Soft - Collinear Effective Theory

- EFT treatment of Soft & Collinear IR physics for hard collisions in QCD (or decays with large E released)

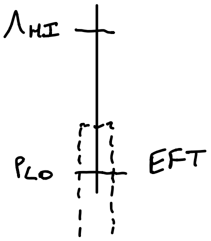
$\Rightarrow$ jets, energetic hadrons, soft partons / hadrons

eg. $e^+e^- \to$ 2-jets, $e^- p \to e^- X$ (DIS), $e^- p \to e^- h X$, $pp \to H + $1-jet, $B \to \pi\pi$, jet substructure, ... [many many more]

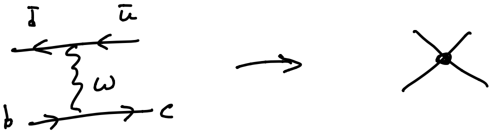
Concepts: Factorization, Wilson Lines, Sum Sudakov Double Logs
Power Corrections, ...

First, Review key EFT Concepts

<u>Decoupling</u> Effects from heavy or <u>offshell</u> particles are suppressed / decouple $\quad p_{LO} \ll \Lambda_{HI}$

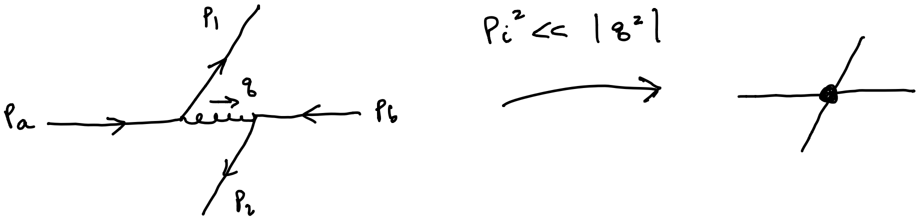


eg



$$\sim \frac{(\bar\psi\psi)(\bar\psi\psi)}{M_W^2} \qquad M_W^2 \gg p_i^2$$

eg.



$$\sim \frac{1}{q^2} (\bar\psi \cdot\cdot \psi)(\bar\psi \cdot\cdot \psi)$$

say $p_i^2 = 0$ on-shell, $\quad q = p_a - p_1 = n_a E_a - n_1 E_1$

$n_a = (1, \hat{z}) \qquad \bar{n}_a = (1, -\hat{z}) \qquad q^2 = -2E_aE_1\, n_a \cdot n_1$

$n_1 = (1, \hat{n}) \qquad \bar{n}_1 = (1, -\hat{n}_1) \qquad\quad = -2E_aE_1\,(1 - \hat{z}\cdot\hat{n})$

large if energies big & deflection angles large

$\boxed{q^2 \sim Q^2 \quad \text{"hard"}}$

Construct $\mathcal{L}_{eff}$

- degrees of freedom ? low energy / nearly onshell modes
  → what fields
- symmetries → constrain interactions / operators
  [Lorentz, Gauge theory, Global, ...]
- expansions, leading order description
  → power counting

$$\mathcal{L}_{EFT} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \dots$$

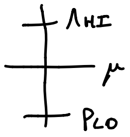
∞ # operators, but only specific subset needed at given order

[often in mass dimension of operators, but not in SCET]

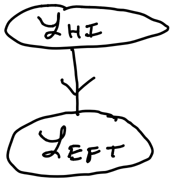
Matching

$$\mathcal{L}_{EFT}^{(k)} = \sum_i \underset{\substack{\uparrow \\ \text{short. dist.} \\ \text{(offshell)}}}{C_i(\mu)} \; \underset{\substack{\uparrow \\ \text{long dist.} \\ (\sim \text{on-shell})}}{O_i^{(k)}(\mu)}$$

- $\mathcal{L}_{HI}$ & $\mathcal{L}_{EFT}$ have same IR, differ in UV
- $C_i(\mu)$ does not depend on IR scales (masses in EFT, Λ_{QCD}, IR regulators, ...)

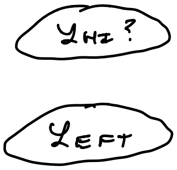


splits HI from LO



"top-down EFT" if we know $\mathcal{L}_{HI}(\Lambda_{HI}, P_{LO})$ we can (perturbatively) construct $\mathcal{L}_{EFT}$.

Calculate C, construct O [H_{weak}, HQET, NRQCD, SCET, ...]



"bottom-up EFT" form $\sum_i C_i O_i$ complete basis

exploit symmetries

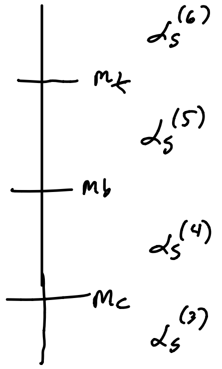
[eg. SM as EFT, Chiral Lagrangians ...]

Renormalization

- parameters g, C in QFT must be defined by a renormalization scheme, also $\mathcal{O}$ ($\overline{MS}$, Wilsonian Cutoff, ...)
- scheme depend on cutoff / renormalization scale "μ" $g(\mu)$, $C(\mu)$ → See QCD lectures by J. Qiu

eg. $\alpha_s^{(n_f)}(\mu)$ in QCD

$$\mu \frac{d}{d\mu} \alpha_s^{(n_f)}(\mu) = -\frac{\beta_0^{(n_f)}}{2\pi} \left[\alpha_s^{(n_f)}(\mu)\right]^2 + \dots$$

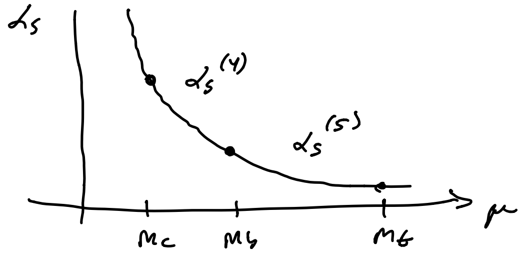


$$\beta_0^{(n_f)} = 11 - \frac{2}{3} n_f$$

Hmwk for anyone unfamiliar with this

Renormalization Group

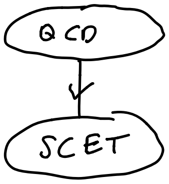
- sums logs between mass scales $\alpha_s^k \ln^k(m_b/m_t)$



[Typically]

- Power counting handles powers $\dfrac{p_{LO}}{\Lambda_{HI}} \ll 1$
- Renormalization group handles logs $\ln\left(\dfrac{p_{LO}}{\Lambda_{HI}}\right)$ which may be large $\alpha_s \ln(\dots) \sim 1$

SCET

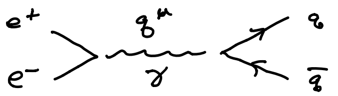


few 100 MeV

hard collision

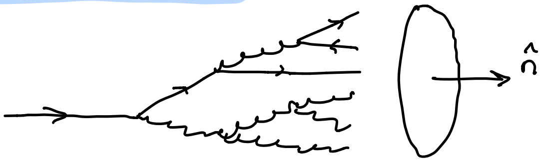
$$\Lambda_{QCD}^2, p_{IR}^2 \ll Q^2$$

degrees of freedom consider $e^+e^- \to 2$ jets



$$Q^2 = q^2$$

Jets / collinear



due to collinear (& soft) enhancements in QCD

- collimated radiation in direction $\hat n$
- $E_{Jet} \sim Q$

Let $n^\mu = (1, \hat n)$

$\bar n^\mu = (1, -\hat n)$

$n^2 = \bar n^2 = 0$, $n\cdot\bar n = 2$

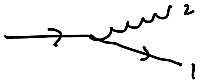
$$p^\mu = \underbrace{\bar n\cdot p}_{p^-}\,\frac{n^\mu}{2} + \underbrace{n\cdot p}_{p^+}\,\frac{\bar n^\mu}{2} + p_\perp^\mu$$

$$p^2 = n\cdot p\,\bar n\cdot p + \underbrace{p_\perp^2}_{-\vec p_\perp^{\,2}}$$

Collinear ?

1 massless particle : $p^\mu = \bar n\cdot p\,\frac{n^\mu}{2}$, $\bar n\cdot p \sim Q$

2 massless :



$$p_i^\mu = \bar n\cdot p_i\,\frac{n^\mu}{2} + p_{i\perp}^\mu + n\cdot p_i\,\frac{\bar n^\mu}{2}$$

$i = 1, 2$

$\bar n\cdot p_i \sim Q$ large

$p_{i\perp}^\mu \ll Q$ collimated

say $p_{i\perp} \sim \lambda Q$ $\boxed{\lambda \ll 1}$

dimensionless power counting parameter

on-shell $n\cdot p_i = -\frac{p_{i\perp}^2}{\bar n\cdot p_i}$ > $n\cdot p_i \sim \lambda^2 Q$ nearly on-shell

n particles : same

$\boxed{\text{n-Collinear}}$: $p^\mu \sim Q(\overset{n\cdot p}{\lambda^2}, \overset{\bar n\cdot p}{1}, \overset{\perp}{\lambda})$

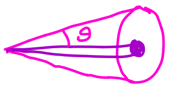
Collinear Fields : quark ξ_n

gluon A_n^μ

- energetic hadron : $p_\perp \sim \Lambda_{QCD}$ ⇒ $\lambda \sim \frac{\Lambda_{QCD}}{Q}$

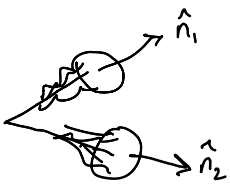
energetic quarks & gluons confine into single hadron

