Quantum Field Theory III 8.325

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Ch1 Standard Model Intro & Review

Focus On: Symmetries and Quantum Numbers

We'll discuss electromagnetism, QCD, and weak interactions with the goal of
- providing a review of material from QFT-II (8.324)
- introducing topics to be done in detail in later chapters (a course overview)

Gauge Symmetry

$U(1)$ electromagnetism for fermion $\psi(x)$ with charge Q

QED

invariant under $\psi(x) \to e^{iQ\alpha(x)}\psi(x) = U(x)\,\psi(x)$, $U^\dagger U = 1$

$U = 1 + iQ\alpha(x) + \cdots$, often α- infinitesimal

$i\partial^\mu$ messes up $U(1)$, need $QA_\mu \to QA_\mu + \frac{i}{e}U(\partial_\mu U^{-1})$

ie $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$ [exact]

use $iD^\mu\psi = (i\partial^\mu + eQA^\mu)\psi$, $iD^\mu\psi \to U\, iD^\mu\psi$

here e is coupling and Q is charge ($Q=-1$ for e^-)

Can Form $[iD^\mu, iD^\nu]\,\psi = ieQ\,F^{\mu\nu}\psi$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $F^{\mu\nu} \to F^{\mu\nu}$ invariant

$$\mathcal{L}_{QED} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$\mathcal{L}_{QED} \to \mathcal{L}_{QED}$ invariant

<u>SU(3) color</u> for QCD

$\psi(x) \to U(x)\,\psi(x) \qquad U(x) = e^{i\alpha^A(x)T^A} \qquad A = 1,\dots,8$

$T^A = \frac{\lambda^A}{2}$ Gell-Mann matrices , $[T^A, T^B] = i f^{ABC} T^C$

Now $A_\mu \equiv A_\mu^A T^A \quad , \quad F_{\mu\nu} \equiv F_{\mu\nu}^A T^A$

$A_\mu \to U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^{-1}$ [QED analog...]

$F_{\mu\nu} \to U\, F_{\mu\nu}\, U^{-1}$ $\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}]$ invariant

$iD_\mu\psi \to U\, iD_\mu\psi$ where $iD_\mu\psi = (i\partial_\mu + g\,T^A A_\mu^A)\,\psi$

"charges" are SU(3) matrices

$[iD^\mu, iD^\nu]\,\psi = ig\,F^{\mu\nu}\,\psi$

$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g\,f^{ABC} A_\mu^B A_\nu^C$

not neutral

Also $iD_\mu A_\nu^A = (i\partial^\mu \delta^{AC} + ig\,f^{ABC} A_\mu^B)\,A_\nu^C$

gluons are charged

so $\mathcal{L}_{QCD} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu}^A F^{\mu\nu A}$

- renormalizable , dim ops ≤ 4 , no $g'\,\bar\psi\,\sigma^{\mu\nu}F_{\mu\nu}\psi$ term. The standard reasoning is that we impose a cutoff Λ for UV divergences (hopefully gauge inv.) then demand that all divergences can be absorbed into our parameters when $\Lambda \to \infty$ (ie. cutoff removed, so if $d = 4-2\epsilon$, $\epsilon \to 0$ in dim. reg.)

The "renormalization group" will allow us to make a stronger statement for finite Λ. (Ch. 2.)

- Free Quarks have never been isolated. We'll see why this doesn't bother us, and how it impacts the meaning of "g" and "m".
  (asymptotic freedom Ch. 2, confinement Ch 8)

- We skipped the gauge inv. dim-4 operator $\theta\, \epsilon^{\mu\nu\lambda\tau} F^A_{\mu\nu} F^A_{\lambda\tau}$ which can be written as a total derivative $\partial^\mu K_\mu$. It will matter for QCD vacuum discussion (Ch. 7)

<u>SU(3) × SU(2) × U(1)</u> S.M. gauge group

Lets infer the charge (representations) of the fields and use gauge symmetry to find $\mathcal{L}$.

