RG Evolution & Matching

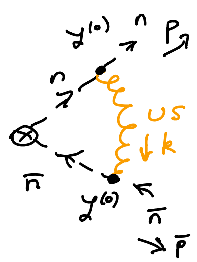
UV renormalization in SCET [now]

compare renormalized QCD to " SCET & extract C's [later]

$e^+e^- \to$ dijets $\qquad \bar\chi_n \gamma_\perp^\mu \chi_{\bar n} = (\bar\xi_n W_n)\, \gamma_\perp^\mu\, (W_{\bar n}^\dagger \xi_{\bar n})$

(use Feyn. Gauge, offshell IR regulator $p^2, \bar p^2 \neq 0$)

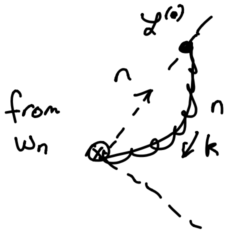
finite terms



$$= \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(\frac{\cancel{(-p^2)(-\bar p^2)}}{\mu^2(-Q^2)}\right) + \cdots\right]$$

$$\uparrow \quad \int \frac{d^dk \quad n\cdot\bar n}{\left(n\cdot k + \frac{p^2}{Q}\right)\left(\bar n\cdot k + \frac{\bar p^2}{Q}\right)k^2}$$

from W_n



$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} - \frac{2}{\epsilon}\ln\left(\frac{\cancel{(-p^2)}}{\mu^2}\right) + \cdots\right]$$

$$\uparrow \quad \int d^dk \left[\frac{\bar n\cdot(k+p)}{\bar n\cdot k\,(k+p)^2\,k^2} - \frac{\bar n\cdot p}{\bar n\cdot k\,(\bar n\cdot p\, n\cdot k + p^2)\,k^2}\right]$$

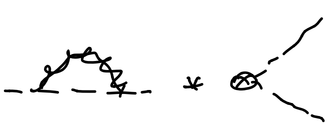
naive collinear integrand $\qquad$ 0-bin subtraction

0-bin: collinear modes in SCET$_I$ have 0-bin subtractions from region $k^\mu \sim Q\lambda^2$ to avoid double counting IR region described by usoft mode.

[part of proper multipole expansion]

from $W_{\bar n}^\dagger$

$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} - \frac{2}{\epsilon}\ln\left(\frac{\cancel{(-\bar p^2)}}{\mu^2}\right) + \cdots\right]$$



$$= -\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon} + \cdots\right]$$

in sum $\frac{\ln(-p^2)}{\epsilon}$ & $\frac{\ln(-\bar{p}^2)}{\epsilon}$ ~~cancel~~ [mixed UV × IR]
[crossed out above]

$$\text{sum} = \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2-i0} + \frac{3}{\epsilon} + \cdots\right]$$

$$C^{bare} = Z_C\, C$$

$\overline{MS}$ counter term

$$(Z_C - 1) \otimes = -\frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2-i0} + \frac{3}{\epsilon}\right]$$

$$0 = \mu\frac{d}{d\mu}C^{bare} = \mu\frac{d}{d\mu}\left[Z_C(\mu,\epsilon)\,C(\mu)\right]$$
$$= \left[\mu\, d/d\mu\, Z_C\right]C + Z_C\left[\mu\, d/d\mu\, C\right]$$

$$\mu\frac{d}{d\mu}C(\mu) = \Big[\underbrace{-Z_C^{-1}\,\mu\, d/d\mu\, Z_C}_{\gamma_C}\Big]\,C(\mu)$$

$\underline{O(\alpha_s)}$ $\quad Z_C^{-1} \to 1 \qquad \mu\, d/d\mu\, \alpha_s = -2\epsilon\,\alpha_s + O(\beta_0\alpha_s^2)$

{recall $\alpha_s^{bare} = \mu^{2\epsilon}\alpha_s(\mu) Z_\alpha$ implies this}

$$\mu\frac{d}{d\mu}Z_C = \frac{C_F}{4\pi}\alpha_s\,(-2\epsilon)\left(-\cancel{\frac{2}{\epsilon^2}} - \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2} - \frac{3}{\epsilon}\right)$$
$$+ \frac{C_F\alpha_s}{4\pi}\left(-\cancel{4/\epsilon}\right) \quad \Leftarrow \text{from } \mu\, d/d\mu \ln\mu^2 = 2$$

$$\gamma_C = -\frac{\alpha_s(\mu)}{4\pi}\left[4C_F\ln\frac{\mu^2}{-Q^2} + 6C_F\right] \qquad \text{finite}$$

cusp anomalous dimension

when we square the amplitude we get
hard function $\quad H = |C(Q,\mu)|^2$

$$\mu\frac{d}{d\mu}H(Q,\mu) = (\gamma_C + \gamma_C^*)\,H = -\frac{\alpha_s(\mu)}{2\pi}\left[8C_F\ln\frac{\mu}{Q} + 6C_F\right]H(Q,\mu)$$

$\underbrace{\text{leading dble logs}}$ $\alpha_s \ln \sim 1$

part of NLL, also need 2-loop cusp $\alpha_s^2 \ln\frac{\mu}{Q}$ term

$\Rightarrow$

$$H(Q,\mu_1) = H(Q,\mu_0)\; U_H(Q,\mu_0,\mu_1)$$

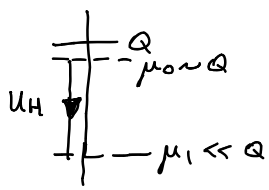
boundary condition

$$= H(Q,\mu_0)\, \exp\left[-\#\, \alpha_s \ln^2\left(\frac{\mu_1}{Q}\right) + \cdots \right]$$

← frozen coupling result

$$= H(Q,\mu_0)\, \exp\left[-\frac{\#}{\alpha_s(\mu_0)}\, f\left(\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_0)}\right) + \cdots \right]$$

← running coupling result



Details on Hmwk

Sudakov Form Factor

no emission until μ_1

$\bar\chi_n \Gamma \chi_{\bar n}$ SCET operator restricts radiation (collinear & soft emissions below μ_1)

