD}/\mu_J = \frac{\Lambda_{QCD}}{\mu_S}\frac{\mu_S}{\mu_J}$ not independent

Current $J^\mu = \overset{QCD}{\bar\Psi\,\Gamma^\mu\,\Psi} \rightarrow (\bar\xi_n W_n)_\omega \Gamma^\mu (W_{\bar n}^\dagger \xi_{\bar n})_{\bar\omega}$

$= (\bar\xi_n W_n)_\omega \Gamma^\mu (Y_n^\dagger Y_{\bar n})_{\bar\omega} (W_{\bar n}^\dagger \xi_{\bar n})$ field redefn

color singlet





Kinematics $q^\mu = p_{X_n}^\mu + p_{X_{\bar n}}^\mu + p_s^\mu$

large $\bar n \cdot q = Q = \bar n \cdot p_{X_n} + \overset{small}{\cdots}$ $\omega = Q$

$n \cdot q = Q = n \cdot p_{X_{\bar n}} + \cdots$ $\bar\omega = Q$

} momentum conservation is strong enough that there are no convolutions in $\omega, \bar\omega$

Cross-Section

QCD $\sigma = \sum_X^{res} (2\pi)^4 \delta^4(q - p_X) \; L_{\mu\nu} \langle 0 | J^{\nu\dagger}(0) | X \rangle \langle X | J^\mu(0) | 0 \rangle$

↑ restricted to dijet X states

SCET allows us to move restrictions into operators

$|X\rangle = |X_n\rangle |X_{\bar n}\rangle |X_s\rangle$

$\bar 3$ rep 3-rep

$$\sigma = N_0 \sum_{\vec n} \sum_{X_n, X_{\bar n}, X_s}^{res'} (2\pi)^4 \delta^4(q - p_{X_n} - p_{X_{\bar n}} - p_s) \langle 0 | \overline{Y}_{\bar n} Y_n | X_s \rangle \langle X_s | Y_n^\dagger \overline{Y}_{\bar n}^\dagger | 0 \rangle$$

$$* \; |C(Q,\mu)|^2 \langle 0 | \bar{n}\!\!\!/ \, \chi_{n,Q} | X_n \rangle \langle X_n | \bar\chi_n | 0 \rangle$$
$$\langle 0 | \bar\chi_{\bar n, Q} | X_{\bar n} \rangle \langle X_{\bar n} | n\!\!\!/ \, \chi_{\bar n} | 0 \rangle$$

all orders in α_s

$+ \cdots \leftarrow$ "other" power corr.

res' : we must still measure enough things about X to ensure its a dijet

Measure $M^2, \bar{M}^2$

$$1 = \int dM^2\, d\bar{M}^2 \;\; \delta\big(M^2 - (p_n + k_s^a)^2\big) \;\; \delta\big(\bar{M}^2 - (p_{\bar{n}} + k_s^b)^2\big)$$

↑ soft momenta in hemisphere ⓐ ↑ soft in ⓑ

$\frac{d\sigma}{dM^2 d\bar{M}^2}$ has these δ's under $\sum_X$

$$\delta\big(M^2 - p_n^2 - p_n^-(k_s^a)^+ + \cdots\big) = \delta\big(M^2 - Q(p_n^+ + k_s^{a+})\big)$$
$$= \frac{1}{Q}\,\delta\big(p_n^+ + k_s^{a+} - M^2/Q\big)$$

n-collinear jet function $J(p_n^2)$

$\bar{n}$- " " " $J(p_{\bar{n}}^2)$

soft function $S(k_s^{a+}, k_s^{b-})$ ← sensitive to use of hemispheres

Factorization Thm (... Algebra ...)

$$\frac{d\sigma}{dM^2 d\bar{M}^2} = \sigma_0\, H(Q,\mu) \int d\ell^+ d\ell^- \;\; J_n(M^2 - Q\ell^+, \mu)\; J_{\bar{n}}(\bar{M}^2 - Q\ell^-, \mu)\; S(\ell^+, \ell^-, \mu)$$

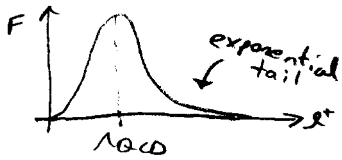
- same jet fn as $b \to s\gamma$

- $$S_{\rm hemi}(\ell^+, \ell^-) = \frac{1}{N_c} \sum_{X_s} \delta(\ell^+ - k_s^{a+})\, \delta(\ell^- - k_s^{b-}) \langle 0|\bar{Y}_{\bar{n}}\, Y_n |X_s\rangle\langle X_s| Y_n^\dagger \bar{Y}_{\bar{n}}^\dagger |0\rangle$$

encodes both momentum scale $\ell^\pm \sim M^2/Q$ and $\Lambda_{QCD} \sim \ell^\pm$