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{Fermi} + \mathcal{L}_{Higgs} + \mathcal{L}_{Y_R}$$

① $\mathcal{L}_{gauge} = -\frac{1}{4} \underbrace{B^{\mu\nu} B_{\mu\nu}}_{U(1)} \; -\frac{1}{4} \underbrace{W^a_{\mu\nu} W^{a\mu\nu}}_{SU(2)} \; -\frac{1}{4} \underbrace{F^A_{\mu\nu} F^{\mu\nu A}}_{SU(3)}$

$a = 1,2,3$ $\qquad A = 1, .., 8$

generators $T^a = \frac{\sigma^a}{2}$

$[T^a, T^b] = i \epsilon^{abc} T^c$

adjoint reps
so for 3 distinct sectors, no U(1) charge for gluon etc.

② $\mathcal{L}_{Fermi} = \bar{\psi}\, i \not{D}\, \psi$

What ψ are observed?

ψ choices $\begin{pmatrix} u \\ d \end{pmatrix}$ $\begin{pmatrix} c \\ s \end{pmatrix}$ $\begin{pmatrix} t \\ b \end{pmatrix}$ quarks come in 6 flavors

$\begin{pmatrix} \nu_e \\ e \end{pmatrix}$ $\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}$ $\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$ leptons

family $i=1$ $i=2$ $i=3$

$$iD^\mu = i\partial^\mu + g_1 Y B^\mu + g_2 T^a W^{\mu a} + g\, T^A A^{\mu A}$$

need reps — in appropriate reps

- $\boxed{\text{color}}$ quarks are charged, triplets [fundamental of $SU(3)$]
  same T^A for all of them, color is flavor blind
  leptons are singlets

- $\boxed{SU(2)}$ breaks parity, acts on left-handed fields only
  $\psi_L = P_L \psi$, $P_L = \frac{(1-\gamma_5)}{2}$, $P_L P_L = P_L$

  charged matter in doublets $\begin{pmatrix} u_L \\ d_L \end{pmatrix} = Q_L$, $\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} = L_L$
  (similar for other families, so study $i=1$)

  $\psi_R = P_R \psi$, $P_R = \frac{(1+\gamma_5)}{2}$ in singlets , u_R, d_R, e_R

Note: Assignment of gauge trnsfm must be consistent with (proper orthochronous homogeneous) Lorentz group. Labelling irreducible reps (A,B), we have Weyl spinor $(\frac{1}{2},0)$ as L-hand part of Dirac and $(0,\frac{1}{2})$ as R-hand part. Hence $(\frac{1}{2},0)$ & $(0,\frac{1}{2})$ can be gauged differently

What about ν_R? Its in $\mathcal{L}_{\nu R}$, which can be dropped if $m_{\nu i}=0$ since these particles decouple [colorless, R-handed, neutral] But not all $m_{\nu i}=0$! [Ch. 4.]

Mass Term? $\quad m\bar{\psi}\psi = m\psi^\dagger\gamma^0(P_L P_L + P_R P_R)\psi$

$$= m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$$

but $\psi_L \to U\psi_L$, so mass term <u>violates</u> our $SU(2)$
$\psi_R \to \psi_R$ gauge inv.!

Same for $m_W^2 W_\mu^a W^{\mu a}$ for W-boson.

Motivates $\mathcal{L}_{Higgs}$

- $\boxed{U(1)}$ charges. This is not electromagnetism because $L_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$ can only have single $U(1)$ charge (else breaks $SU(2)$)

E.M. hides inside $SU(2)\times U(1)$. Notice that experimental e.m. charge of (upper) − (lower) components in

$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ is $\frac{2}{3} - (-\frac{1}{3}) = +1 = 0 - (-1)$

$$\Delta Q = \Delta T_3$$

[linear combo of generators] $\to$ $Q = T^3 + Y$

$\uparrow \pm\frac{1}{2}$ for L-doublet
0 for R-singlet

$Y = -\frac{1}{2}$	L_L
$\frac{1}{6}$	Q_L
$\frac{2}{3}$	u_R
$-\frac{1}{3}$	d_R
-1	e_R

The charges seem somewhat arbitrary, but they are constrained by anomalies [Ch. 5.]. We assumed $Q(u_L) = +\frac{2}{3}$, $Q(d_L) = -\frac{1}{3}$

How do we measure these? [Ch. 7.]