- jet of hadrons : $1 \gg \lambda \gg \frac{\Lambda_{QCD}}{Q}$



$\theta \sim \lambda$

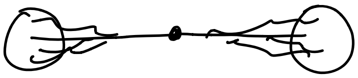
2-jets



$n_1, \bar n_1$ $\xi_{n_1}, A_{n_1}^\mu$

$n_2, \bar n_2$ $\xi_{n_2}, A_{n_2}^\mu$

back-to-back jets: $n_2 = \bar{n} = (1, -\hat{n})$ $\bar{n}_2 = n$



$n_1 = n = (1, \hat{n})$

$\bar{n}_1 = \bar{n}$

$$\begin{array}{ll} & (+,-,\perp) \\ \hline \xi_n,\ A_n & (\lambda^2, 1, \lambda) \\ \xi_{\bar{n}},\ A_{\bar{n}} & (1, \lambda^2, \lambda) \end{array}$$

$\uparrow L1$

$\downarrow L2$

Soft $p_s^\mu \sim Q\lambda^\alpha$ all components small & homogeneous

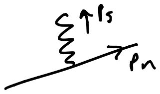
soft + soft = soft

soft + hard = hard

collinear + hard = hard

n_1-collinear + n_2-collinear = hard $\leftarrow$ hard interaction produces jets

collinear + soft ?



$$(p_n + p_s)^2 = 2p_n \cdot p_s = \underset{\lambda^0 * \lambda^\alpha}{\bar{n}\cdot p_n\ n\cdot p_s} + \overset{\text{suppressed}}{\cdots} \sim Q^2\lambda^\alpha$$

Value of α depends on what we measure

eg1   Mass in (large enough) region a, $M_a^2 = \left(\sum_{i\in a} p_i^\mu\right)^2$

[mass of R = 1 jet, hemisphere mass, ...]

demand $M_a^2 \sim Q^2\lambda^2 \ll Q^2$ [collimated jet has $E_J \gg M_J$]

Collinear + collinear $(p_n + p_n')^2 = 2\,\underset{\substack{+\\-\\\perp}}{p_n}\cdot\underset{\substack{-\\+\\\perp}}{p_n'} \sim Q^2\lambda^2$ contributes

collinear + soft $(p_n + p_s)^2 \sim Q^2\lambda^\alpha$

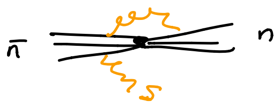
$\therefore$ $\alpha = 2$ to contribute   "ultrasoft"

eg2   Transverse Momenta, broadening $B_\perp = \sum_{i\in a} \underset{\sim\lambda}{|\vec{p}_i^{\,\perp}|} \ll Q$

Σ collinear ✓

soft $\Rightarrow$ $\alpha = 1$ "soft"

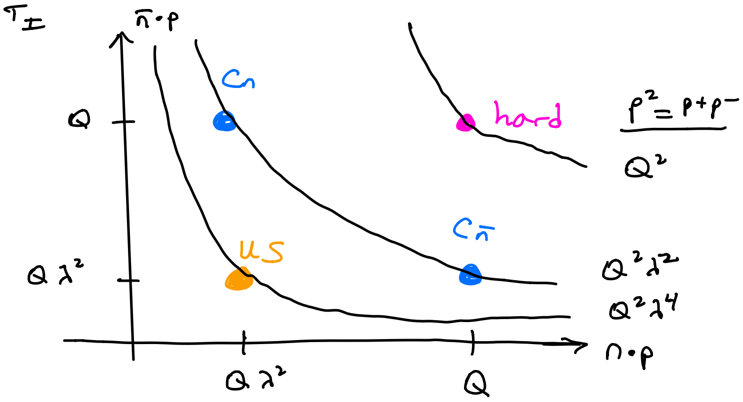
DOF Picture $e^+e^- \to 2$ jets (cm frame)



[virtual too]

① $SCET_I$

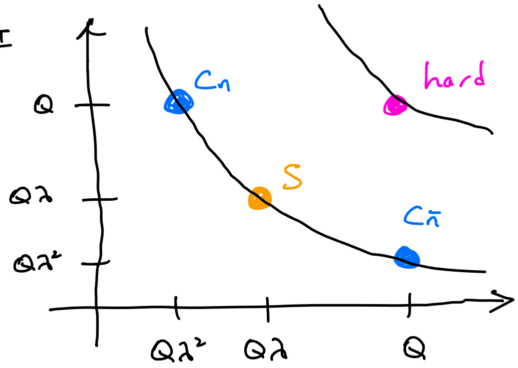
$\alpha = 2$



- modes cover regions of momentum space, extend into IR

② $SCET_{II}$

$\alpha = 1$



- power counting requires multiple fields for same particle
- relative scaling of modes is important
  [boost invariant, unlike absolute scaling]
- modes not classified by p^2 alone

Study $SCET_I$, come back to $SCET_{II}$

Field Power Counting

Use free kinetic term

ξ_n propagator

$$p^2 = n\cdot p \; \bar n \cdot p + p_\perp^2$$

$$\lambda^2 \times \lambda^0 + (\lambda)^2 \quad \text{same size}$$

$$\frac{i\slashed{p}}{p^2+i0} = \frac{i\slashed{n}}{2}\frac{\bar n\cdot p}{p^2+i0} + \dots = \frac{i\slashed{n}}{2}\frac{1}{n\cdot p + \frac{p_\perp^2}{\bar n \cdot p} + i0\,\mathrm{sign}(\bar n\cdot p)} + \dots$$

must have

$$\underbrace{\int d^4x}_{\lambda^{-4}} \; \underbrace{e^{-ip\cdot x}}_{\lambda^0} \; \langle 0|\, T\, \xi_n(x)\, \bar{\xi}_n(0)\, |0\rangle = \frac{i\not{n}}{2} \underbrace{\frac{\bar{n}\cdot p}{p^2+i0}}_{\lambda^{-2}} \qquad (*)$$

$(d^4p \sim \lambda^4)$

thus $\boxed{\xi_n \sim \lambda}$ [differs from $\frac{3}{2}$ mass dimension]

Note: $(*)$ implies $\not{n}\,\xi_n = 0$ since $\not{n}^2 = n^2 = 0$

take $\xi_n = \underbrace{\frac{\not{n}\bar{\not{n}}}{4}}_{\text{projection op.}}\psi$ for spin ["good components"]

spinors $u_n = \frac{\not{n}\bar{\not{n}}}{4}u(p)$

$$\sum_s u_n^s \bar{u}_n^s = \frac{\not{n}\bar{\not{n}}}{4}\sum_s u^s\bar{u}^s\,\frac{\bar{\not{n}}\not{n}}{4} = \frac{\not{n}}{2}\,\bar{n}\cdot p \quad \checkmark$$

$$u_+(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{p^-} \\ \cancel{\sqrt{p^+}e^{i\phi_p}} \\ \sqrt{p^-} \\ \cancel{\sqrt{p^+}e^{i\phi_p}} \end{pmatrix},$$

$\frac{\not{n}\bar{\not{n}}}{4}u(p)$ <u>kills</u> <u>small terms</u>

Dirac Rep: $\frac{\not{n}\bar{\not{n}}}{4} = \frac{1}{2}\begin{pmatrix} 1 & \sigma^3 \\ \sigma^3 & 1 \end{pmatrix}$

similar for $u_-(p)$ & antiquarks $v_+(p)$, $v_-(p)$

A_n^μ same propagator as QCD

$p^\mu \sim (\lambda^2, 1, \lambda) \sim i\partial_n^\mu$ $\qquad$ $iD_n^\mu = i\partial_n^\mu + g A_n^\mu$

Want $i\partial_n^\mu \sim A_n^\mu$ so $\boxed{A_n^\mu \sim (\overset{+}{\lambda^2}, \overset{-}{1}, \overset{\perp}{\lambda})}$ true in any gauge

[or derive from free propagator]

Soft Similar analysis

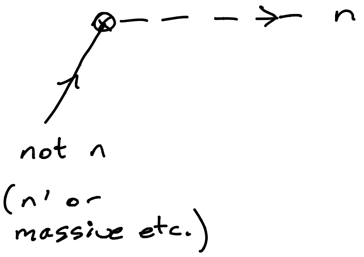
$p_s \sim \lambda^\alpha$

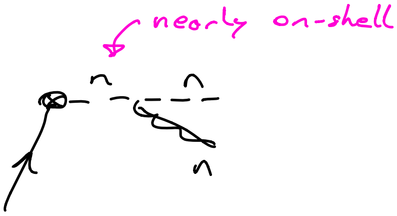
$$\boxed{\begin{aligned} A_s^\mu &\sim p_s^\mu \sim \lambda^\alpha \\ \psi_s &\sim \lambda^{3\alpha/2} \end{aligned}}$$

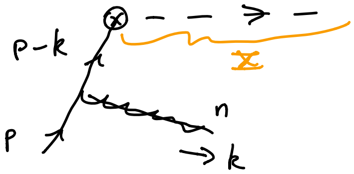
$$\underbrace{\int d^4x}_{\lambda^{-4\alpha}}\; \bar{q}_s\, \underbrace{i\not{\partial}}_{\lambda^\alpha}\, q_s$$

Collinear Wilson Lines

- $\bar n\cdot A_n \sim \lambda^0$?   no suppression for building operators







$$(p-k)^2 = \cancel{p^2} + \cancel{k^2} - 2p\cdot k = -\underbrace{\bar n\cdot k}_{\lambda^0}\ \underbrace{n\cdot q}_{\lambda^0 \text{ since not } n\text{-collinear}} + \cdots$$

$\therefore$ offshell

integrate it out

$$= \underline{\overline{X}}\ \frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i0}\ (igT^a \not\epsilon_n^a)\ u(p)$$

$\not\epsilon_n^a \leftarrow \frac{\not n}{2}\,\bar n\cdot\epsilon_n^a + \cdots$

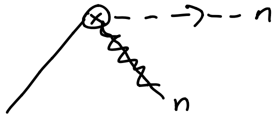
expand Homework

since $\not A = \underbrace{\bar n\cdot A}_{\lambda^0}\frac{\not n}{2} + \cdots$

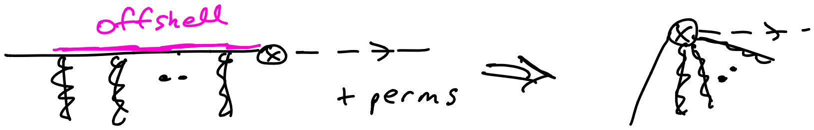
$$= \underline{\overline{X}}\ \frac{(-g)\ \bar n\cdot A_n^a}{-\bar n\cdot k + i0}\ T^a\ u(p)$$