Soft Function OPE

$$S_{hemi}(\ell^+,\ell^-) = \int d\ell'^{\pm} \; \underset{\substack{\uparrow \\ \text{power tail} \\ \frac{(\ln \ell^+/\mu)^k}{\ell^+}}}{S^{pert}_{hemi}(\ell^+-\ell^{+\prime},\ell^- -\ell^{-\prime})} \; \underset{\sim \Lambda_{QCD} \text{ effects}}{F(\ell^{+\prime},\ell^{-\prime})}$$



Thrust $\quad T = \max\limits_{\hat t} \dfrac{\sum_i |\vec p_i \cdot \hat t\,|}{\sum_i |\vec p_i|}$ $\qquad \frac{1}{2} \le T \le 1$

$0 \le \tau \le 1/2$

$\tau = 1-T$

for dijets $\quad \tau = \dfrac{M^2+\bar M^2}{Q^2} \leftarrow$ symmetric projection

$$\frac{d\sigma}{d\tau} = \sigma_0 \, H(Q,\mu)\, Q \int d\ell \;\; J_\tau(Q^2\tau - Q\ell,\mu)\; S_\tau(\ell,\mu)$$

$p^2 \sim Q^2 \qquad$ jet $\; p^2 \sim Q^2\tau \qquad$ usoft $\; p^2 \sim Q^2\tau^2$

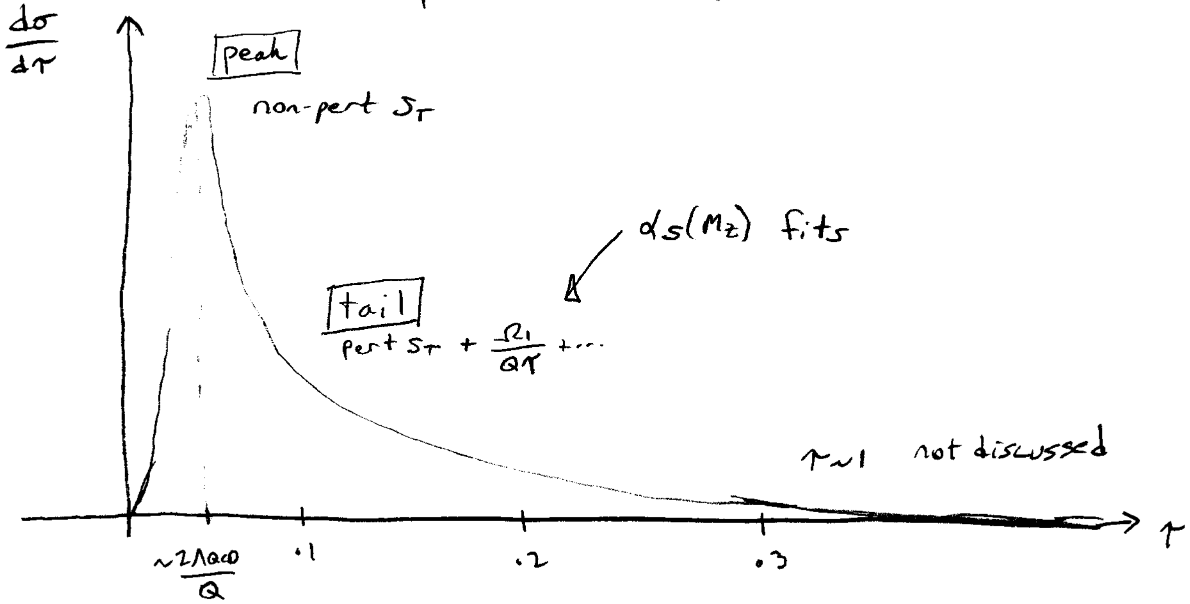
$$Q^2 \gg Q^2\tau \gg Q^2\tau^2$$

$$\mu_h^2 \gg \mu_J^2 \gg \mu_S^2 \;\overset{\gg}{\underset{\sim}{\text{or}}}\; \Lambda_{QCD}^2$$

schematically: $\quad \dfrac{d\sigma}{d\tau} \sim \sum\limits_{n,m} \underbrace{\dfrac{\alpha_s^n \ln^m \tau}{\tau}}_{} \;+\;$ non-perturbative effects in F