Now

$$iD^\mu Q_L = i\partial^\mu Q_L + \frac{g_1}{6}B^\mu Q_L + g_2 W^{\mu a}\left(\frac{\sigma^a}{2}\right)Q_L + g A^{\mu A}\left(\frac{\lambda^A}{2}\right)Q_L$$

$$iD^\mu e_R = i\partial^\mu e_R - g_1 B^\mu e_R$$

etc.

Lets add a family index $i = 1,2,3$ and summarize

Field	SU(3)	SU(2)	U(1)	Lorentz
$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$	3	2	1/6	$(\frac{1}{2},0)$
u_R^i	3	1	2/3	$(0,\frac{1}{2})$
d_R^i	3	1	$-1/3$	$(0,\frac{1}{2})$
$L_L^i = \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix}$	1	2	$-1/2$	$(\frac{1}{2},0)$
e_R^i	1	1	-1	$(0,\frac{1}{2})$
ν_R^i	1	1	0 sterile	$(0,\frac{1}{2})$
A_μ^A	8	1	0	$(\frac{1}{2},\frac{1}{2})$
W_μ^a	1	3	0	$(\frac{1}{2},\frac{1}{2})$
B_μ	1	1	0	$(\frac{1}{2},\frac{1}{2})$
$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$	1	2	$Y_H = ?$	$(0,0)$
	leave SU(3) unbroken	break SU(2)		

To generate masses we add H (Higgs Mechanism) and spontaneously break $SU_L(2) \times U_Y(1) \to U_Q(1)$
(Quantization of electroweak theory in Ch. 3.)

Lets sketch the basic idea so we can find Y_H

and see how gauge symmetry restricts $\mathcal{L}_{Higgs}$.

Complex Scalar $H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$$\mathcal{L}_{Higgs} = (D_\mu H)^\dagger (D^\mu H) + \mu^2 (H^\dagger H) - \lambda (H^\dagger H)^2 + \mathcal{L}_{Yukawa}^{\mp 4H}$$

$\uparrow$ sign carefully chosen, complete the square

$$= (D_\mu H)^\dagger (D_\mu H) - \lambda \underbrace{\left(H^\dagger H - \frac{\mu^2}{2\lambda} \right)^2} + \text{const.} + \mathcal{L}_{Yukawa}$$

potential $V(H)$ which is <u>not</u> minimized at $\langle 0|H|0\rangle = 0$. Not the minimum energy vacuum!

Higgs Mechanism (Ch. 3)

$\langle 0|h_1|0\rangle = 0$, $\langle 0|h_2|0\rangle = v/\sqrt{2}$ $\leftarrow$ breaks Symmetry $SU_W(2) \times U_Y(1)$

$v = \sqrt{\mu^2/\lambda}$ to minimize potential

[h_2 charged under $SU_L(2) \times U_Y(1)$]

$$QH = \begin{pmatrix} (\frac{1}{2} + Y_H)\, h_1 \\ (-\frac{1}{2} + Y_H)\, h_2 \end{pmatrix}$$

do not want to break electromagnetism $\rightarrow$ pick $Y_H = \frac{1}{2}$

Most General

$$\mathcal{L}_{Yukawa} = -g_e^{ij}\, \underbrace{\bar{e}_R^i \underbrace{H^\dagger L_L^j}_{SU(2)\text{ singlet}}} - g_d^{ij}\, \bar{d}_R^i H^\dagger Q_L^j + g_u^{ij}\, \bar{u}_R^i \underbrace{H^T \epsilon Q_L^j}_{SU(2)\text{ singlet}}$$