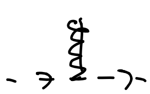
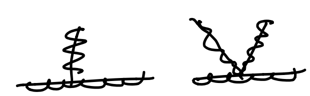
Back to $\mathcal{L}^{(0)}_{SCET_I}$

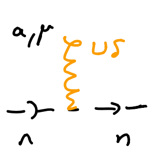
Feyn. Rules

ξ_n  $\dfrac{i\not{\bar n}}{2} \dfrac{\theta(\bar n\cdot p)}{n\cdot p + \frac{p_\perp^2}{\bar n\cdot p} + i0} + \dfrac{i\not{\bar n}}{2}\dfrac{\theta(-\bar n\cdot p)}{n\cdot p + \frac{p_\perp^2}{\bar n\cdot p} - i0} = \dfrac{i\not{\bar n}}{2}\dfrac{\bar n\cdot p}{p^2 + i0}$

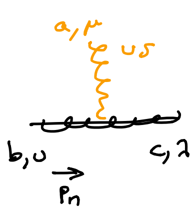
particle   antiparticle

 $= \ldots$

 $= \ldots$,  $= \ldots$, 

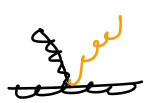


$= i g T^a \dfrac{\not{\bar n}}{2} n^\mu$,

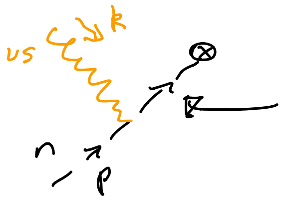


$= g f^{abc} n^\mu \bar n\cdot p_n\, g^{\nu\lambda}$

[Feyn. Gauge for collinear]

 $\propto n^\mu$ too

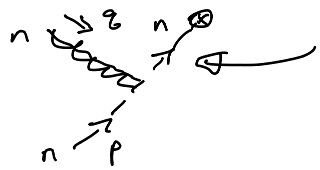
Usofts have eikonal coupling $\propto n^\mu$ to collinears



$$\propto \frac{\bar n\cdot p}{\bar n\cdot p\, n\cdot(p+k) + p_\perp^2 + i0} = \frac{\bar n\cdot p}{\bar n\cdot p\, n\cdot k + p^2 + i0} \underset{\substack{\text{on-shell}\\ p^2=0}}{=} \frac{1}{n\cdot k + i0}$$

eikonal propagator

[Usofts do not change $p_n^\perp$, $\bar n\cdot p_n$, neither soft nor collinear can change direction n]



$$\propto \frac{\bar n\cdot(p+q)}{(p+q)^2+i0} \qquad \text{for collinears}$$

Ultrasoft - Collinear Factorization

put $n\cdot A_{us}$ into usoft Wilson lines

$$Y_n(x) = P\exp\left(ig\int_{-\infty}^{0} ds\; n\cdot A_{us}(x+ns)\right)$$

$$[n\cdot D_{us}\, Y_n] = 0\,, \qquad Y_n^\dagger Y_n = \mathbb{1} = Y_n Y_n^\dagger$$

Field Redefinition:
$$\xi_n(x) = Y_n(x)\,\xi_n'(x)$$
$$A_n^\mu(x) = Y_n(x)\,A_n'^\mu(x)\,Y_n^\dagger(x) \qquad \left[\begin{array}{l}\text{same for}\\ \text{ghost } c_n\end{array}\right]$$

$$W_n = \sum_{\text{perms}} \exp\left(\frac{-g}{i\bar n\cdot\partial_n}\,\bar n\cdot A_n\right) \xrightarrow[\substack{\text{use}\\ \text{multipole}\\ \text{expn}}]{} Y_n W_n' Y_n^\dagger\,, \qquad \chi_n \to Y_n \chi_n'$$
$$\mathcal{B}_{n\perp} \to Y_n \mathcal{B}_{n\perp}' Y_n^\dagger$$

$$\mathcal{L}_{nq}^{(0)} = \bar\xi_n' \frac{\bar{n}\!\!\!/}{2}\left[Y_n^\dagger\, i n\cdot D_{us} Y_n + Y_n^\dagger(Y_n g n\cdot A_n' Y_n^\dagger) Y_n + \cdots\right]\xi_n'$$
$$= \bar\xi_n' \frac{\bar{n}\!\!\!/}{2}\left[i n\cdot\partial + g n\cdot A_n' + i D\!\!\!\!/_{n\perp}' \frac{1}{i\bar n\cdot D_n'} i D\!\!\!\!/_{n\perp}'\right]\xi_n'$$

$$\mathcal{L}_{nq}^{(0)}(\xi_n, A_n, n\cdot A_{us}) = \mathcal{L}_{nq}^{(0)}(\xi_n', A_n', 0)$$

same for $\mathcal{L}_{ng}^{(0)}$, so decoupled in $\mathcal{L}^{(0)}$

Reappear in currents

eg 1 $\quad (\bar{\chi}_n \Gamma \chi_{\bar{n}}) \longrightarrow \bar{\chi}_n' \, (Y_n^\dagger Y_{\bar{n}}) \, \Gamma \, \chi_{\bar{n}}'$

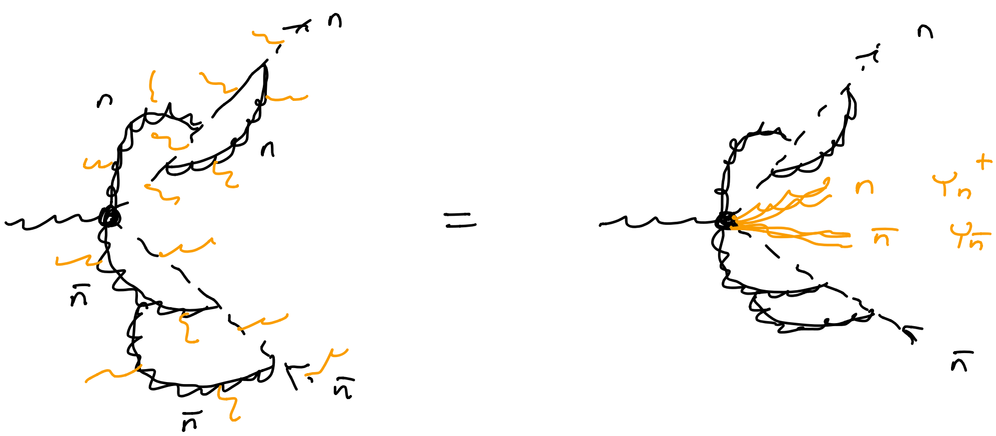
$\qquad\qquad$ (n-collin) (usoft) ($\bar{n}$-collin)