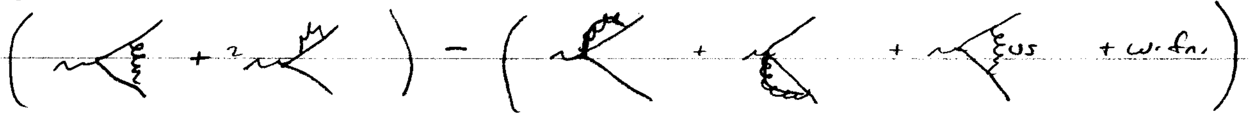
$\nwarrow$ sum logs

+ power corrections



Pert. Results

- match quark form factor



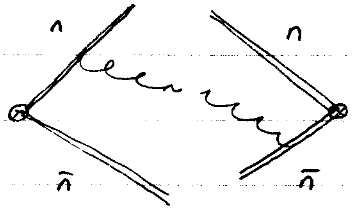
$$C(Q,\mu) = 1 + \frac{C_F\,\alpha_s(\mu)}{4\pi}\left[\,3\ln^2\!\left(\frac{-Q^2}{\mu^2}\right) - \ln\!\left(-\frac{Q^2}{\mu^2}\right) - 8 + \frac{\pi^2}{6}\right]$$

$$H = |C|^2$$

(Renormalized)
- Jet Function

$$J_n(s,\mu) = \delta(s) + \frac{\alpha_s(\mu)\,C_F}{4\pi}\left[\#\,\delta(s) + \#\left[\frac{\mu^2\theta(s)}{s}\right]_+ + \#\left[\frac{\mu^2\ln(\mu^2/s)\,\theta(s)}{s}\right]_+\right]$$

- Pert. Soft Fn



etc.

$$S^{pert}(\ell^+,\ell^-) = \left\{\delta(\ell^+) + \frac{\alpha_s C_F}{4\pi}\left[\#\,\delta(\ell^+) + 0\left[\frac{\mu}{\ell^+}\theta(\ell^+)\right] + \#\left[\frac{\mu}{\ell^+}\ln\!\left(\frac{\ell^+}{\mu}\right)\right]_+\right]\right.$$

$$\left.\times\left\{\delta(\ell^-) + \frac{\alpha_s C_F}{4\pi}\left[\text{ditto } \ell^+\to\ell^-\right]\right\}\right.$$

C renormalizes multiplicatively $\quad C^{bare} = Z_C\,C = C + (Z_C-1)\,C$

$$\mu\, d/d\mu\; C(Q,\mu) = \gamma_C(Q,\mu)\, C(Q,\mu)$$

J, S renormalize like PDF, with convolutions

eg. $\quad J_n^{bare}(s) = \int ds'\; Z_J(s-s')\; J_n(s',\mu)$

$$\mu\, d/d\mu\; J_n(s,\mu) = \int ds'\; \gamma_J(s-s')\; J_n(s',\mu)$$

$\uparrow$ invariant mass evolution

Coefficient Renormalization $=$ $\left(\begin{matrix}\text{Operator}\\ \text{Renormalization}\end{matrix}\right)^{-1}$ "consistency conditions"

$$|Z_C|^2\,\delta(s)\,\delta(\bar s) = \int ds'\,d\bar s'\; Z_J^{-1}(s-s')\,Z_J^{-1}(\bar s-\bar s')\,Z_S^{-1}\!\left(\frac{s'}{Q},\frac{\bar s'}{Q}\right)$$

$\boxed{\text{RGE}}$

$$\gamma_J(s,\mu) = -2\,\Gamma^{cusp}[\alpha_s]\,\frac{1}{\mu^2}\left[\frac{\mu^2\theta(s)}{s}\right]_+ + \gamma[\alpha_s]\,\delta(s)$$

all order structure

(γ_S similar, two variables factorize)

Fourier Transform $\quad y = y - i0$

$$\gamma_J(y) = \int ds\; e^{-isy}\,\gamma_J(s)$$

$$J(y) = \int ds\; e^{-isy}\,J(s) \qquad\qquad \mu\,d/d\mu\; J(y,\mu) = \gamma_J(y,\mu)\,J(y,\mu)$$

<u>simple</u>

$$\gamma_J(y,\mu) = 2\,\Gamma^{cusp}[\alpha_s]\,\ln\!\left(iy\mu^2e^{\gamma_E}\right) + \gamma[\alpha_s]$$

$$\left[\frac{\ln^k(s/\mu)}{s}\right]_+ \longleftrightarrow \ln^{k+1}\!\left(iy\mu^2e^{\gamma_E}\right)$$

$$d\ln\mu = \frac{d\alpha_s}{\beta[\alpha_s]}\;,\qquad \ln\mu/\mu_0 = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\beta[\alpha_s]}$$