$+ \text{h.c.}$

$Y = 1 - \frac{1}{2} - \frac{1}{2} = 0$

for any family indices i, j

Note: $\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, 2 is pseudo-real rep in $SU(2)$

Let $U = e^{i\alpha^a \sigma^a}$ then $\epsilon U \epsilon = -U^*$ $\Rightarrow$ $U^T \epsilon U = \epsilon$

(pseudo-real)

$\uparrow$

check this for

$U = 1 + i\alpha^a\sigma^a + \ldots$

$\epsilon^2 = -\mathbb{1}$

$\uparrow$ this makes g_u^{ij} term gauge invariant

Form of $\mathcal{L}_{Higgs}$ is fairly constrained by gauge inv.
— (still not as nice as $\mathcal{L}_{gauge} + \mathcal{L}_{Fermi}$)

<u>Masses:</u>

take $H = \begin{pmatrix} 0 \\ h^0 \end{pmatrix}$ $\quad h^0 \to \frac{v}{\sqrt{2}}$

$$D^\mu H = \frac{g_1}{2} B^\mu \begin{pmatrix} 0 \\ h^0 \end{pmatrix} + g_2 W^{\mu a} \left(\frac{\sigma^a}{2}\right) \begin{pmatrix} 0 \\ h^0 \end{pmatrix} = \frac{h^0}{2} \begin{pmatrix} g_2 \overbrace{(W^{\mu 1} - i W^{\mu 2})}^{W^{+\mu}} \\ \underbrace{g_1 B^\mu - g_2 W^{\mu 3}}_{\text{massive } Z^\mu} \end{pmatrix}$$

- $(D^\mu H)^\dagger (D_\mu H)$ has $\underbrace{\left(\frac{g_2^2 v^2}{8}\right)}_{\text{mass for } W^{\mu 1}} W^{\mu 1} W_{\mu 1}$ etc

- $g_e \bar{e}_R H^\dagger L_L$ + h.c. has $g_e h_0 \bar{e}_R e_L$ + h.c
  $\hookrightarrow g_e \frac{v}{\sqrt{2}} (\bar{e} e)$ mass for e^-

[more on masses later Ch3 & Ch4. Ghosts, family indices etc]

<u>Remark</u> We constructed $\mathcal{L}_{SM}$ based on gauge symmetry but we could instead have started by listing the bosons & fermions and constructing all operators with $d \le 4$

$$\partial^\mu \phi \, \partial_\mu \phi, \; \phi \, \partial_\mu \phi \, A^\mu, \; \phi \bar{\psi} \psi, \; \phi^2 A^\mu A_\mu, \; \bar{\psi} \not{A} \psi, \text{ etc.}$$

then imposing gauge invariance would give relations between the coefficients of all these operators.

Symmetries of the SM

Discrete Symmetries

CPT Theorem

P $\quad (x^0, \vec{x}) \rightarrow (x^0, -\vec{x}) = x_P \qquad P\psi(x)P^{-1} = \gamma^0 \psi(x_P)$

T $\quad (x^0, \vec{x}) \rightarrow (-x^0, \vec{x}) = x_T$

C $\quad$ particles $\rightarrow$ antiparticles $\qquad C\psi(x)C^{-1} = \mathcal{C}\,(\bar{\psi})^T$

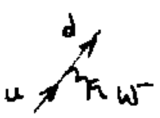
$\uparrow$ representation dependent, $\mathcal{C} = -i\gamma^2\gamma^0$ in Peskin