factorized up to global color & spin indicies

eg 2 $\quad (\bar{\chi}_n \Gamma \chi_n) \longrightarrow \bar{\chi}_n' \, (\cancel{Y_n^\dagger} \cancel{Y_n}) \, \Gamma \, \chi_n$

cancel here

Sums up ∞ class of diagrams



1-loop Matching Example

$e^+e^- \to$ dijets

QCD $\downarrow$ SCET

$\mathcal{L}_{QCD} + \quad J^\mu = \bar{\psi}\gamma^\mu\psi$

$\downarrow$

$\mathcal{L}_{SCET}^{(0)} + \mathcal{L}_{hard}^{(0)} = C \, \bar{\chi}_n \gamma_\perp^\mu \chi_{\bar{n}}$

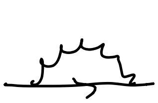
find C at $O(\alpha_s)$

[Feyn. Gauge again]

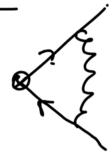
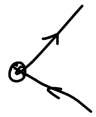
(1-loop ren. QCD) $-$ (1-loop ren. SCET) $= C^{(1\text{-loop})} \langle O_{SCET}^{(0)} \rangle$

- Must use some IR regulator in QCD & SCET
- Result for C will be independent of IR reg. choice.

$p^2 = \bar{p}^2 \neq 0$

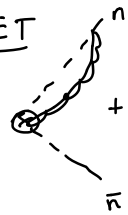
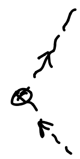
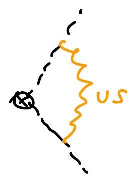
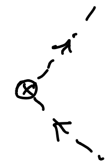
 $= Z_\psi - 1 = Z_\xi - 1 =$  $= \dfrac{\alpha_s C_F}{4\pi}\left[-\dfrac{1}{\epsilon} - \ln\dfrac{\mu^2}{-p^2} - 1\right]$

QCD

 $+$   $= \dfrac{\alpha_s(\mu) C_F}{4\pi}\Big[-2\ln^2\Big(\dfrac{p^2}{Q^2}\Big) - 3\ln\Big(\dfrac{p^2}{Q^2}\Big) - \dfrac{2\pi^2}{3} - 1\Big]$

$\dfrac{1}{\epsilon_{UV}}$ $\qquad -\dfrac{1}{\epsilon_{UV}}$ $\leftarrow$ cancel since cons. current

SCET

 $+$  $+$   $+$  $+\ (Z_c^{\overline{MS}} - 1)$ 

$$= \frac{\alpha_s(\mu) C_F}{4\pi}\Big[\underbrace{2\ln^2\Big(\frac{\mu^2}{-p^2}\Big) + 3\ln\frac{\mu^2}{-p^2}}_{\text{collinear graphs}} - \underbrace{\ln^2\Big(\frac{\mu^2 Q^2}{-p^4}\Big)}_{\text{usoft graph}} + \underbrace{7 - \frac{5\pi^2}{6}}_{\text{both}}\Big]$$

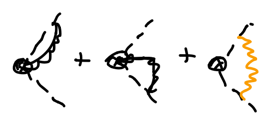
$$= \frac{\alpha_s(\mu) C_F}{4\pi}\Big[\ln^2\Big(\frac{\mu^2}{-Q^2}\Big) \underbrace{- 2\ln^2\Big(\frac{p^2}{Q^2}\Big) - 3\ln\Big(\frac{p^2}{Q^2}\Big)}_{\ln p^2 \text{ IR divergences agree}} + 3\ln\frac{\mu^2}{-Q^2} + 7 - \frac{5\pi^2}{6}\Big]$$

$$\text{QCD} - \text{SCET} = \frac{\alpha_s(\mu) C_F}{4\pi}\Big[-\ln^2\Big(\frac{\mu^2}{-Q^2 - i0}\Big) - 3\ln\Big(\frac{\mu^2}{-Q^2 - i0}\Big) - 8 + \frac{\pi^2}{6}\Big]$$

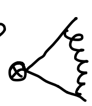
$$C(Q,\mu) = 1 + \frac{\alpha_s(\mu) C_F}{4\pi}\Big[\quad " \qquad " \qquad " \qquad " \quad\Big]$$

Dim. Reg. Trick [useful for many EFT matching calculations]

use $1/\epsilon_{IR}$ for IR divergences & $1/\epsilon_{UV}$ for UV

 $=$  $\propto \Big(\dfrac{1}{\epsilon_{UV}} - \dfrac{1}{\epsilon_{IR}}\Big)$,  $\propto \dfrac{1}{\epsilon_{UV}^2} - \dfrac{1}{\epsilon_{IR}^2}$, $\dfrac{1}{\epsilon_{UV}} - \dfrac{1}{\epsilon_{IR}}$

$(Z_c^{\overline{MS}} - 1)$  $= \dfrac{\alpha_s C_F}{4\pi}\Big[\dfrac{-2}{\epsilon_{UV}^2} - \dfrac{2}{\epsilon_{UV}}\ln\dfrac{\mu^2}{-Q^2} - \dfrac{3}{\epsilon_{UV}}\Big]$ as before

QCD  $= \dfrac{\alpha_s C_F}{4\pi}\Big[\dfrac{-2}{\epsilon_{IR}^2} - \dfrac{2}{\epsilon_{IR}}\ln\dfrac{\mu^2}{-Q^2} - \dfrac{3}{\epsilon_{IR}} - \ln^2\dfrac{\mu^2}{-Q^2} - 3\ln\dfrac{\mu^2}{-Q^2} - 8 + \dfrac{\pi^2}{6}\Big]$