All orders Solution

$$\ln\left[\frac{J(s,\mu)}{J(s,\mu_0)}\right] = \omega(\mu,\mu_0)\,\ln\!\left(iy\mu_0^2e^{\gamma_E}\right) + K(\mu,\mu_0)$$

↖ same structure for H, J, S

$$\omega = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s\;\frac{2\,\Gamma^{cusp}[\alpha_s]}{\beta[\alpha_s]}$$

$$K = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\beta[\alpha_s]}\,\gamma[\alpha_s] + 2\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha}{\beta[\alpha]}\,2\,\Gamma^{cusp}[\alpha]\int_{\alpha_s(\mu_0)}^{\alpha}\frac{d\alpha'}{\beta[\alpha']}$$

determine ω, K order by order

$y \leftrightarrow \tau$

$$\ln \frac{d\sigma}{dy} = \underset{LL}{(\ln y)(\alpha_s \ln)^k} + \underset{NLL}{(\alpha_s \ln)^k} + \underset{NNLL}{\alpha_s (\alpha_s \ln)^k} + \cdots$$

Momentum Space Answer with resummation

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = H(Q,\mu_Q)\, U_H(Q,\mu_Q,\mu_J)\, J_T(Q^2\tau - s') \otimes U_J(s'-Q\ell,\mu_S,\mu_J) \otimes S_T^{pert}(\ell-\ell',\mu_S) \otimes F(\ell')$$

where $\mu_Q \sim Q, \quad \mu_J \sim Q\sqrt{\tau}, \quad \mu_S \sim Q\tau$

$$U_J(s,\mu,\mu_0) = \frac{e^K (e^{\gamma_E})^\omega}{\mu_0^2\, \Gamma(-\omega)} \left[\frac{(\mu_0^2)^{1+\omega}\, \theta(s)}{s^{1+\omega}}\right]_+$$

boundary at ∞ rather than 1

Consistency says $\quad \gamma_J[\alpha_s] + \gamma_S[\alpha_s] = -\frac{1}{2}\gamma_H[\alpha_s]$

Final Example: Drell-Yan $pp \to X\,\ell^+\ell^-$

- prototype LHC process (pp in, measure leptons, replace $\ell^+\ell^-$ by jets, ..., etc)

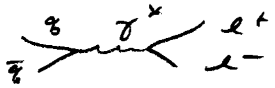
Kinematics

$$p\ p \to X\,(\underbrace{\ell^+\ell^-})$$

$$p_A + p_B = p_X + q$$

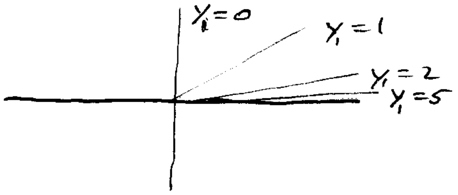
$E_{cm}^2 = (p_A+p_B)^2$ collision energy

q^2 hard scale of partonic collision



$\tau \equiv q^2/E_{cm}^2 \le 1$

$Y = \frac{1}{2}\ln\left(\frac{p_b\cdot q}{p_a\cdot q}\right)$ total lepton rapidity (angular variable)



$$\left.\begin{aligned} x_a &\equiv \sqrt{\tau}\,e^{Y} \\ x_b &= \sqrt{\tau}\,e^{-Y} \end{aligned}\right\}$$ analogs of Bjorken var in DIS

$$\tau \le x_{a,b} \le 1$$

parton fractions ($\xi_i = x_i$ tree level)

$$x_a \le \xi_a \le 1$$
$$x_b \le \xi_b \le 1$$

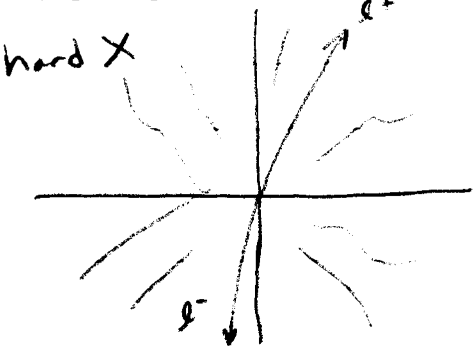
$$p_X^2 \le E_{cm}^2\,(1-\sqrt{\tau})^2$$

Cases:

Cases:	Inclusive	$\tau \sim 1$	$p_X^2 \sim q^2 \sim E_{cm}^2$	$x_{a,b} \sim 1$	$\xi_{a,b} \sim 1$
	Endpoint	$\tau \to 1$	$p_X^2 \ll q^2 \to E_{cm}^2$ ↑ usoft	$x_{a,b} \to 1$	$\xi_{a,b} \to 1$
	(Small x	$\tau \to 0$	take $\xi_a, \xi_b \to 0$)		
	"Isolated"				

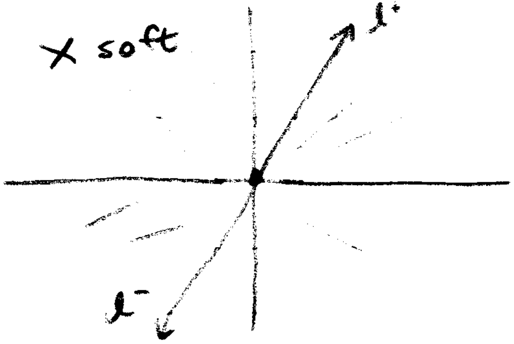
Inclusive

hard X

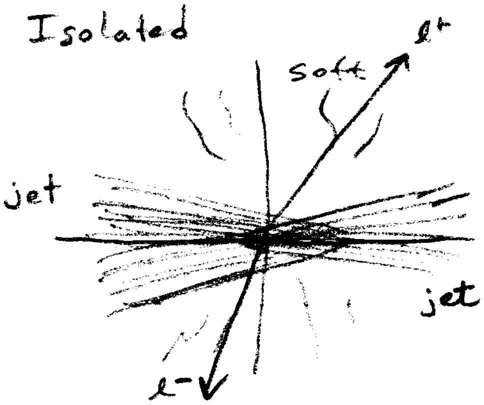


Endpoint

X soft

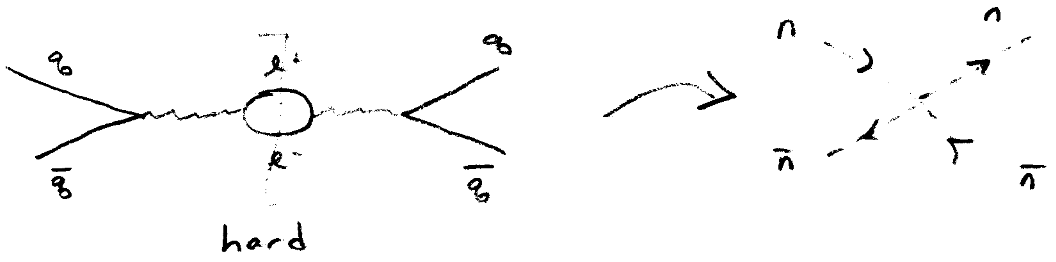


Isolated



Inclusive $p_n\, p_{\bar n} \to X_{hard}\ (\ell^+\ell^-)$

Factorization: SCET$_I$ problem (hard-collinear Factorization)



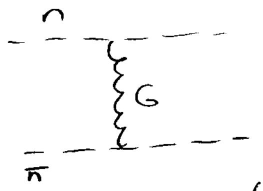
4-quark operator in SCET, which after Fierzing is

$$\Big[(\bar\xi_n W_n)\frac{\bar{n}\!\!\!/}{2}(W_n^\dagger \xi_n)\Big]\Big[(\bar\xi_{\bar n} W_{\bar n})\frac{n\!\!\!/}{2}(W_{\bar n}^\dagger \xi_{\bar n})\Big]$$

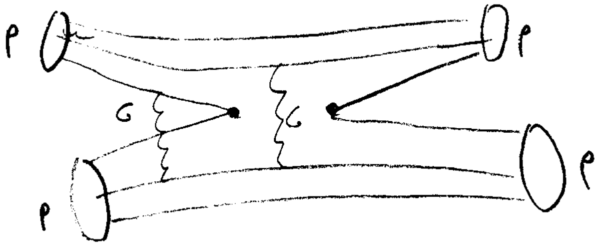
- $T^A \otimes T^A$ octet structure vanishes under $\langle p_n|\cdots|p_n\rangle$
- $\xi_n \to Y_n \xi_n$, $\xi_{\bar n} \to Y_{\bar n}\xi_{\bar n}$ etc, no coupling to soft gluons, they cancel out
- $\langle p_n|\bar\chi_{n,\omega}\frac{\bar{n}\!\!\!/}{2}\chi_{n,\omega'}|p_n\rangle$ give PDF
  $\langle p_{\bar n}|\bar\chi_{\bar n,\bar\omega}\frac{n\!\!\!/}{2}\chi_{\bar n,\bar\omega'}|p_{\bar n}\rangle$ " "