- universal,
  independent of $p, m, \ldots$

$\rightarrow$



keep going



Gives Wilson line

$$W_n(y,-\infty) = P\exp\left(ig\int_{-\infty}^{0} ds\ \bar n\cdot A_n(s\bar n + y)\right)$$

More Hmwk

$\boxed{W_n \sim \lambda^0}$

SCET operator $(\bar{\xi}_n W_n)\ (\Gamma \psi)$

generic, operator "building block"

- parton fields
- jet fields

quark $\underline{\chi_n \equiv W_n^\dagger \xi_n}$

gluon $\underline{\mathcal{B}_{n\perp}^\mu \equiv \frac{1}{g}\left[W_n^\dagger i D_{n\perp}^\mu W_n\right]} = \left[\frac{1}{g\, i\bar{n}\cdot\partial_n} W_n^\dagger \underline{[i\bar{n}\cdot D_n, iD_\perp^\mu]} W_n\right]$

field strength + adjoint Wilson line

$$= A_{n\perp}^\mu - \frac{k_\perp^\mu}{\bar{n}\cdot k}\,\bar{n}\cdot A_n + \cdots$$

[vanishes if $A^\mu \to k^\mu$, g. inv.]

Gauge Symmetry symmetry trnsfm. must leave us <u>within</u> the EFT

$U(x) = e^{i\alpha^A(x) T^A}$ $\quad i\partial^\mu U_n(x) \sim P_n^\mu U_n(x)$ collinear

$\quad i\partial^\mu U_{us}(x) \sim P_{us}^\mu U_{us}(x)$ ultrasoft

- $\xi_n \to U_n \xi_n$ $\quad iD_n^\mu \to U_n\, iD_n^\mu\, U_n^\dagger$ for A_n

  $q_{us} \to q_{us}$ $\quad$ [else not ultrasoft], $W_n \to U_n W_n$

- $q_{us} \to U_{us} q_{us}$ $\quad iD_{us} \to \ldots$

  $\xi_n \to U_{us}\xi_n$ $\quad A_n^\mu \to U_{us} A_n^\mu U_{us}^\dagger$, $W_n \to U_{us} W_n U_{us}^\dagger$

$\chi_n = W_n^\dagger \xi_n \to W_n^\dagger \cancel{U_n^\dagger} \cancel{U_n} \xi_n$ $\quad$ protected by g. inv.
eg. stays together when we add loop corrections

build operators out of <u>n-collinear</u> <u>gauge invariant</u> building blocks χ_n, $\mathcal{B}_{n\perp}^\mu$

Wilson lines needed to ensure gauge invariance in presence of operators where gluons that only couple in on-shell manner to single colored field.

trades $\bar{n}\cdot A_n \rightarrow W_n$

$W_n^\dagger W_n = \mathbb{1} = W_n W_n^\dagger$

collinear gauge singlet

$[i\bar{n}\cdot D_n W_n] = 0$ $\quad\therefore\quad$ $i\bar{n}\cdot D_n W_n \Phi = W_n\, i\bar{n}\cdot\partial_n \Phi$

$W_n^\dagger\, i\bar{n}\cdot D_n W_n = i\bar{n}\cdot\partial_n$ as operator

$i\bar{n}\cdot D_n = W_n\, i\bar{n}\cdot\partial_n W_n^\dagger$

Hard - Collinear Factorization

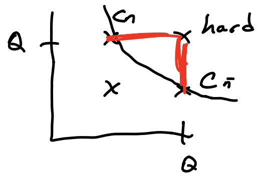
$\mathcal{L}^{\text{hard}} = C \otimes \mathcal{O}$

What do Wilson Coefficients depend on?

$i\bar{n}\cdot\partial_n \sim \lambda^0$

Allows $\underbrace{C(i\bar{n}\cdot\partial_n)}_{\text{gauge inv.}}\chi_n = \int d\omega\, C(\omega)\, \underbrace{\delta(\omega - i\bar{n}\cdot\partial_n)\chi_n}_{\text{operator} \equiv \chi_{n,\omega}}$

Hard & collinear modes communicate through $\sim\lambda^0$ momenta



constrained by gauge inv.

& momentum conservation

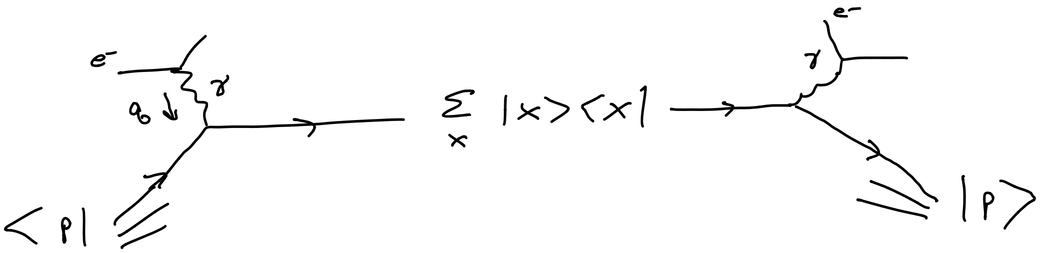
↑L2

↓L3

DIS

$e^- p \rightarrow e^- X$ Inclusive Factorization

[full analysis requires more knowledge, eg $\mathcal{L}$, cover few key parts]



$q = (0,0,0,Q) = \frac{Q}{2}(\bar{n}-n)$ $\qquad$ $q^2 = -Q^2$ spacelike

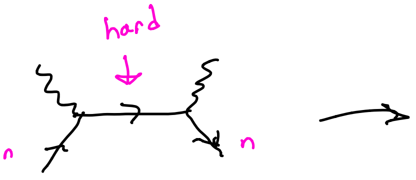
Bjorken $x = \dfrac{Q^2}{2p\cdot q}$ $\qquad$ Breit frame, where proton is n-collinear

$p_X = p + q$ = hard

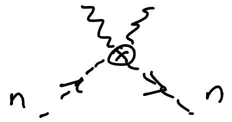
Proton $p_p^\mu = \frac{n^\mu}{2}\,\bar n\cdot p_p + \frac{\bar n^\mu}{2}\underbrace{\frac{M_p^2}{\bar n\cdot p_p}}_{\text{small}}$, big $\bar n\cdot p_p = \frac{Q}{x} \sim \lambda^0$

$\lambda = \frac{\Lambda_{QCD}}{Q} \ll 1$

QCD



SCET



$O_q = \underset{\lambda}{\bar\chi_n}\,\frac{\bar{n}\!\!\!/}{2}\,\underset{\lambda}{\chi_n}$

$O \sim \lambda^2$ twist-2

Add arbitrary pert. α_s^k corrections:

also gluon $O_g = \mathcal{B}_{n\perp}^\mu\,\mathcal{B}_{n\perp\mu}$

$$\mathcal{L}_{hard} = \int d\omega\,\cancel{d\omega'}\;C(\omega,\cancel{\omega'},Q)\;\bar\chi_n\frac{\bar{n}\!\!\!/}{2}\,\cancel{\delta(\omega' + i\bar n\cdot\partial_n)}\,\delta(\omega - i\bar n\cdot\partial_n)\,\chi_n$$

forward $\langle p|\cdots|p\rangle$ matrix element fixes $\omega=\omega'$

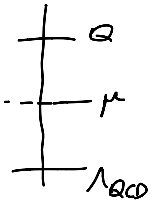
$$\sigma \sim \int d\omega\;\mathrm{Im}\,C(\omega,Q)\,\langle p|\,\bar\chi_n\frac{\bar{n}\!\!\!/}{2}\,\delta(\omega - i\bar n\cdot\partial_n)\chi_n\,|p\rangle$$

↑ momentum of quark in proton

both dimensionless

$$\sim \int \frac{d\xi}{\xi}\;\underbrace{H\Big(\frac{x}{\xi},\frac{Q}{\mu},\alpha_s(\mu)\Big)}_{\text{Hard}}\;\underbrace{f_{q/p}\Big(\xi,\frac{\mu}{\Lambda_{QCD}}\Big)}_{\text{Collinear}}\ ,\qquad \xi = \frac{\omega}{\bar n\cdot p}$$

parton dist'n



$\frac{Q}{\omega} = \frac{Q}{\xi\,\bar n\cdot p} = \frac{x}{\xi}$

Factorization

More Hard Operators

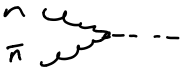
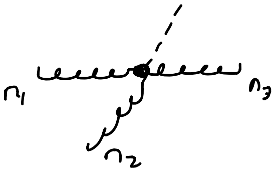
power counting, symmetry & matching calc imply O are built from χ_n

$\mathcal{B}_{n\perp}^\mu$

$\mathcal{P}_\perp^\mu$ } often suppressed

& Soft Fields

[Note: true at any order
other collinear ops
elliminated by operator identities
& eqtns. of motion.]