- set $\epsilon_{IR} = \epsilon_{UV}$ & assume $1/\epsilon_{IR}$ match. Don't need EFT calc.
- Result for $C(Q,\mu)$ is IR finite part of pure dim. reg QCD result (agrees with earlier $p^2 \neq 0$ result as expected)

Factorization for $e^+e^- \to$ dijets

$\cancel{\mathcal{L}_G}$

hemisphere jet masses M_a^2, M_b^2

QCD $$\sigma = \sum_{\substack{X \\ \text{dijet}}} (2\pi)^4 \delta^4(q - p_X) L_{\mu\nu} \langle 0 | J^{\nu\dagger}(0) | X \rangle \langle X | J^\mu(0) | 0 \rangle$$

$$J^\mu = \bar\psi \gamma^\mu \psi = C \, \bar\chi_n \gamma_\perp^\mu (Y_n^\dagger Y_{\bar n}) \chi_{\bar n} + \mathcal{O}(\lambda), \quad |X\rangle = |X_n\rangle |X_{\bar n}\rangle |X_{us}\rangle$$

$$\sigma = N_0 \sum_{X_n, X_{\bar n}, X_{us}} (2\pi)^4 \delta^4(q - p_{X_n} - p_{X_{\bar n}} - p_{X_{us}}) \langle 0 | Y_n^\dagger Y_{\bar n} | X_{us} \rangle \langle X_{us} | Y_{\bar n}^\dagger Y_n | 0 \rangle$$

$$* \; |C(Q)|^2 \langle 0 | \bar{n}\!\!\!/ \, \chi_{n,Q} | X_n \rangle \langle X_n | \bar\chi_n | 0 \rangle$$

$$* \; \langle 0 | \bar\chi_{\bar n, Q} | X_{\bar n} \rangle \langle X_{\bar n} | n\!\!\!/ \, \chi_{\bar n} | 0 \rangle$$

$$* \int dM_a^2 \, dM_b^2 \; \delta(M_a^2 - (p_{X_n} + p_{X_s}^a)^2) \, \delta(M_b^2 - (p_{X_{\bar n}} + p_{X_s}^b)^2) + \mathcal{O}(\lambda^2)$$

$\leftarrow$ $M_a^2 \sim M_b^2 \ll Q^2$ ensures $X =$ dijet

Factorize Measurement, Simplify, ...

$$\frac{d\sigma}{dM_a^2 \, dM_b^2} = \sigma_0 \underbrace{|C(Q)|^2}_{H(Q)} \int dk^+ d\ell^+ dk^- d\ell^- \; \delta(M_a^2 - Q(k^+ + \ell^+)) \, \delta(M_b^2 - Q(k^- + \ell^-))$$

$$* \; \mathrm{Im}\left[\frac{-i}{\pi Q} \int d^4x \, e^{ik^+ x^-/2} \langle 0 | T \, \bar\chi_{n,Q}(0) \frac{\bar{n}\!\!\!/}{4N_c} \chi_n(x) | 0 \rangle \right]$$

$$* \; \mathrm{Im}\left[\frac{-i}{\pi Q} \int d^4y \, e^{ik^- y^+/2} \langle 0 | T \, \bar\chi_{\bar n, Q} \frac{n\!\!\!/}{4N_c} \chi_{\bar n}(y) | 0 \rangle \right]$$

Jet Functions $$* \underbrace{\sum_{X_s} \frac{1}{N_c} \delta(\ell^+ - p_{X_s}^a) \, \delta(\ell^- - p_{X_s}^b)}_{} \; \mathrm{tr} \langle 0 | Y_n^\dagger Y_{\bar n} | X_s \rangle \langle X_s | Y_{\bar n}^\dagger Y_n | 0 \rangle$$

$J(Qk^+)$, $J(Qk^-)$

soft function $= S(\ell^+, \ell^-)$

$$\frac{d\sigma}{dM_a^2 \, dM_b^2} = \sigma_0 \, H(Q, \mu) \int d\ell^+ d\ell^- \; J(M_a^2 - Q\ell^+, \mu) \, J(M_b^2 - Q\ell^-, \mu) \, S(\ell^+, \ell^-, \mu)$$

[J, S : δ-fns at tree-level, have real & virtual graphs @ 1-loop]

Note: thrust $\tau = 1 - T = \dfrac{M_a^2 + M_b^2}{Q^2} \ll 1$ so simple projection

Non-perturbative : leading corrections $\mathcal{O}\left(\dfrac{\Lambda_{QCD}}{M^2/Q}\right)$ from $\downarrow$ F

$$S(\ell^+, \ell^-, \mu) = \int d\ell^{+\prime} d\ell^{-\prime} \, S^{\text{pert}}(\ell^+ - \ell^{+\prime}, \ell^- - \ell^{-\prime}, \mu) \; F(\ell^{+\prime}, \ell^{-\prime})$$

Hmwk: Calculate $J(s,\mu)$ at 1-loop

Sum Logs: $\alpha_s^k \ln^j(M_{a,b}^2/Q^2)$ with RGE for H, J, S

$$
\begin{aligned}
\int_0^{M^{cut}} d\sigma \sim\ & 1 + \alpha_s L^2 + \alpha_s^2 L^4 + \alpha_s^3 L^6 + \cdots && \} \ LL \\
& + \alpha_s L + \alpha_s^2 L^3 + \alpha_s^3 L^5 + \cdots && \} \ NLL \\
& + \alpha_s + \alpha_s^2 L^3 + \alpha_s^3 L^4 + \cdots && \\
& \qquad + \alpha_s^2 L^2 + \alpha_s^3 L^3 + \cdots && \} \ NNLL \\
& \qquad + \alpha_s^2 L + \alpha_s^3 L^2 + \cdots && \\
& \qquad + \alpha_s^2 + \alpha_s^3 L + \cdots && \} \ N^3LL \\
& \qquad\qquad + \underbrace{\alpha_s^3}_{N^3LL'} + \cdots &&
\end{aligned}
$$