$$\frac{1}{\sigma_0}\frac{d\sigma}{dq^2\,dY} = \sum_{i,j}\int_{x_a}^{1}\frac{d\xi_a}{\xi_a}\int_{x_b}^{1}\frac{d\xi_b}{\xi_b}\, H_{ij}^{incl}\Big(\frac{x_a}{\xi_a},\frac{x_b}{\xi_b},q^2,\mu\Big)\, f_i(\xi_a,\mu)\, f_j(\xi_b,\mu)$$
$$* \Big[1 + \mathcal{O}\Big(\frac{\Lambda_{QCD}}{\sqrt{q^2}}\Big)\Big]$$

- One more (important) caveat, "Glauber Gluons"



$$p_G^\mu \sim (\overset{+}{\lambda^2}, \overset{-}{\lambda^2}, \overset{\perp}{\lambda})$$



These gluons cancel out at Leading Order (Proving this would take us too far afield)

Threshold Limit only certain terms in H_{ij}^{incl} contribute (most singular in $1-\tau$)

$$H_{ij}^{incl} \rightarrow S_{q\bar{q}}^{thr}\left[\sqrt{q^2}\left(1-\frac{\tau}{\xi_a\xi_b}\right),\mu\right] H_{ij}(q^2,\mu) \left[1+\mathcal{O}(1-\tau)^0\right]$$

& $ij = u\bar{u}, d\bar{d}, \ldots$ quarks
no glue

$\xi_{a,b} \rightarrow 1$ so one parton in each proton carries all the momentum, not the dominant LHC region

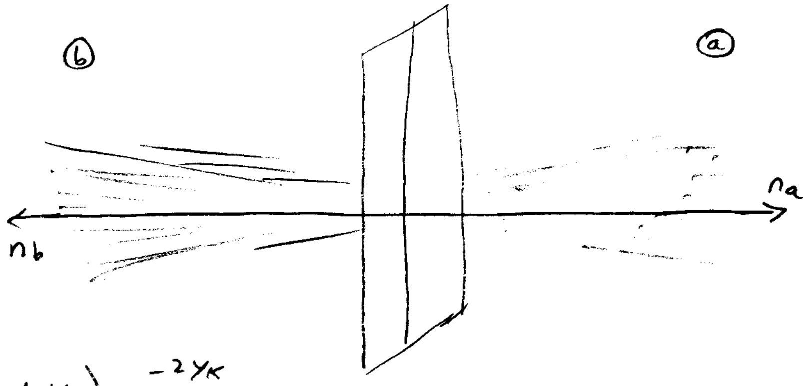
Isolated DY - allow forward jets to carry away part of Ecm, so $\xi_{a,b} \not\rightarrow 1$

- restrict central region to still only have soft radiation (signal region is bkgnd free, no jets, ie jet veto)

need to <u>observe</u> something to guarantee this.

<u>Observable</u> $P_X = B_a + B_b$

- two hemispheres, ⊥ to the beam axis



$$B_a^+ = n_a \cdot B_a = \sum_{k\in a} n_a \cdot p_k$$
$$= \sum_{k\in a} E_k\,(1+\tanh y_k)\, e^{-2y_k}$$

plus momenta for n-collinear radiation should be small

Take $B_a^+ \leq Q\, e^{-2y_{cut}} \ll Q$ $\qquad Q = \sqrt{q^2}$

$B_b^+ \equiv n_b \cdot B_b \leq$ " $\ll Q$

does the trick
(inclusive variable for jet veto)

n-collinear : proton ⓐ <u>and</u> jet ⓐ

we do not simply get a PDF from the hard-collinear-soft factorization

[Glauber's again cancel]

$$\frac{1}{\sigma_0}\frac{d\sigma}{dq^2\, dY\, dB_a^+ dB_b^+} = \sum_{ij} H_{ij}(q^2,\mu) \int dk_a^+ dk_b^+ \; Q^2\, B_i[\omega_a(B_a^+ - k_a^+), x_a, \mu]$$

$$* \; B_j[\omega_b(B_b^+ - k_b^+), x_b, \mu]$$

$$* \; S_{ihemi}(k_a^+, k_b^+, \mu)$$

$$* \left[1 + \mathcal{O}\left(\frac{\Lambda_{QCD}}{Q} , \frac{\sqrt{B_{a,b}^+ \omega_{a,b}}}{Q} \right) \right]$$

where $\omega_{a,b} = x_{a,b}\, E_{cm}$

B_i = "beam function"