Example	Operators		
$e^+e^- \to 2$ jets	$\bar\chi_n \, \gamma_\perp^\mu \, \chi_{\bar n}$		Amplitude
$gg \to H$	$\mathcal{B}^\mu_{n\perp} \, \mathcal{B}_{\bar n \perp\mu} \, H$		Ampl.
[quark PDF	$\bar\chi_n \frac{\bar{n}\!\!\!/}{2} \delta(\omega - i\bar n\cdot\partial)\chi_n$		Ampl.^{2}]
gluon PDF	$\text{tr}\left[\mathcal{B}^\mu_{n\perp}\, \delta(\omega - i\bar n\cdot\partial)\mathcal{B}_{n\perp\mu}\right]$		Ampl^2
$pp \to H + 1$-jet	(remove top)	 	Ampl

- $\mathcal{B}^{a_1\mu_1}_{n_1\perp} \quad \mathcal{B}^{a_2\mu_2}_{n_2\perp} \quad \mathcal{B}^{a_3\mu_3}_{n_3\perp} \quad H \quad T_{\mu_1\mu_2\mu_3} \quad (if^{a_1a_2a_3})$
- $\mathcal{B}^{a\,\mu}_{n_1\perp} \quad \bar\chi^{\bar\alpha}_{n_2} \quad \chi^{\beta}_{n_3} \quad H \quad T^{\alpha\bar\beta}_{\mu_1}$

$\uparrow$ no $d^{a_1a_2a_3}$ by charge conjugation

---

how many operators ? Helicity Methods Skip this

Helicity basis: natural in SCET since we have direction to use $\hat n$

$$\mathcal{B}^a_{n\pm} \equiv - \epsilon^\mu_\mp(n,\bar n)\,\mathcal{B}^\perp_{n\mu} \,, \qquad \epsilon_\mp = \frac{1}{\sqrt2}(0,1,\pm i,0)$$

$$J^{\bar\alpha\beta}_{n_1n_2\pm} \propto \epsilon^\mu_\mp(n_1,n_2)\,\bar\chi^{\bar\alpha}_{n_1\pm}\,\gamma_\mu\,\underbrace{\chi^\beta_{n_2\pm}}_{\left(\frac{1\pm\gamma_5}{2}\right)\chi_n}$$

Allowed

$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	
+	+	+	
+	+	−	
−	−	+	} Wilson Coeff fixed by Parity
−	−	−	

$\mathcal{B}$	J	
+	+	
−	+	
+	−	} fixed by charge Conj.
−	−	

4 non-trivial coefficients

[note: no evenescent operators in leading power SCET due to helicity conservation]

Easy to exploit modern spinor-helicity results.

[see 1508.02397 for more on helicity operators in SCET.]

SCET $\mathcal{L}$ SCET$_{\rm I}$ $(\alpha=2)$

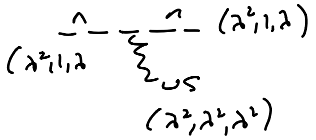
For interactions that are isolated and <u>purely</u> n-collinear or <u>purely</u> ultrasoft we just have full QCD $\mathcal{L}$ for each sector.

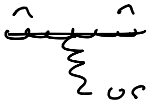
usoft : nothing to expand

n-collinear boost everything $\overset{+\ -\ \perp}{(\lambda^2,1,\lambda)} \to (\lambda,\lambda,\lambda)$ same

Key thing SCET describes is interactions between sectors

For $\mathcal{L}^{(0)}$

- 



usoft leave collinear ~ on-shell

- hard interactions produce collinear quarks with $\not n\,\xi_n = 0$
  [hard int. breaks boost arguement]

$$\psi = \left(\frac{\not n\bar{\not n}}{4} + \frac{\bar{\not n}\not n}{4}\right)\psi = \xi_n + \varphi_n$$

$$\mathcal{L}_{QCD} = \bar\psi i\not D\psi = \bar\xi_n \frac{\bar{\not n}}{2} in\cdot D\,\xi_n + \bar\varphi_n \frac{\not n}{2} i\bar n\cdot D\,\varphi_n + \bar\xi_n i\not D_\perp \varphi_n + \bar\varphi_n i\not D_\perp \xi_n$$

e.o.m. $\delta/\delta\bar\varphi_n \Rightarrow$ $\varphi_n = \frac{1}{i\bar n\cdot D}\, \underset{\lambda}{i\not D_\perp}\, \frac{\bar{\not n}}{2}\, \underset{\lambda}{\xi_n}$ smaller than ξ_n for hard production

$$\mathcal{L}_{QCD} = \bar\xi_n\left(in\cdot D + i\not D_\perp \frac{1}{i\bar n\cdot D} i\not D_\perp\right)\frac{\bar{\not n}}{2}\xi_n$$ <u>still</u> <u>QCD</u>

Expand

- couple only to ξ_n in path integral $J\xi_n$
- $in\cdot D = \underset{\lambda^2}{in\cdot\partial} + \underset{\lambda^2}{g\,n\cdot A_n} + \underset{\lambda^2}{g\,n\cdot A_s}$ multipole expansion

label comment

$iD_\perp = \underset{\lambda}{i\partial_{n\perp}} + \underset{\lambda}{g A_{n\perp}} + \dots$

$A^\perp_{us} \ll A_{n\perp}$

$i\partial^\perp_{us} \ll i\partial^\perp_n$

similarly

$i\bar n\cdot D = i\bar n\cdot D_n = i\bar n\cdot\partial_n + g\,\bar n\cdot A_n + \dots$

$\bar n\cdot A_{us} \ll \bar n\cdot A_n$

$i\bar n\cdot\partial_{us} \ll i\bar n\cdot\partial_n$

$$\mathcal{L}^{(0)}_{nq} = \bar\xi_n \left(in\cdot D + i D\!\!\!/_{n\perp} \frac{1}{i\bar n\cdot D_n} i D\!\!\!/_{n\perp}\right) \frac{\bar n\!\!\!/}{2}\, \xi_n$$

↑ gives $\dfrac{(\bar n\!\!\!/ /2)}{n\cdot p + \frac{p_\perp^2}{\bar n\cdot p} + i0\,\mathrm{sign}(\bar n\cdot p)}$ ✓

↖ bit more work for particle vs. antiparticle see EFTx

gluons

$$\mathcal{L}^{(0)}_{ng} = \mathcal{L}^{(0)}_{ng}\left[n\cdot D,\ D_{n\perp},\ \bar n\cdot D_n\right] \text{ too}$$

(+ gauge fixing & ghosts)

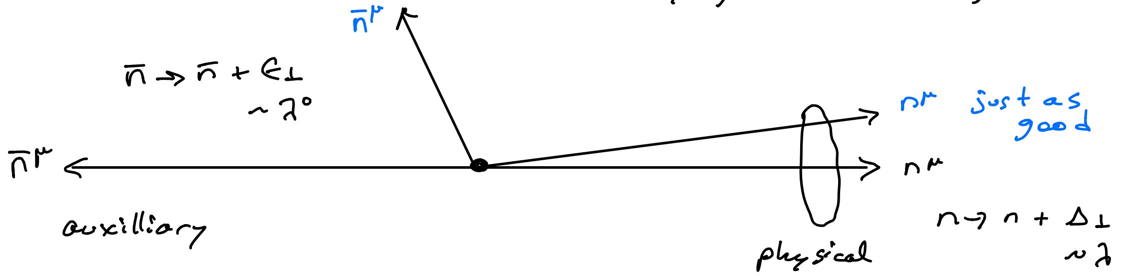
If we drop $n\cdot A_{us}$ these are QCD Lagrangians

Higher Orders eg. $\mathcal{L}^{(1)} = \underset{\lambda}{(\bar\xi_n W_n)}\ \underset{\lambda^2}{i D\!\!\!/^{us}_\perp} \left(W_n^\dagger \frac{1}{i\bar n\cdot D_n} \underset{\lambda}{i D\!\!\!/_{n\perp}} \frac{\bar n\!\!\!/}{2} \underset{\lambda}{\xi_n}\right) = \lambda^5$

eg. $\mathcal{L}^{(1)} = \underset{\lambda}{(\bar\xi_n W_n)}\ g\, \underset{\lambda}{\mathcal{B}\!\!\!/_{n\perp}}\ \underset{\lambda^3}{q_{us}} + h.c. \qquad = \lambda^5$

Gauge Inv ✓

Reparameterization Inv (RPI) freedom to choose n & $\bar n$ satisfying $n^2 = \bar n^2 = 0$, $n\cdot\bar n = 2$



RPI_{III} $\quad n \to \kappa\, n$, $\quad \bar n \to \dfrac{\bar n}{\kappa}$ $\qquad$ Numerator (# n's − # $\bar n$'s) = Denominator (# n's − # $\bar n$'s)

Each collinear sector has its own RPI symmetry

[protects $\mathcal{L}^{(k)}$ coeff from loop corrections, relates operator coeffs.]

$$\mathcal{L}^{(0)}_{SCET_I} = \mathcal{L}^{(0)}_{us} + \sum_n \left(\mathcal{L}^{(0)}_{nq} + \mathcal{L}^{(0)}_{ng}\right) + \mathcal{L}^{(0)}_{Glauber}$$

↗ just full QCD q_{us}, A^μ_{us}

↗ sum over distinct RPI equivalence classes $n_1\cdot n_2 \gg \lambda^2$

↖ extra term for ≥ two collinear directions, only factorization violating term (more later)

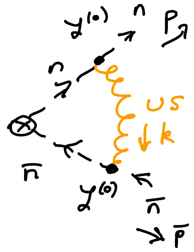
RG Evolution & Matching

UV renormalization
in SCET
[now]

compare renormalized QCD
to '' SCET
& extract C's [later]

$e^+e^- \to$ dijets $\quad \bar\chi_n \gamma_\perp^\mu \chi_{\bar n} = (\bar\xi_n W_n)\, \gamma_\perp^\mu\, (W_{\bar n}^\dagger \xi_{\bar n})$

(use Feyn. Gauge, offshell IR regulator $p^2, \bar p^2 \neq 0$)



$$= \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \ln\left(\frac{\cancel{(-p^2)(-\bar p^2)}}{\mu^2(-Q^2)}\right) + \underset{\text{finite terms}}{\cdots} \right]$$

$$\int \frac{d^dk \;\; n\cdot\bar n}{\left(n\cdot k + \frac{p^2}{Q}\right)\left(\bar n\cdot k + \frac{\bar p^2}{Q}\right) k^2}$$