known to this order

no large logs at these scale

$H(Q,\mu_H)$ ——— $\mu_H^2 \simeq Q^2$

U_H

$J(s,\mu_J)$ ——— $\mu_J^2 \simeq M_a^2$

U_J

$S(\ell^+,\ell^-,\mu_S)$ ——— $\mu_S^2 \simeq M_a^4/Q^2$

large logs in U_H & U_J

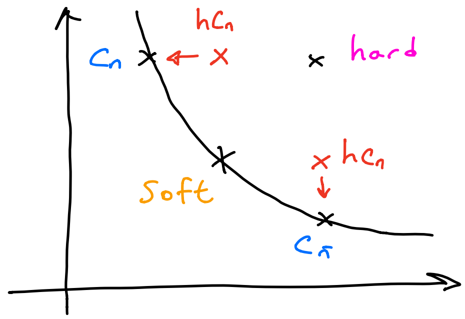
$$
\frac{d\sigma}{dM_a^2\, dM_b^2} = \sigma_0\, H(Q,\mu_H)\, \underline{U_H}(\mu_H,\mu_S) \int d\ell^+ d\ell^-\, S(\ell^+,\ell^-,\mu_S)
$$
$$
* \int ds\, J(s,\mu_J)\, \underline{U_J}(M_a^2 - Q\ell^+ - s, \mu_J, \mu_S)
$$
$$
* \int ds'\, J(s',\mu_J)\, \underline{U_J}(M_b^2 - Q\ell^- - s', \mu_J, \mu_S)
$$

U_H = Sudakov Form Factor

[U_J see Hmwk solution, has Sudakov dble logs too]

$$
\frac{\uparrow L^3}{\downarrow L^4}
$$

$SCET_{II}$



$$q = p_n + p_s \sim Q(\overset{+}{\lambda}, \overset{-}{1}, \overset{\perp}{\lambda})$$

$$q^2 = Q^2\lambda \gg Q^2\lambda^2 \;!$$

offshell

$q \sim Q(\lambda, 1, \sqrt{\lambda})$ on-shell scaling

hard-collinear mode

Constructing $SCET_{II}$ operators using $SCET_I$:

1) Match QCD to $SCET_I$ (hc_n, $hc_{\bar n}$, soft)
2) Factorize with field redefinition
3) Match $SCET_I$ to $SCET_{II}$ (c_n, $c_{\bar n}$, soft)

eg. $e^+e^- \to$ dijet $p_\perp$

$$J_{SCET_I} = \bar\chi_n^{hc}\,\Gamma\,\chi_{\bar n}^{hc}$$

$$J_{SCET_I} = \bar\chi_n^{hc}\,(Y_n^\dagger Y_{\bar n})\,\Gamma\,\chi_{\bar n}^{hc}$$

$$\Downarrow$$

$$J_{SCET_{II}} = \bar\chi_n\,(S_n^\dagger S_{\bar n})\,\Gamma\,\chi_{\bar n}$$

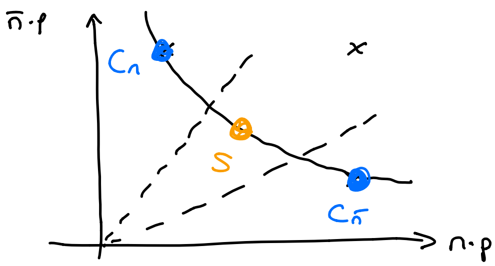
soft Wilson lines S_n & $S_{\bar n}$

- can also be obtained by matching QCD $\to$ $SCET_{II}$, but more work
- For T-products in $SCET_I$ with ≥ 2 operators having both usoft & collinear fields can get non-trivial coefficient from hc_n ($hc_{\bar n}$) $\quad \int dp^-\,dk^+\, J(p^-,k^+)\; C_n(p^-)\, S(k^+)$
  (usually subleading power)

$$\mathcal{L}^{(0)}_{SCET_{II}} = \mathcal{L}^{(0)}_{soft} + \sum_n \left(\mathcal{L}^{(0)}_{\xi n} + \mathcal{L}^{(0)}_{gn}\right) + \mathcal{L}^{(0)}_{Glauber}$$

already decoupled

same $SCET_I = SCET_{II}$



- $SCET_{II}$ also has 0-bin subtractions

eg. $C_n = C_n^{\text{naive}} - C_{ns}^{0}$

$k^\mu \sim (\lambda^2, 1, \lambda)$ ↑ take $k^\mu \sim \lambda$ in integrand & expand

$C_{\bar{n}} = C_{\bar{n}}^{\text{naive}} - C_{\bar{n}s}^{0}$

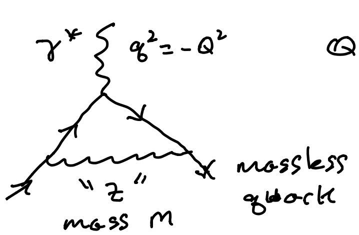
Rapidity Divergences

variable that distinguishes modes is rapidity Y

$e^{2Y} = p^- / p^+$ $\qquad e^{2Y} \sim \underset{C_n}{\lambda^{-2}}, \underset{S}{\lambda^0}, \underset{C_{\bar{n}}}{\lambda^2}$

Sometimes (but not always) we may have rapidity divergences from our separation of modes. Same p^2, so not regulated by dim. reg.

Simple Example: Massive Sudakov Form Factor



$Q^2 \gg m^2 \qquad J^\mu = \bar{\psi}\gamma^\mu\psi \qquad p^- = \bar{p}^+ = Q$

$$\langle q(\bar{p}) | J^\mu | q(p) \rangle = F(Q^2, m^2)\, \bar{u}\gamma^\mu u$$

$\lambda = \dfrac{m}{Q}$, Z can be:

C_n	$Q(\lambda^2, 1, \lambda)$
$C_{\bar{n}}$	$Q(1, \lambda^2, \lambda)$
S	$Q(\lambda, \lambda, \lambda)$

$$J^\mu_{SCET_{II}} = (\bar{\xi}_{\bar{n}} W_{\bar{n}})(S_{\bar{n}}^\dagger S_n)\gamma^\mu(W_n^\dagger \xi_n)$$