$$B_q(\omega b^+, \omega/P^-, \mu) = \frac{\theta(\omega)}{\omega} \int \frac{dy^-}{4\pi}\, e^{ib^+ y^-/2} \langle p_n(P^-) |\, \bar{\chi}_n(y^- \tfrac{n}{2})\, \delta(\omega - \bar{P}) \tfrac{\bar{n}\!\!\!/}{2}\, \chi_n(0)\, | p_n(P^-) \rangle$$

recall jet fn $\langle 0 |\, \bar{\chi}_{n,\omega}(y^- \tfrac{n}{2})\, \tfrac{\bar{n}\!\!\!/}{2}\, \chi_n(0) \,| 0 \rangle$

PDF $\langle p |\, \bar{\chi}_{n,\omega}(0)\, \tfrac{\bar{n}\!\!\!/}{2}\, \chi_n(0)\, | p \rangle$

beam function is mix of both

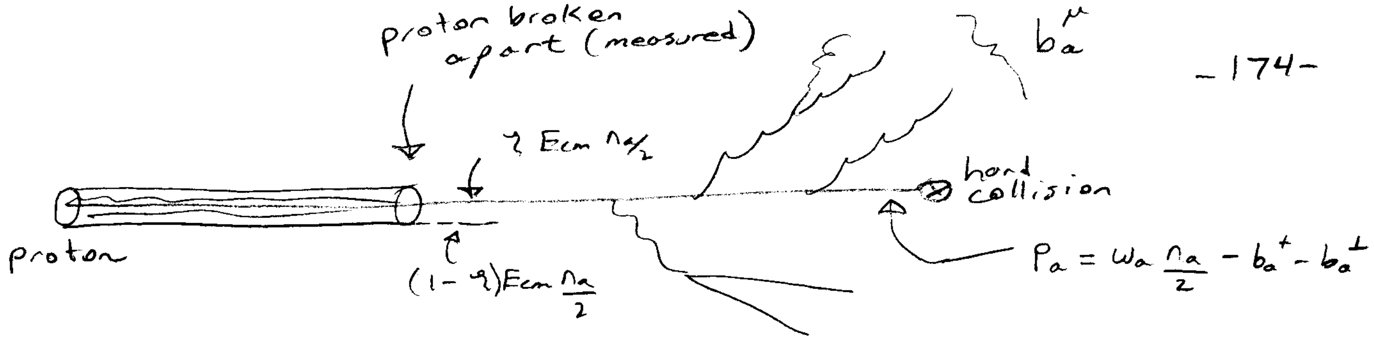
proton = $SCET_{II}$ collinear

jet = $SCET_I$ collinear (B_q is in $SCET_I$)

Match $SCET_I \to SCET_{II}$:

$$B_i(t, x, \mu) = \sum_j \int_x^1 \frac{d\xi}{\xi}\, \mathcal{I}_{ij}\left(t, \frac{x}{\xi}, \mu\right) f_j(\xi, \mu) \left[1 + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{t}\right) \right]$$

$\uparrow$ f_g & f_q contribute to B_q (B_g)



$$b_a^\mu = (\eta - x) E_{cm} \frac{n_a}{2} + b_a^+ \frac{\bar{n}_a}{2} + b_{a\perp}$$

$$P_a^2 = -\underbrace{\omega_a b_a^+}_{t_a \gg \Lambda_{QCD}} - \vec{b}_{\perp a}^{\,2} \le 0$$ spacelike active parton participates in hard collision

Tree-Level

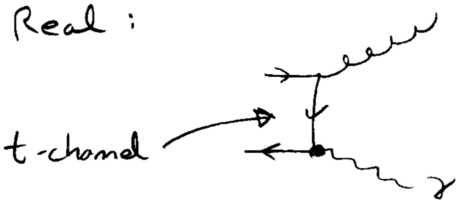
$$B_i(t,x,\mu) = \delta(t) \, f_i(x,\mu)$$

Order α_s

Real & Virtual Contractions

QCD

Real:



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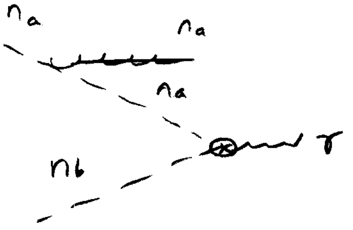
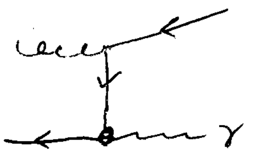