↑ L3

↓ L4

from W_n

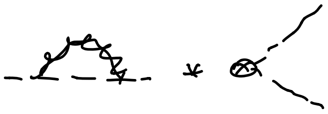
$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} - \frac{2}{\epsilon} \ln\left(\frac{\cancel{(-p^2)}}{\mu^2}\right) + \cdots \right]$$

$$\int d^dk \left[\underbrace{\frac{\bar n\cdot(k+p)}{\bar n\cdot k\,(k+p)^2\, k^2}}_{\text{naive collinear integrand}} - \underbrace{\frac{\bar n\cdot p}{\bar n\cdot k\,(\bar n\cdot p\, n\cdot k + p^2)\, k^2}}_{\text{0-bin subtraction}} \right]$$

0-bin: collinear modes in $SCET_I$ have 0-bin subtractions
from region $k^\mu \sim Q\lambda^2$ to avoid double counting
IR region described by usoft mode.
[part of proper multipole expansion]

from $W_{\bar n}^\dagger$

$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} - \frac{2}{\epsilon} \ln\left(\frac{\cancel{(-\bar p^2)}}{\mu^2}\right) + \cdots \right]$$



$$= -\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon} + \cdots \right]$$

in sum $\frac{\ln(-p^2)}{\epsilon}$ & $\frac{\ln(-\bar{p}^2)}{\epsilon}$ cancel [mixed UV × IR]
[crossed out above]

$$\text{sum} = \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2-i0} + \frac{3}{\epsilon} + \cdots\right]$$

$$C^{bare} = Z_c\, C$$

$\overline{MS}$ counter term

$(Z_c - 1)$  $= \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2-i0} - \frac{3}{\epsilon}\right]$

$$0 = \mu\frac{d}{d\mu}C^{bare} = \mu\frac{d}{d\mu}\left[Z_c(\mu,\epsilon)\,C(\mu)\right]$$
$$= \left[\mu\, d/d\mu\, Z_c\right]C + Z_c\left[\mu\, d/d\mu\, C\right]$$

$$\mu\frac{d}{d\mu}C(\mu) = \Big[\underbrace{-Z_c^{-1}\,\mu\, d/d\mu\, Z_c}_{\gamma_c}\Big]\,C(\mu)$$

anomalous dimension

$\underline{O(\alpha_s)}$ $\quad Z_c^{-1} \to 1 \qquad \mu\, d/d\mu\, \alpha_s = -2\epsilon\,\alpha_s + O(\beta_0\alpha_s^2)$ [follows from $\alpha_s^{bare} = \mu^{2\epsilon}\alpha_s(\mu) Z_\alpha$]

$$\mu\frac{d}{d\mu}Z_c = \frac{C_F}{4\pi}\alpha_s\,(-2\epsilon)\left(-\cancel{\frac{2}{\epsilon^2}} - \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2} - \frac{3}{\epsilon}\right)$$
$$+ \frac{C_F\alpha_s}{4\pi}\left(-\cancel{4/\epsilon}\right)$$ ← from $\mu\, d/d\mu\, \ln\mu^2 = 2$

$$\gamma_c = -\frac{\alpha_s(\mu)}{4\pi}\left[\underbrace{4}_{}\, C_F \ln\frac{\mu^2}{-Q^2} + 6\,C_F\right] \qquad \text{finite}$$

cusp anomalous dimension

when we square the amplitude we get
hard function $H = |C(Q,\mu)|^2$

$$\mu\frac{d}{d\mu}H(Q,\mu) = (\gamma_c + \gamma_c^*)\,H = -\frac{\alpha_s(\mu)}{2\pi}\left[8C_F\ln\frac{\mu}{Q} + \underbrace{6C_F}_{}\right]H(Q,\mu)$$

⇒ Details on Hmwk

leading dble logs $\alpha_s \ln \sim 1$

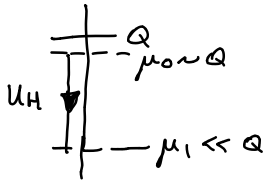
part of NLL, also need 2-loop cusp term $\alpha_s^2 \ln\frac{\mu}{Q}$

$$H(Q,\mu_1) = H(Q,\mu_0)\ U_H(Q,\mu_0,\mu_1)$$

boundary condition

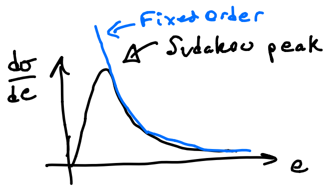
$$= H(Q,\mu_0)\exp\left[-\#\ \alpha_s \ln^2\left(\frac{\mu_1}{Q}\right) + \cdots\right]$$ ← frozen coupling result

$$= H(Q,\mu_0)\exp\left[-\frac{\#}{\alpha_s(\mu_0)}\ f\left(\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_0)}\right) + \cdots\right]$$ ← running coupling result



Sudakov Form Factor no emission until μ_1

$\bar{\chi}_n \Gamma \chi_{\bar{n}}$ SCET operator restricts radiation (collinear & soft emissions below μ_1)



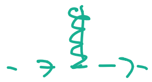
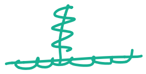
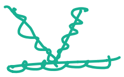
Back to $\mathcal{L}^{(0)}_{SCET_I}$

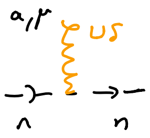
Feyn. Rules

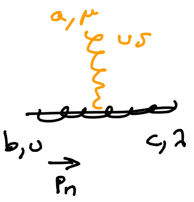
ξ_n 
$$\frac{i\not{\bar{n}}}{2}\frac{\theta(\bar{n}\cdot p)}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p} + i0} + \frac{i\not{\bar{n}}}{2}\frac{\theta(-\bar{n}\cdot p)}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p} - i0} = \frac{i\not{\bar{n}}}{2}\frac{\bar{n}\cdot p}{p^2 + i0}$$

particle antiparticle

 = ...

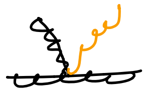
 = ... ,  = ... ,  



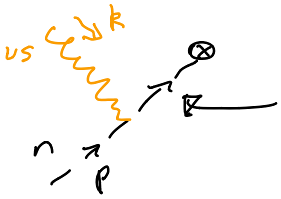
$= i g T^a \frac{\not{\bar{n}}}{2} n^\mu$,


$= g f^{abc} n^\mu \bar{n}\cdot p_n g^{\nu\lambda}$

[Feyn. Gauge for collinear]

 $\propto n^\mu$ too

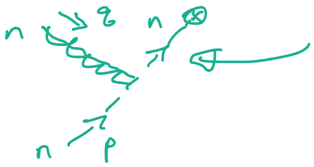
Usofts have eikonal coupling $\propto n^\mu$ to collinears



$$\propto \frac{\bar{n}\cdot p}{\bar{n}\cdot p\, n\cdot(p+k) + p_\perp^2 + i0} = \frac{\bar{n}\cdot p}{\bar{n}\cdot p\, n\cdot k + p^2 + i0} \underset{\substack{\uparrow \\ \text{on-shell} \\ p^2=0}}{=} \frac{1}{n\cdot k + i0}$$

eikonal propagator

[usofts do not change $p_n^\perp$, $\bar{n}\cdot p_n$, neither soft nor collinear can change direction n]



$$\propto \frac{\bar{n}\cdot(p+q)}{(p+q)^2 + i0}$$ for collinears

Ultrasoft - Collinear Factorization

put $n\cdot A_{us}$ into usoft Wilson lines

$$Y_n(x) = P\exp\left(ig\int_{-\infty}^{0} ds\ n\cdot A_{us}(x+ns)\right)$$

$$[n\cdot D_{us}\, Y_n] = 0\ , \qquad Y_n^\dagger Y_n = \mathbb{1} = Y_n Y_n^\dagger$$

Field Redefinition: $\xi_n(x) = Y_n(x)\,\xi_n'(x)$

$$A_n^\mu(x) = Y_n(x)\, A_n'^\mu(x)\, Y_n^\dagger(x)$$ [same for ghost c_n]

$$W_n = \sum_{\text{perms}} \exp\left(\frac{-g}{i\bar{n}\cdot\partial_n}\bar{n}\cdot A_n\right) \underset{\substack{\text{use} \\ \text{multipole} \\ \text{expn}}}{\longrightarrow} Y_n W_n' Y_n^\dagger$$

$$\left(\chi_n \to Y_n\chi_n'\ ,\ \mathcal{B}_{n\perp} \to Y_n \mathcal{B}_{n\perp}' Y_n^\dagger\right)$$

$$\mathcal{L}_{n\xi}^{(0)} = \bar{\xi}_n' \frac{\bar{n}\!\!\!/}{2}\left[Y_n^\dagger\, in\cdot D_{us} Y_n + Y_n^\dagger (Y_n g\, n\cdot A_n' Y_n^\dagger) Y_n + \cdots\right]\xi_n'$$

$$= \bar{\xi}_n' \frac{\bar{n}\!\!\!/}{2}\left[in\cdot\partial + g\,n\cdot A_n' + i D\!\!\!\!/_{n\perp}' \frac{1}{i\bar{n}\cdot D_n'} i D\!\!\!\!/_{n\perp}'\right]\xi_n'$$

$$\mathcal{L}_{n\xi}^{(0)}(\xi_n, A_n, n\cdot A_{us}) = \mathcal{L}_{n\xi}^{(0)}(\xi_n', A_n', 0)$$

same for $\mathcal{L}_{ng}^{(0)}$, so decoupled in $\mathcal{L}^{(0)}$

Reappear in currents

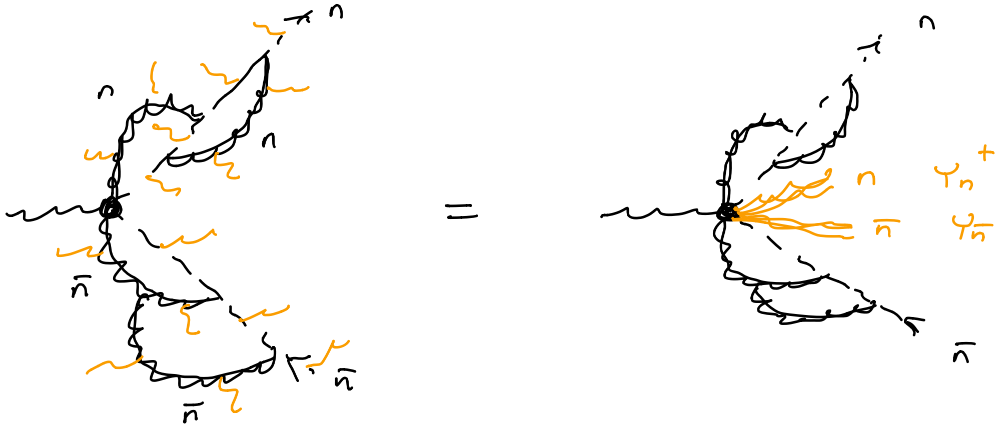
eg1 $(\bar{\chi}_n \Gamma \chi_{\bar{n}}) \longrightarrow \bar{\chi}_n' (Y_n^\dagger Y_{\bar{n}}) \Gamma \chi_{\bar{n}}'$