$\hookrightarrow F(Q^2, m^2) = \mathcal{H}\, C_{\bar{n}}\, S\, C_n$

Add regulator to Wilson lines

[just one possible regulator]

$$S_n = \sum_{\text{perms}} \exp\left(\frac{-g}{i n\cdot\partial_s} \frac{w\, \nu^{\eta/2}}{|2 i \partial_z|^{\eta/2}} n\cdot A_s \right)$$

$$W_n = \sum_{\text{perms}} \exp\left(\frac{-g}{i\bar{n}\cdot\partial_n} \frac{w^2\, \nu^{\eta}}{|\bar{n}\cdot i\partial_n|^{\eta}} \bar{n}\cdot A_n \right)$$

$|2i\partial_z| = |\bar{n}\cdot i\partial|$ up to power corr for W_n

Dim. reg like rapidity regulator $\quad \frac{1}{\eta}$ like $1/\epsilon$

$\qquad\qquad\qquad\qquad\qquad\qquad \ln\nu \qquad \ln\mu$

$$w^{bare} = w(\eta,\nu)\,\nu^{\eta/2}\,, \qquad \nu\frac{\partial}{\partial\nu}\, w(\eta,\nu) = -\frac{\eta}{2}\, w(\eta,\nu)$$

$$w(0,\nu) \equiv 1$$

w is book keeping parameter >

- Renormalize by 1st $\eta \to 0$, add $\frac{f(\epsilon)}{\eta}$ counterterm, then $\epsilon \to 0$ & $1/\epsilon$ counterterms

 & 

most IR div. integrals (scalar) integral is

$$\int d^dk \; \frac{1}{(k^2-m^2)(k^2+k^+p^-)\,k^-} \; \frac{w^2\nu^\eta}{|k^-|^\eta}$$

$$\text{Full } C_n = \frac{\alpha_s C_F\, w^2}{\pi}\Big[\underbrace{\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\frac{\mu}{m}\right)^{2\epsilon}}{2\eta} + \frac{1}{2\epsilon}\ln\frac{\nu}{p^-} + \frac{3}{8\epsilon}}_{\equiv\, (Z_n - 1)} + \ln\!\left(\frac{\mu}{m}\right) + \ln\!\left(\frac{\nu}{p^-}\right)\ln\!\left(\frac{\mu}{m}\right) + \text{constant}\Big]$$

$$C_n\!\left(m,\mu,\frac{\nu}{p^-}\right) = Z_{\xi_n}^{-1/2}\, Z_n^{-1}\, C_n^{bare}$$

same $+ \leftrightarrow -$

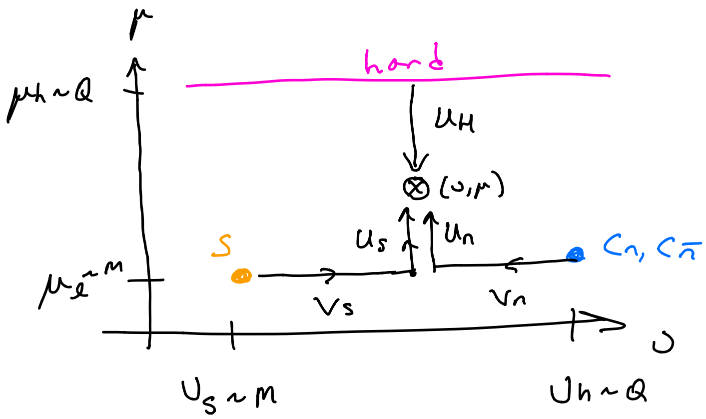
eg. $\displaystyle\int d^dk \; \frac{1}{(k^2-m^2)\,(k^+)(k^-)} \; \frac{w^2\nu^\eta}{|2k_z|^\eta}$

$$\text{Full } S = \frac{\alpha_s C_F\, w^2}{\pi}\Big[\underbrace{-\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\frac{\mu}{m}\right)^{2\epsilon}}{\eta} + \frac{1}{\epsilon}\ln\!\left(\frac{\mu}{\nu}\right) + \frac{1}{2\epsilon^2}}_{\equiv\, (Z_S - 1)} + \ln^2\frac{\mu}{m} - 2\ln\frac{\nu}{m}\ln\frac{\mu}{m} + \text{constant}\Big]$$

$$\text{Sum} = \frac{\alpha_s C_F}{\pi}\Big[\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu}{Q} + \frac{1}{\epsilon} + \ln^2\frac{\mu}{m} + 2\ln\frac{\mu}{m}\ln\frac{m}{Q} + 2\ln\frac{\mu}{m} + \text{constant}\Big]$$

- rapidity div. cancels btwn sectors
- same overall counterterm Z_C as SCET$_I$
- logs in C_n minimized for single μ,ν choice, & same for S

μ-RGE & ν-RGE to sum logs



$$\mu \frac{d}{d\mu} S = \gamma_\mu^S \, S$$

$$\nu \, d/d\nu \, S = \gamma_\nu^S \, S$$

etc.

- path independence $\left[\frac{d}{d\ln\mu}, \frac{d}{d\ln\nu}\right] = 0$

$\boxed{\mu}$

$$\gamma_\mu^S = -Z_S^{-1} \mu \, d/d\mu \, Z_S = \frac{\alpha_s(\mu) C_F}{\pi} \, 2 \ln \frac{\mu}{\nu}$$

$$\gamma_\mu^n = -Z_n^{-1} \mu \, d/d\mu \, Z_n = \frac{\alpha_s(\mu) C_F}{\pi} \left[\ln \frac{\nu}{Q} + \frac{3}{4} \right] = \gamma_\mu^{\bar n}$$

$\boxed{\nu}$

$$\gamma_\nu^S = -Z_S^{-1} \nu \, d/d\nu \, Z_S = -\frac{\alpha_s(\mu) C_F}{\pi} \, 2 \ln \frac{\mu}{M}$$

$$\gamma_\nu^n = -Z_n^{-1} \nu \, d/d\nu \, Z_n = \frac{\alpha_s(\mu) C_F}{\pi} \ln \frac{\mu}{M} = \gamma_\nu^{\bar n}$$

$$\gamma_\mu^S + \gamma_\mu^n + \gamma_\mu^{\bar n} = -\gamma_H \quad , \quad \gamma_\nu^S + \gamma_\nu^n + \gamma_\nu^{\bar n} = 0$$

since $Z_S^{-1} \left[\frac{d}{d\ln\mu}, \frac{d}{d\ln\nu}\right] Z_S = 0 \;\rightarrow\; \mu \, d/d\mu \, \gamma_\nu^S = \nu \frac{d}{d\nu} \gamma_\mu^S$ etc

which we can check

<u>solutions</u> are evolution kernels U_S, U_n, V_S, V_n

eg. $$U_S^{LL}(\mu, \mu_S; \nu_S) = \exp\left[-\frac{8\pi C_F}{\beta_0^2} \left(\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu_S)} - \frac{1}{\alpha_s(\nu_S)} \ln \frac{\alpha_s(\mu)}{\alpha_s(\mu_S)} \right) \right]$$