- $\int d^4x \; \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under P, C, T
  for QED
  & for QCD $\left(\begin{array}{l} \theta F\tilde{F} \text{ term in QCD} \\ \text{breaks P, CP, but } \theta \text{ v. small} \end{array} \right)$

eg. $\quad m\bar{\psi}\psi(x) \xrightarrow{P} m\bar{\psi}\psi(x_P) \qquad d^4x = d^4x_P$

$$\bar{\psi}\gamma^\mu\psi \xrightarrow{C} -\bar{\psi}\gamma^\mu\psi$$
$$A^\mu \rightarrow -A^\mu$$

- Weak Interactions violate P, C and also CP



$\bar{\psi}_1\gamma^\mu P_L\psi_2 \xrightarrow{P} \bar{\psi}_1\gamma^\mu P_R\psi_2$ $\qquad$ so violated

$\xrightarrow{C} \bar{\psi}_2\gamma^\mu P_R\psi_1$ $\qquad$ " "

$\xrightarrow{CP} \bar{\psi}_2\gamma^\mu P_L\psi_1$

hence $\bar{\psi}_1\gamma^\mu P_L\psi_2$ + h.c. is CP invariant

but $\lambda\,\bar{\psi}_1\gamma^\mu P_L\psi_2$ + h.c. " " " ONLY if coupling $\lambda = \lambda^*$

In Ch. 4 we'll prove there is one imaginary coupling in SM.

Classifying Global Symmetries

- Exact Symmetry

- Approximate Symmetry. Broken by "small" terms in $\mathcal{L}_{SM}$

  e.g. $SU(2)$ isospin $\begin{pmatrix} u \\ d \end{pmatrix}$ is broken by $\frac{m_u - m_d}{\Lambda}$, $\alpha_{em} \simeq \frac{1}{137}$

  and is a good symmetry for bound hadrons in QCD

  What is Λ? $\bar{\Psi}\,(i\partial\!\!\!/ - m_u)\,\Psi$

  $\uparrow\ p \sim (1\,\text{fm})^{-1} \sim 200$ MeV $\sim \Lambda$

1fm

- "Spontaneously Broken" by vacuum expectation value, a hidden symmetry. $\mathcal{L}$ is symmetric, but ground state is not. Symmetry still has implications for dynamics. [Ch. 3]

- Anomalous. Classical Symmetry is not a symmetry of the Quantum theory.
  (breaking here can still be small) [Ch. 5]

Lets look at classical symmetries of $\mathcal{L}_{SM}$.

Global Symmetries of $\mathcal{L}$

fields ϕ^i, invariance of $\mathcal{L}$ under

$$\phi^i \to U^{ij}(\epsilon)\,\phi^j$$
$$\to (\delta^{ij} + i\epsilon T^{ij})\,\phi^j \qquad \epsilon \text{ - infinitesimal}$$

Construct Conserved current (Noether)

$$\pi^\mu_i \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}, \qquad \partial_\mu \pi^\mu_i = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = \frac{\partial \mathcal{L}}{\partial \phi^i}$$

using equation of motion from stationary action

$$J^\mu = \pi^\mu_i \,(i T^{ij})\,\phi_j$$

[Check: $$\partial_\mu J^\mu = (\partial_\mu \pi^\mu_i)(i T^{ij}\phi_j) + \pi^\mu_i (i T^{ij}\partial_\mu \phi_j)$$
$$= \frac{\partial \mathcal{L}}{\partial \phi^i}(i T^{ij}\phi_j) + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^i)}(i T^{ij})\,\partial_\mu \phi_j$$
$$= \delta_\epsilon \mathcal{L} = 0 \;]$$

In QFT the charge $Q = \int d^3x\, J^0$, $\pi^j \equiv \pi^j_{\mu=0}$

and $$[\phi^i(\vec{x},t), \pi^j(\vec{y},t)] = i\,\delta^{ij}\,\delta^3(\vec{x}-\vec{y})$$