(n-collin) (usoft) ($\bar{n}$-collin)

factorized up to global color & spin indicies

eg2 $(\bar{\chi}_n \Gamma \chi_n) \longrightarrow \bar{\chi}_n' (\cancel{Y_n^\dagger} \cancel{Y_n}) \Gamma \chi_n$ cancel here

Sums up ∞ class of diagrams



Matching Not Discussed → but see notes at the end if interested

Factorization for $e^+e^- \to$ dijets

$\cancel{\mathcal{L}_G}$

hemisphere jet masses M_a^2, M_b^2

QCD $\quad \sigma = \sum\limits_{\substack{X \\ \text{dijet}}} (2\pi)^4 \delta^4(q-p_X) L_{\mu\nu} \langle 0 | J^{\nu\dagger}(0) | X \rangle \langle X | J^\mu(0) | 0 \rangle$

$$J^\mu = \bar\psi \gamma^\mu \psi = C \, \bar\chi_n \gamma_\perp^\mu (Y_n^\dagger Y_{\bar n}) \chi_{\bar n} + \mathcal{O}(\lambda), \quad |X\rangle = |X_n\rangle |X_{\bar n}\rangle |X_{us}\rangle$$

$M_a^2 \sim M_b^2 \ll Q^2$ ensures X = dijet

$$\sigma = N_0 \sum_{X_n, X_{\bar n}, X_{us}} (2\pi)^4 \delta^4(q - p_{X_n} - p_{X_{\bar n}} - p_{X_{us}}) \langle 0 | Y_n^\dagger Y_{\bar n} | X_{us} \rangle \langle X_{us} | Y_{\bar n}^\dagger Y_n | 0 \rangle$$

$$* \; |C(Q)|^2 \langle 0 | \bar{n}\!\!\!/ \, \chi_{n,Q} | X_n \rangle \langle X_n | \bar\chi_n | 0 \rangle$$

$$* \; \langle 0 | \bar\chi_{\bar n, Q} | X_{\bar n} \rangle \langle X_{\bar n} | n\!\!\!/ \, \chi_{\bar n} | 0 \rangle$$

$$* \int dM_a^2 \, dM_b^2 \; \delta\big(\underbrace{M_a^2 - (p_{X_n} + p_{X_s}^a)^2}\big) \, \delta\big(\underbrace{M_b^2 - (p_{X_{\bar n}} + p_{X_s}^b)^2}\big) + \mathcal{O}(\lambda^2)$$

$$M_a^2 = p_{X_n}^2 + Q \ell_s^+ \qquad\qquad M_b^2 = p_{X_{\bar n}}^2 + Q \ell_s^-$$

⋮

$$\boxed{\text{Factorization}}$$

$$\frac{d\sigma}{dM_a^2 \, dM_b^2} = \sigma_0 \underbrace{H(Q,\mu)}_{\substack{\text{hard} \\ \text{function}}} \int d\ell^+ d\ell^- \underbrace{J(M_a^2 - Q\ell^+, \mu) \, J(M_b^2 - Q\ell^-, \mu)}_{\text{jet functions}} \underbrace{S(\ell^+, \ell^-, \mu)}_{\text{soft fn.}}$$

$$H = |C(Q,\mu)|^2$$

$$J(s) = \mathrm{Im} \left[\frac{-i}{\pi Q} \int d^4x \, e^{\frac{i}{2} \frac{s x^-}{Q}} \langle 0 | T \, \bar\chi_{n,Q}(0) \frac{\bar{n}\!\!\!/}{4N_c} \chi_n(x) | 0 \rangle \right]$$

$$S(\ell^+, \ell^-) = \sum_{X_s} \frac{1}{N_c} \delta(\ell^+ - p_{X_s}^a) \, \delta(\ell^- - p_{X_s}^b) \, \mathrm{tr} \langle 0 | Y_n^\dagger Y_{\bar n} | X_s \rangle \langle X_s | Y_{\bar n}^\dagger Y_n | 0 \rangle$$

Non-perturbative : leading corrections $\mathcal{O}\left(\frac{\Lambda_{QCD}}{M^2/Q}\right)^k$ from ↓ F

$$S(\ell^+, \ell^-, \mu) = \int dk^+ dk^- \, S^{pert}(\ell^+ - k^+, \ell^- - k^-, \mu) \, F(k^+, k^-)$$

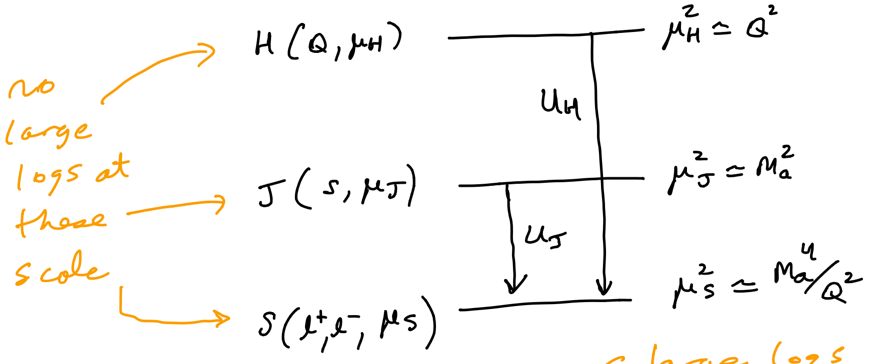
soft
nonpert.
corrections
dominate
hadronization

$\boxed{\text{Hmwk}}$ Calculate $J(s,\mu)$ at 1-loop

Sum Logs: $\alpha_s^k \ln^j (M_{a,b}^2/Q^2)$ with RGE for H, J, S

$$\int_0^{m_{cut}} d\sigma \sim \begin{array}{llll} 1 + \alpha_s L^2 & + \alpha_s^2 L^4 & + \alpha_s^3 L^6 & + \cdots \Big\} LL \\ + \alpha_s L & + \alpha_s^2 L^3 & + \alpha_s^3 L^5 & + \cdots \Big\} NLL \\ + \alpha_s & + \alpha_s^2 L^3 & + \alpha_s^3 L^4 & + \cdots \\ & + \alpha_s^2 L^2 & + \alpha_s^3 L^3 & + \cdots \Big\} NNLL \\ & + \alpha_s^2 L & + \alpha_s^3 L^2 & + \cdots \\ & + \alpha_s^2 & + \alpha_s^3 L & + \cdots \Big\} N^3LL \\ & & + \underset{N^3LL'}{\underline{\alpha_s^3}} & + \cdots \end{array}$$

Known to this order



large logs in U_H & U_J

$$\frac{d\sigma}{dM_a^2\, dM_b^2} = \sigma_0\, H(Q,\mu_H)\, \underline{U_H}(\mu_H,\mu_S) \int d\ell^+ d\ell^-\, S(\ell^+,\ell^-,\mu_S)$$
$$* \int ds\, J(s,\mu_J)\, \underline{U_J}(M_a^2 - Q\ell^+ - s, \mu_J, \mu_S)$$
$$* \int ds'\, J(s',\mu_J)\, \underline{U_J}(M_b^2 - Q\ell^- - s', \mu_J, \mu_S)$$

U_H = Sudakov Form Factor

[U_J see Hmwk solution, has Sudakov dble logs too]

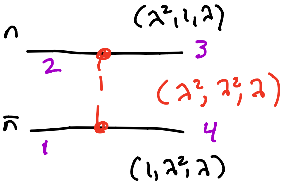
$\Rightarrow$ $N^3LL' + O(\alpha_s^3)$ predictions ~ 1% level precision fits for $\alpha_s(M_Z)$

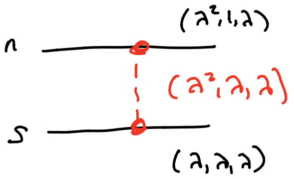
Glauber Exchange

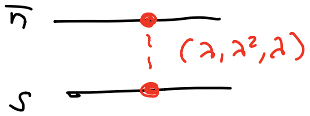
see arXiv: 1601.04695

- Glauber modes with $p^+p^- \ll \vec{p}_\perp^{\,2} \sim \lambda^2$ offshell

  $\frac{1}{\vec{p}_\perp^{\,2}}$ potentials, instantaneous in z & t







- mediates Forward scattering $s \gg -t$

  Forward: $\bar{n}\cdot p_2 = \bar{n}\cdot p_3$, $n\cdot p_1 = n\cdot p_4$

Match from QCD, integrating Glauber out:

$$\mathcal{L}_G^{(0)} = \sum_n \sum_{i,j=q,g} O_n^{iB} \frac{1}{P_\perp^2} O_s^{j_n B} + \sum_{n,n'} \sum_{i,j=q,g} O_n^{iB} \frac{1}{P_\perp^2} O_s^{BC} \frac{1}{P_\perp^2} O_{n'}^{jC}$$