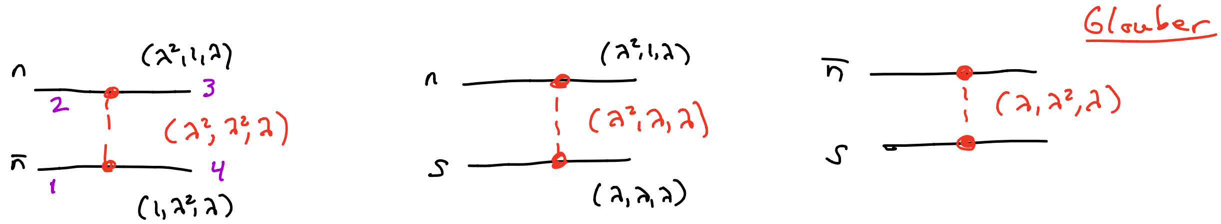
$$V_S^{LL}(\nu, \nu_S; \mu) = \exp\left[\frac{2 C_F}{\beta_0} \ln\left(\frac{\alpha_s(\mu)}{\alpha_s(M)} \right) \ln\left(\frac{\nu^2}{\nu_S^2} \right) \right]$$

see arXiv: 1202.0814 for further details

Glauber Exchange

see arXiv: 1601.04695

- modes with $p^+ p^- \ll \vec{p}_\perp^{\,2} \sim \lambda^2$ offshell
- needed? (not seen in standard matching calcs) add it & see
- mediates Forward scattering $s \gg -t$



$\frac{1}{\vec{p}_\perp^{\,2}}$ potentials, instantaneous in z & t

Forward: $\bar{n}\cdot p_2 = \bar{n}\cdot p_3$, $n\cdot p_1 = n\cdot p_4$

Match from QCD, integrating Glauber out:

$$\mathcal{L}_G^{(0)} = \sum_n \sum_{i,j=q,g} O_n^{iB} \frac{1}{P_\perp^2} O_s^{j_n B} + \sum_{n,n'} \sum_{i,j=q,g} O_n^{iB} \frac{1}{P_\perp^2} O_s^{BC} \frac{1}{P_\perp^2} O_{n'}^{jC}$$

(2-rapidities) (3-rapidities)

$$O_n^{qB} = \bar{\chi}_n T^B \frac{\bar{n}\!\!\!/}{2} \chi_n , \qquad O_n^{gB} = \frac{i}{2} f^{BCD} \mathcal{B}_{n\perp\mu}^C \frac{\bar{n}}{2}\cdot(i\overrightarrow{\partial}_n - i\overleftarrow{\partial}_n) \mathcal{B}_{n\perp}^{D\mu}$$

similar $O_{\bar{n}}$'s

$$O_s^{q_n B} = 8\pi\alpha_s\, \bar{\psi}_s^{\,n} T^B \frac{n\!\!\!/}{2} \psi_s^n , \qquad O_s^{g_n B} = 8\pi\alpha_s \frac{i}{2} f^{BCD} \mathcal{B}_{s\perp\mu}^{nC} \frac{n}{2}\cdot(i\overrightarrow{\partial}_s - i\overleftarrow{\partial}_s) \mathcal{B}_{s\perp}^{nD\mu}$$

$$O_s^{BC} = 8\pi\alpha_s \Big\{ P_\perp^\mu S_n^T S_{\bar{n}} P_{\perp\mu} - P_{\perp\mu}\, g \tilde{\mathcal{B}}_{s\perp}^{n\mu} S_n^T S_{\bar{n}} - S_n^T S_{\bar{n}}\, g \tilde{\mathcal{B}}_{s\perp}^{\bar{n}\mu} P_{\perp\mu}$$
$$- g \tilde{\mathcal{B}}_{s\perp}^{n\mu} S_n^T S_{\bar{n}} \tilde{\mathcal{B}}_{s\perp\mu}^{\bar{n}} - \frac{n_\mu \bar{n}_\nu}{2} S_n^T i g \tilde{G}_s^{\mu\nu} S_{\bar{n}} \Big\}^{BC}$$

Here $\psi_s^n = S_n^\dagger q_s$, $\mathcal{B}_{s\perp}^{n\mu} = \frac{1}{g}\left[S_n^\dagger i D_{s\perp}^\mu S_n\right]$

tildes: $\tilde{\mathcal{B}}_{s\perp}^{n\,AB} = -i f^{ABC} \mathcal{B}_{s\perp}^{nC}$

S_n adjoint wilson line

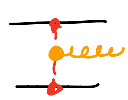
- suppressed: rapidity regulator $|k_z|^{-\eta}$, multipole expansion
- 0-bin subtractions (more soon)

<u>Note</u>
- construction involves using SCET p.c. theorem
- universal for $i, j = q, g$
- no hard coefficient or loop corrections to $\mathcal{L}_G^{(0)}$
- only pairs of collinear directions in $\mathcal{L}_G^{(0)}$, rest are T-products
- breaks factorization $\mathcal{L}_G^{(0)}(\{\xi_{n_i}, A_{n_i}\}, q_s, A_s)$
  coupling at $\mathcal{O}(\lambda^0)$ btwn n_i, n_j & s
- encodes known examples of fact. violation
  (Wilson line directions, $i\pi$'s, ...)
- SCET vs. CSS Glauber

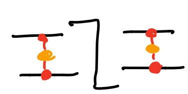
  SCET: expand first, defined as contribution that can be independently calculated