So $$[Q(t), \phi^j(\vec{y},t)] = \int d^3x\, [\pi^i(\vec{x},t), \phi^j(\vec{y},t)]\,(i T^{ik}\phi_k(\vec{x},t))$$
$$= + T^{jk}\,\phi_k(\vec{y},t)$$
$$[Q(t), \pi^j(\vec{y},t)] = + T^{jk}\,\pi_k(\vec{y},t)$$
$$[Q(t), \mathcal{O}(t)] = -i\,\delta_\epsilon\,\mathcal{O}$$

built from ϕ & π's

$$[Q(t), H] = -i\,\delta_\epsilon H = 0 \qquad \text{so} \quad \frac{dQ}{dt} = 0$$

conserved

<u>$U(1)$ Baryon #</u> $\quad \psi_i \to e^{-i\theta}\psi_i$ for each quark "i"

(here $T_{ij} = (-\mathbb{1})_{ij}$)

$\bar\psi i \not{D} \psi \; - g_d \bar{d} H^+ Q_L + g_u \bar{u}_R H^T \epsilon Q_L$ invariant

<u>$J^\mu = ?$</u> $\quad \Pi^\mu_i = \bar\psi_i (i\gamma^\mu)$

$$J^\mu = \sum_{i \in \text{quarks}} \bar\psi_i \gamma^\mu \psi_i$$

an "accidental" symmetry of $\mathcal{L}_{SM}$

($\mathcal{J}$ – B & L anomalous due to weak effects, very small)

<u>Lepton #</u> $\quad \psi_i \to e^{-i\theta}\psi_i \quad$ "i" = lepton

in fact neglecting $\mathcal{L}_{\nu R}$ we can diagonalize $\mathcal{L}_{SM}$

$g_e^{ij} \bar{e}_R^i H^+ L_L^j \Rightarrow g_e^{ii} \bar{e}_R^i H^+ L_L^i$

by unitary redefinitions of e_R, L_L (later)

then $\quad e$ # $\quad \psi_i \to e^{-i\theta_e}\psi_i \quad i \in (e, \nu_e)$

μ # $\quad e^{-i\theta_\mu}\psi_i \quad$ etc

τ # $\quad e^{-i\theta_\tau}\psi_i$

<u>Quark #'s ?</u> $\quad$ Strangeness, Charm, ...

g_u^{ij}, g_d^{ij} $\quad$ either Yukawa or <u>Weak</u>

<u>interactions</u> are not diagonal

but

strangeness $s \to e^{-i\theta_s} s$ is symmetry of QCD, QED

$\langle \pi^+ e^- \bar\nu_e \mid H_{weak} \mid \bar{K}^0 \rangle$

$\underset{u\bar{d}}{} \quad \underbrace{}_{\Delta S = 1} \quad \underset{s\bar{d}}{}$

can use Wigner-Ekhart since often weak interactions act only once.

<u>U(1) axial</u> $\quad m_u, m_d \to 0 \quad (\ll \Lambda) \qquad i = u, d$

Consider $\quad \psi_i \to e^{-i\theta\gamma_5}\psi_i$

$$J^{\mu(5)} = \sum_{i=u,d} \bar{\psi}_i \gamma^\mu \gamma_5 \psi_i$$

Or $\quad \begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} e^{-i\theta\gamma_5}\, u \\ e^{+i\theta\gamma_5}\, d \end{pmatrix}, \quad J^\mu = \bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d$

These U(1)'s are anomalous due to strong & QED effects

<u>SU(2) isospin</u> $\quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} \to \exp(-i\tau\cdot\theta)\,\psi \qquad m_u, m_d \ll \Lambda$

$\qquad\qquad\qquad\qquad$ ↑ Pauli matrices $\qquad \alpha_{em} = 0$

3 θ's : θ^a

$$J_\mu^a = \sum_i \bar{\psi}_i \gamma_\mu \tau^a \psi_i = \bar{\psi}\gamma_\mu \tau^a \psi$$