(2-rapidities) (3-rapidities)

$$O_n^{qB} = \bar{\chi}_n T^B \frac{\bar{n}\!\!\!/}{2} \chi_n \,, \qquad O_n^{gB} = \frac{i}{2} f^{BCD} \mathcal{B}_{n\perp\mu}^{C} \frac{\bar{n}}{2}\cdot(i\partial_n - i\overleftarrow{\partial}_n) \mathcal{B}_{n\perp}^{D\mu}$$

similar $O_{\bar{n}}$'s

$$O_s^{q_n B} = 8\pi\alpha_s \bar{\psi}_s^n T^B \frac{n\!\!\!/}{2} \psi_s^n \,, \qquad O_s^{g_n B} = 8\pi\alpha_s \frac{i}{2} f^{BCD} \mathcal{B}_{s\perp\mu}^{nC} \frac{n}{2}\cdot(i\partial_s - i\overleftarrow{\partial}_s) \mathcal{B}_{s\perp}^{nD\mu}$$

$$O_s^{BC} = 8\pi\alpha_s \Big\{ P_\perp^\mu \mathcal{S}_n^T \mathcal{S}_{\bar{n}} \mathcal{P}_{\perp\mu} - \mathcal{P}_{\perp\mu}\, g \tilde{\mathcal{B}}_{s\perp}^{n\mu} \mathcal{S}_n^T \mathcal{S}_{\bar{n}} - \mathcal{S}_n^T \mathcal{S}_{\bar{n}}\, g \tilde{\mathcal{B}}_{s\perp}^{\bar{n}\mu} \mathcal{P}_{\perp\mu}$$
$$- g \tilde{\mathcal{B}}_{s\perp}^{n\mu} \mathcal{S}_n^T \mathcal{S}_{\bar{n}} \tilde{\mathcal{B}}_{s\perp\mu}^{\bar{n}} - \frac{n_\mu \bar{n}_\nu}{2} \mathcal{S}_n^T\, ig \tilde{G}_s^{\mu\nu} \mathcal{S}_{\bar{n}} \Big\}^{BC}$$

Here $\psi_s^n = S_n^+ q_s$, $\mathcal{B}_{s\perp}^{n\mu} = \frac{1}{g}[S_n^+ iD_{s\perp}^\mu S_n]$

tildes: $\tilde{\mathcal{B}}_{s\perp}^{n\,AB} = -i f^{ABC} \mathcal{B}_{s\perp}^{nC}$

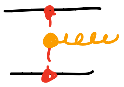
$\mathcal{S}_n$ adjoint wilson line

Note has: rapidity regulator $|k_z|^{-\eta}$, multipole expansion

0-bin subtractions

Note

- construction involves using SCET p.c. theorem
- universal for $i, j = q, g$
- no hard coefficient or loop corrections to $\mathcal{L}_G^{(0)}$
- only pairs of collinear directions in $\mathcal{L}_G^{(0)}$, rest are T-products
- breaks factorization $\mathcal{L}_G^{(0)}(\{\xi_{n_i}, A_{n_i}\}, q_s, A_s)$ coupling at $\mathcal{O}(\lambda^0)$ btwn n_i, n_j & s
- encodes known examples of fact. violation (Wilson line directions, $i\pi$'s, ...)
- One-gluon Feyn. Rule of O_s^{AB} is Lipatov Vertex



gluon reggeization (Amplitude level)

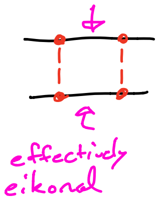
$$\left(\frac{\nu^2}{\nu_0^2}\right)^{-\gamma_{\nu}} = \left(\frac{s}{-t}\right)^{-\gamma_{\nu}}$$

BFKL equation (cross-section level)

$$\nu \frac{\partial}{\partial \nu} \sigma(q_\perp, q_\perp', \nu) = \int d^2k_\perp \, \gamma^{BFKL}(q_\perp, k_\perp) \, \sigma(k_\perp, q_\perp', \nu)$$

→ small-x resummation

- Glauber Loops give $i\pi$



$$\int \frac{d^dk \quad |2k^z|^{-\eta} \, \nu^{2\eta}}{\vec{k}_\perp^2 (\vec{k}_\perp - \vec{q}_\perp)^2 (k^+ - \Delta_1(k_\perp) + i0)(-k^- - \Delta_2(k_\perp) + i0)}$$

$$= \left(\frac{-i}{4\pi}\right) \int \frac{d^{d-2}k_\perp}{\vec{k}_\perp^2 (\vec{k}_\perp - \vec{q}_\perp)^2} \left[-i\pi + \mathcal{O}(\eta) \right]$$

- Wilson Line Directions → Glauber region

$$\frac{1}{n\cdot k \pm i0} \rightarrow \text{sign} \mp i\pi \, \delta(n\cdot k)$$

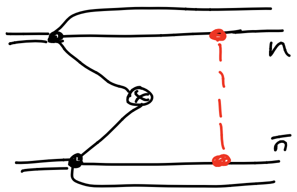
not soft or collinear

$$\frac{1}{\bar{n}\cdot k \pm i0} \rightarrow \text{sign} \mp i\pi \, \delta(\bar{n}\cdot k)$$

Sometimes Glauber contribution must be absorbed into some Wilson line directions to establish factorization (TMDs)

Some Glauber's must cancel to establish factorization

eg

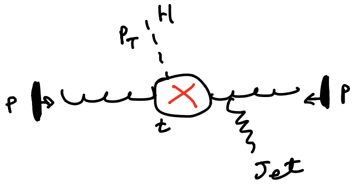


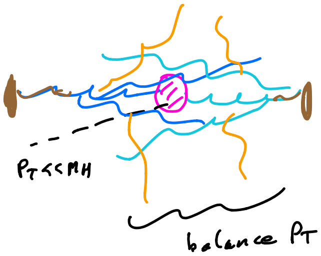
Spectator - Spectator

no soft or collinear analogs at leading power

Final Example

Precision p_T spectrum of the Higgs Boson





$P_n \sim m_H(\lambda^2, 1, \lambda)$

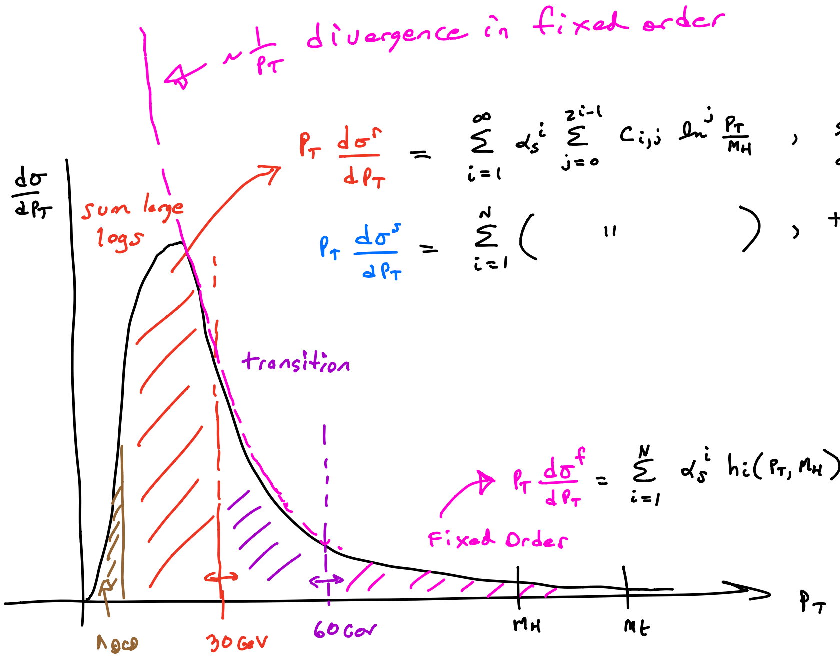
$P_{\bar{n}} \sim m_H(1, \lambda^2, \lambda)$

$p_s^\mu \sim m_H \lambda$

$\boxed{SCET_{II}}$

consider

$p_T^H \sim m_H \lambda \ll Q = m_H$



(LL, NLL etc below)

$$p_T \frac{d\sigma^r}{dp_T} = \sum_{i=1}^{\infty} \alpha_s^i \sum_{j=0}^{2i-1} C_{ij} \ln^j \frac{p_T}{m_H}$$, singular terms $p_T \ll m_H$ all orders in α_s

$$p_T \frac{d\sigma^s}{dp_T} = \sum_{i=1}^{N} \left(\quad " \quad \right)$$, truncated to α_s^N

$$p_T \frac{d\sigma^f}{dp_T} = \sum_{i=1}^{N} \alpha_s^i h_i(p_T, m_H) = \overset{LO}{\alpha_s h_1} + \overset{NLO}{\alpha_s^2 h_2} + \overset{NNLO}{\alpha_s^3 h_3}$$

remove double counting

full result: $$p_T \frac{d\sigma}{dp_T} = p_T \frac{d\sigma^r}{dp_T} + \left(p_T \frac{d\sigma^f}{dp_T} - p_T \frac{d\sigma^s}{dp_T} \right)$$

Resummation (Factorization)

$$\frac{d\sigma}{dp_T^2} = \pi \sigma_0 \int dx_a dx_b\, \delta\!\left(x_a x_b - \frac{m_H^2}{E_{cm}^2}\right) \int d^2b\, e^{i\vec{p}_\perp \cdot \vec{b}}\, W(x_a, x_b, m_H, \vec{b})$$