$\mathcal{I}_{qq}^{(\alpha_s)}$



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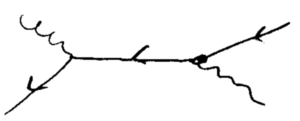
$\mathcal{I}_{qg}^{(\alpha_s)}$



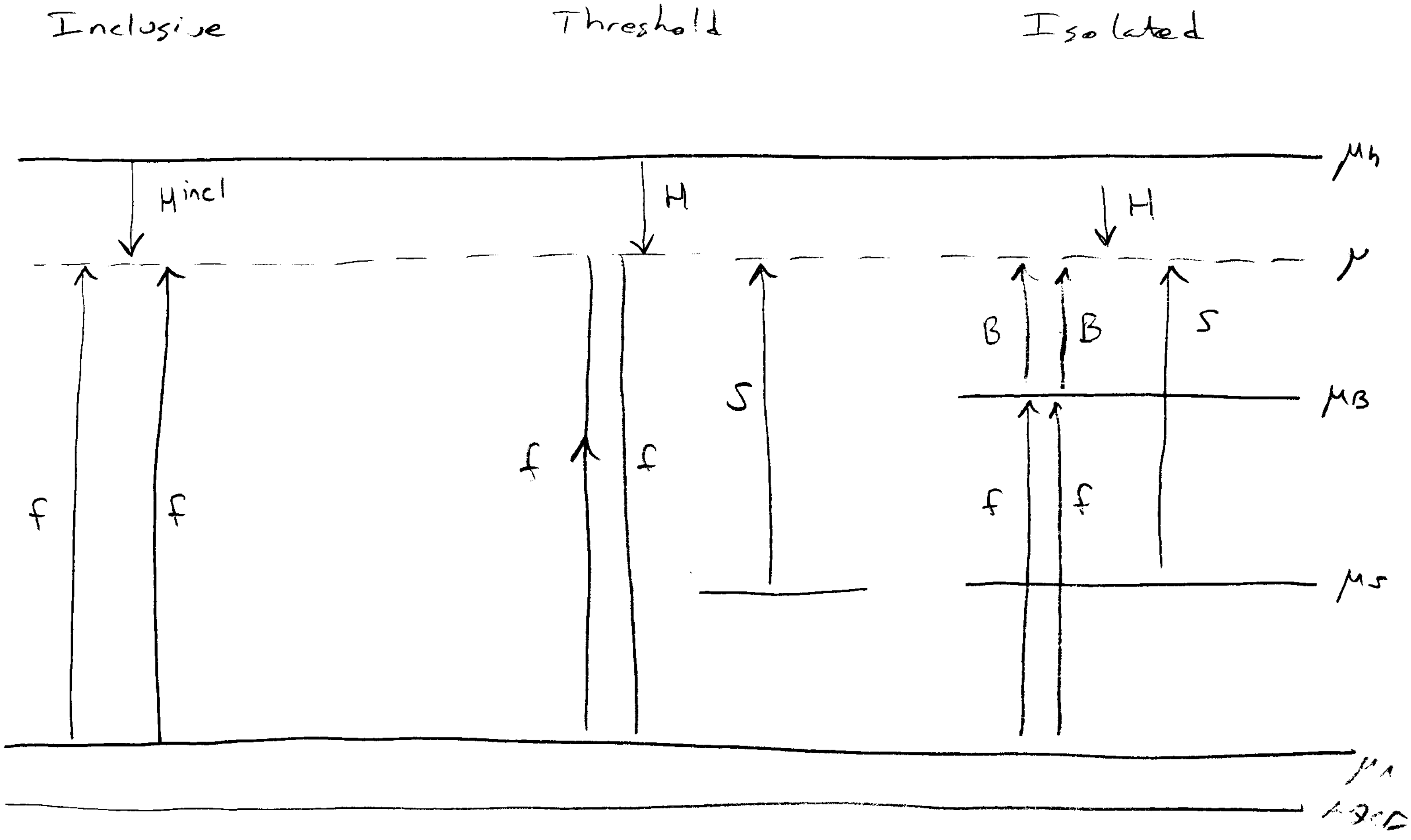
power correction $\sim \frac{t}{s} \sim \frac{\omega B_a^+}{Q^2}$

(would be ~ 1 for inclusive)

$\boxed{\text{RGE}}$ $$\mu \, d/d\mu \; B_i(t, x, \mu) = \int dt' \, \gamma_i(t-t', \mu) \, B_i(t', x, \mu)$$

like the jet function
(invariant mass evolution)

- sums $\ln^2(t/\mu)$
- indep of x & no mixing



consistency of
RGE for isolated core requires B's since
H and S have double logs, but f's do not