  CSS: deform contour, see where we are trapped
  make soft expn once out of trapped region

- One-gluon Feyn. Rule of O_S^{AB} is Lipatov Vertex



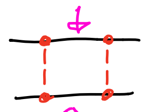
rapidity RGE for $\mathcal{L}_G^{(0)}$ (Amplitude level) gives
gluon reggeization $\left(\frac{\nu^2}{\nu_0^2}\right)^{-\gamma_{n\nu}} = \left(\frac{s}{-t}\right)^{-\gamma_{n\nu}}$

rapidity RGE for  gives BFKL equation

$$\nu \frac{\partial}{\partial \nu} S(q_\perp, q_\perp', \nu) = \int d^2 k_\perp \, \gamma^{BFKL}(q_\perp, k_\perp)\, S(k_\perp, q_\perp', \nu)$$

useful for small-x resummation

— Glauber Loops give $i\pi$

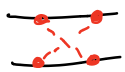


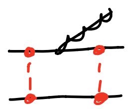
effectively eikonal

$$\int \frac{d^dk \; |2k^z|^{-\eta}\, \nu^{2\eta}}{k_\perp^2 (k_\perp - q_\perp)^2 \left(k^+ - \Delta_1(k_\perp) + i0\right)\left(-k^- - \Delta_2(k_\perp) + i0\right)}$$

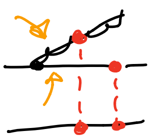
$$= \left(\frac{-i}{4\pi}\right) \int \frac{d^{d-2}k_\perp}{k_\perp^2 (k_\perp - q_\perp)^2} \left[-i\pi + O(\eta) \right]$$

Δ's matter here

 $= 0$ with regulator

 $= 0$ (can't collapse to equal t & z)

not eikonal

 $\neq 0$

effectively eikonal

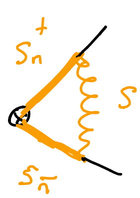
— Wilson Line Directions

$\dfrac{1}{\bar n \cdot k \pm i0}$ in W_n sign matters for $\delta(\bar n \cdot k)$ not collinear

$\dfrac{1}{n \cdot k \pm i0}$ in S_n " " " $\delta(n \cdot k)$ not soft

actually Glauber

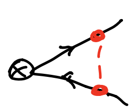
eg 1

$S =$ 

naive $\tilde S = \displaystyle\int \frac{d^dk \; |k_z|^{-\eta}}{(k^2 - m_{SR}^2)(n\cdot k + i0)(\bar n \cdot k - i0)}$

$= (\ldots) + i\pi \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right)$

true $S = \tilde S - S^{(G)} = (\ldots)$ only (0-bin subt.)

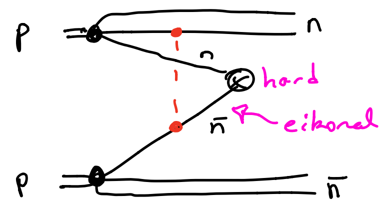
$G =$ 

$G = S^{(G)}$ here, pure $i\pi$

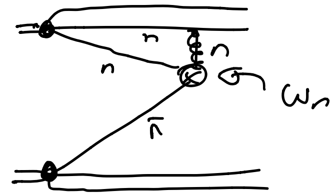
$S + G = \tilde S$

- G carries info about Soft Wilson line directions
- can absorb G into soft if we take proper directions for S_n lines

eg 2



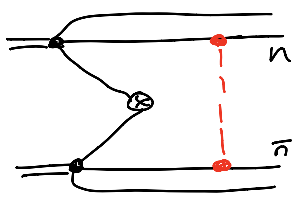
G



$C_n = \tilde C_n - C_n^{(G)}$

- direction dependence in G not C_n
- can absorb G into C_n

eg 3



Spectator - Spectator

no soft or collinear analogs at leading power

cancel for $|A|^2$ with integration over $\Delta p_\perp$ of spectators

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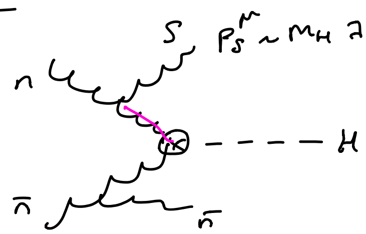
TMD example with rapidity RGE

$gg \to$ Higgs p_T dist'n

$P_n \sim M_H(\lambda^2, 1, \lambda)$



$P_S^\mu \sim M_H \lambda$

soft radiation SCET$_{II}$

$P_T^H \sim M_H \lambda$

$P_{\bar n} \sim M_H(1, \lambda^2, \lambda)$

$$\frac{d\sigma}{dp_{TH}\, dy} = N_0\, H(M_H, \mu) \int d^2p_{1\perp}\, d^2p_{2\perp}\, d^2p_{s\perp}\, \delta\left(\vec p_T^{H\,2} - |\vec p_{1\perp} + \vec p_{2\perp} + \vec p_{s\perp}|^2\right)$$

$$* f^{\mu\nu}_{g/p}\left(\frac{M_H}{E_{cm}} e^{-Y}, \vec p_{1\perp}, \mu, \frac{\nu}{M_H e^{-Y}}\right)$$ $\mu \sim \vec p_\perp$, $\nu \sim M_H$ $\leftarrow \langle p_n | \mathcal{B}^{A\mu}_{n\perp}(x^+, \vec x_\perp) W_T\, \mathcal{B}^{A\nu}_{n\perp}(0) | p_n \rangle$

has 0-bin subtractions $= \dfrac{\tilde f^{\,\text{naive}}_{g/p}}{S^{\text{0-bin}}}$

$$* f_{g/p}^{\mu\nu}\left(\frac{M_H}{E_{cm}} e^{Y}, \vec p_{2\perp}, \mu, \frac{\nu}{M_H e^{Y}}\right)$$

$$* S\left(\vec p_{s\perp}, \mu, \nu/\mu\right)$$ $\nu \sim \mu \sim \vec p_\perp$

Often: $f^{TMD}(x, \vec x_\perp, \mu, Q) = f_{g/p} \sqrt{S} = \dfrac{\tilde f^{\,\text{naive}}_{g/p}}{S^{\text{0-bin}}} \sqrt{S}$