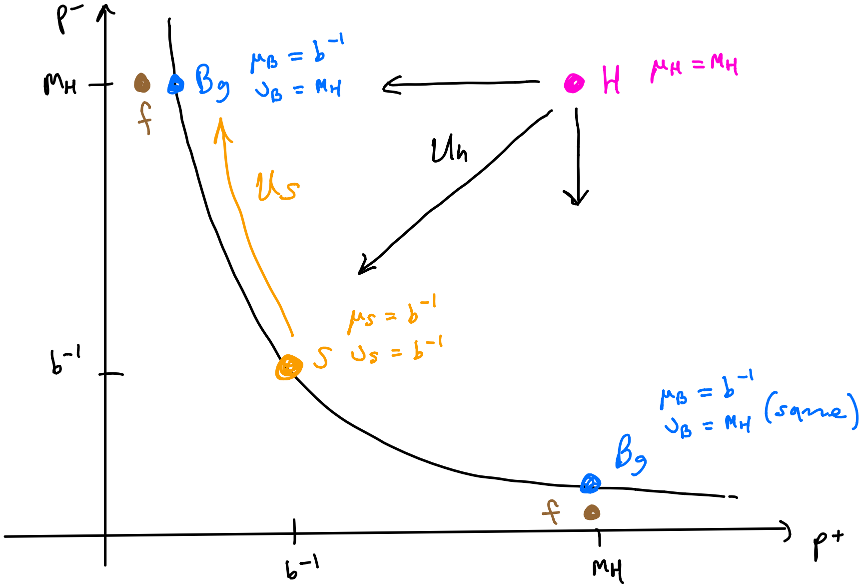
independent of scales to order N^kLL one is working

$$W = H(m_H, \mu_H)\, U_h(\mu_h, \mu_B)\, S_\perp(\vec{b}, \mu_S, \nu_S)\, U_S(b, \mu_B, \mu_S; \nu_B, \nu_S)\, B_g(x_a, \vec{b}, m_H, \mu_B, \nu_B)\, B_g(x_b, \vec{b}, m_H, \mu_B, \nu_B)$$

hard soft beam fns

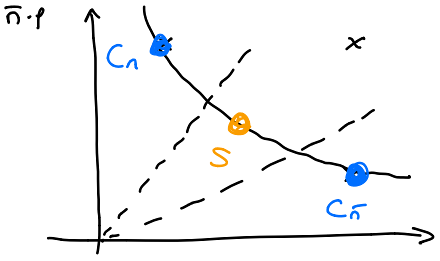
$$B_g = \sum_i \int_x^1 \frac{dz}{z}\, \mathcal{I}_{gi}\!\left(\frac{x}{z}, \vec{b}, m_H, \mu_B, \nu_B\right) f_i(z, \mu_B)$$

PDFs



μ & ν are inv. mass & "rapidity" scales

Discuss transition



$$e^{2Y} = p^-/p^+$$

distinguish modes by rapidity Y

$$e^{2Y} \sim \lambda^{-2},\ \lambda^0,\ \lambda^2$$

C_n S $C_{\bar{n}}$

→ can have rapidity divergences

Log Resummation Orders

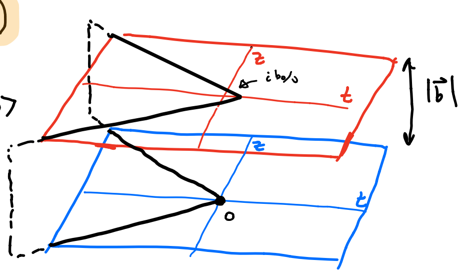
$$\ln W = L \sum_{k=1}^{\infty} (\alpha_s L)^k + \sum_k (\alpha_s L)^k + \alpha_s \sum_k (\alpha_s L)^k + \alpha_s^2 \sum_k (\alpha_s L)^k$$

LL NLL NNLL N^3LL

$L = \ln(b\, m_H)$ needed for all singular L terms at NNLO

Definitions [field thy defns on side board]

$S(b_\perp, \mu, \nu)$



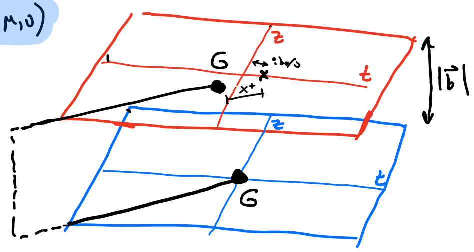
$\langle 0|$operator$|0\rangle$

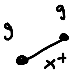
only Wilson Lines

$B_g(x, \vec{b}, m_H, \mu, \nu)$

Wilson lines with $G^{\mu\nu}$'s on ends

$\langle p|$operator$|p\rangle$



Compare to PDF: $f_g(z, \mu)$ $\langle p|$  $|p\rangle$

For studying non-perturbative p_T define

$$f^{TMD}(x, \vec{b}, \mu, M_H) = B_g \sqrt{S} = \frac{\tilde{B}_g^{\text{naive}}}{S_{0\text{-bin}}} \sqrt{S}$$

dependance from two rapidity scales

— The End —

Additional Notes on Matching

1-loop Matching Example

$e^+e^- \to$ dijets

$$\begin{tikzcd} \boxed{QCD} \arrow[d] \\ \boxed{SCET} \end{tikzcd}$$

$$\mathcal{L}_{QCD} + J^\mu = \bar\psi \gamma^\mu \psi$$
$$\downarrow$$
$$\mathcal{L}^{(0)}_{SCET} + \mathcal{L}^{(0)}_{hard} = C\, \bar\chi_n \gamma_\perp^\mu \chi_{\bar n}$$

find C at $O(\alpha_s)$

[Feyn. Gauge again]

$$(\text{1-loop ren. QCD}) - (\text{1-loop ren. SCET}) = C^{(1\text{-loop})} \langle O^{(0)}_{SCET} \rangle$$

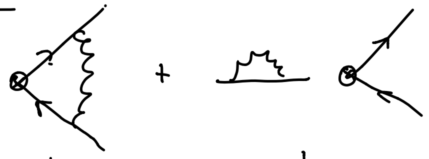
- Must use some IR regulator in QCD & SCET
- Result for C will be independent of IR reg. choice.

$p^2 = \bar p^2 \neq 0$

$= Z_\psi - 1 = Z_\xi - 1 =$

$$= \frac{\alpha_s C_F}{4\pi}\left[-\frac{1}{\epsilon} - \ln\frac{\mu^2}{-p^2} - 1 \right]$$

QCD



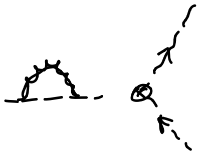
$$= \frac{\alpha_s(\mu) C_F}{4\pi}\left[\underline{-2\ln^2\left(\frac{p^2}{Q^2}\right)} - \underline{3\ln\left(\frac{p^2}{Q^2}\right)} - \frac{2\pi^2}{3} - 1 \right]$$

$\frac{1}{\epsilon_{UV}}$ $\quad$ $-\frac{1}{\epsilon_{UV}}$ ⟸ cancel since cons. current

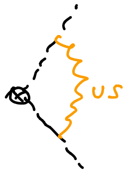
SCET

$+$

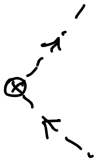
$+$



$+$



$+ \left(Z_C^{\overline{MS}} - 1\right)$



$$= \frac{\alpha_s(\mu) C_F}{4\pi}\Bigg[\underbrace{2\ln^2\left(\frac{\mu^2}{-p^2}\right) + 3\ln\frac{\mu^2}{-p^2}}_{\text{collinear graphs}} - \underbrace{\ln^2\left(\frac{\mu^2 Q^2}{-p^4}\right)}_{\text{usoft graph}} + \underbrace{7 - \frac{5\pi^2}{6}}_{\text{both}} \Bigg]$$

$$= \frac{\alpha_s(\mu) C_F}{4\pi}\left[\ln^2\left(\frac{\mu^2}{-Q^2}\right) \underline{- 2\ln^2\left(\frac{p^2}{Q^2}\right)} \underline{- 3\ln\left(\frac{p^2}{Q^2}\right)} + 3\ln\frac{\mu^2}{-Q^2} + 7 - \frac{5\pi^2}{6} \right]$$

$\ln p^2$ IR divergences agree

$$\text{QCD} - \text{SCET} = \frac{\alpha_s(\mu) C_F}{4\pi}\left[-\ln^2\left(\frac{\mu^2}{-Q^2-i0}\right) - 3\ln^2\left(\frac{\mu^2}{-Q^2-i0}\right) - 8 + \frac{\pi^2}{6}\right]$$

$$C(Q,\mu) = 1 + \frac{\alpha_s(\mu) C_F}{4\pi}\left[\quad '' \quad\quad '' \quad\quad '' \quad\quad '' \quad \right]$$

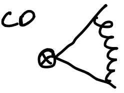
Dim. Reg. Trick

[useful for many EFT matching calculations]

use $1/\epsilon_{IR}$ for IR divergences & $1/\epsilon_{UV}$ for UV

 = QCD  $\propto \left(\frac{1}{\epsilon_{UV}} - 1/\epsilon_{IR}\right)$,  +  +  $\propto$ $1/\epsilon_{UV}^2 - 1/\epsilon_{IR}^2$, $\frac{1}{\epsilon_{UV}} - 1/\epsilon_{IR}$

$(Z_C^{\overline{ms}} - 1)$  $= \frac{\alpha_s C_F}{4\pi}\left[\frac{-2}{\epsilon_{UV}^2} - \frac{2}{\epsilon_{UV}} \ln\frac{\mu^2}{-Q^2} - 3/\epsilon_{UV}\right]$ as before

QCD  $= \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\epsilon_{IR}^2} - \frac{2}{\epsilon_{IR}}\ln\frac{\mu^2}{-Q^2} - \frac{3}{\epsilon_{IR}} - \ln^2\frac{\mu^2}{-Q^2} - 3\ln\frac{\mu^2}{-Q^2} - 8 + \frac{\pi^2}{6}\right]$

- set $\epsilon_{IR} = \epsilon_{UV}$ & assume $1/\epsilon_{IR}$ match. Don't need EFT calc.
- Result for $C(Q,\mu)$ is IR finite part of pure dim. reg QCD result (agrees with earlier $p^2 \neq 0$ result as expected)