<u>Flavor SU(3)</u>

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad m_u, m_d, m_s \ll \Lambda$$

$$J_\mu^A = \bar{\psi}\gamma_\mu \lambda^A \psi$$

<u>Heavy Quark Symmetry</u> $\quad m_c, m_b \to \infty \quad (\gg \Lambda)$

$$\psi = \begin{pmatrix} c\uparrow \\ c\downarrow \\ b\uparrow \\ b\downarrow \end{pmatrix} \qquad U(2) \text{ flavor } \& \ SU(2) \text{ spin} \to U(4) \text{ symmetry}$$

<u>Chiral Symmetry</u> $\quad m_u, m_d \to 0 \qquad SU(2)_L \times SU(2)_R$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta_L^a \qquad \theta_R^a$

$$\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \to \exp(-i\sigma\cdot\theta_L)\,\psi_L, \quad \psi_R \to \psi_R$$

$$\psi_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to \exp(-i\sigma\cdot\theta_R)\,\psi_R, \quad \psi_L \to \psi_L$$

$$J_\mu^{aL} = \bar{\Psi}_L \gamma^\mu \sigma^a \Psi_L$$
$$J_\mu^{aR} = \bar{\Psi}_R \gamma^\mu \sigma^a \Psi_R$$

Spontaneously broken in QCD

$$SU(2)_L \times SU(2)_R \rightarrow SU(2)_{isospin}$$

Similar for $SU(3)_R \times SU(3)_L \rightarrow SU(3)_{flavor}$
$(m_{u,d,s} \to 0)$

Classes of Diagrams & Generating Functions

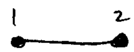
Consider $\lambda \phi^3$ theory with $\mathcal{L}(\phi)$

- Feynman graphs can be generated with

$$Z[J] = \int D\phi \; e^{i\int d^4x (\mathcal{L} + J\phi)} \qquad \text{since}$$

$$\frac{1}{Z[0]}\left(\frac{-i\delta}{\delta J(x_1)}\right)\cdots\left(\frac{-i\delta}{J(x_n)}\right) Z[J]\Big|_{J=0} = \frac{\int \mathcal{D}\phi \; \phi(x_1)\cdots\phi(x_n)\; e^{i\int(\mathcal{L}+J\phi)}}{\int D\phi \; e^{i\int(\mathcal{L}+J\phi)}}$$

$$= \langle 0 | T \, \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

$n=2$  ,  ,  , ...

- Connected Feyn. Graphs are generated by $W[J]$ where

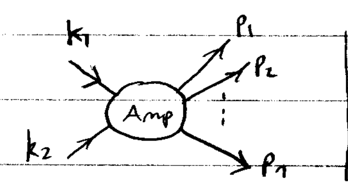
$$Z[J]/Z[0] = e^{iW[J]}$$

$$\left(\frac{-i\delta}{\delta J(x_1)}\right)\cdots\left(\frac{-i\delta}{J(x_n)}\right) W[J] = \langle 0 | T \, \phi(x_1)\cdots\phi(x_n)|0\rangle_{conn}$$

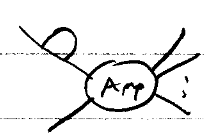
$n=2$  , ...

- Observables like cross sections and decay rates are determined by S-matrix, $S = 1 + iT$

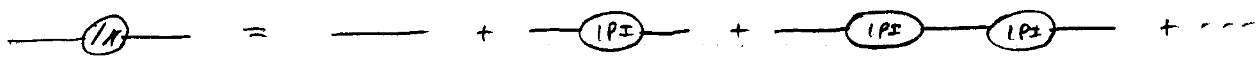
$$\langle p_1 p_2 \cdots p_n | iT | k_1 k_2 \rangle = (2\pi)^4 \, \delta^{(4)}(k_1+k_2-p_1-p_2-\cdots p_n) \; i\mathcal{A}(k_1 k_2 \to p_1 p_2 \cdots)$$

by LSZ $\qquad = (\sqrt{Z})^{2+n}$  connected & amputated graphs

for each extend line

amputated means no loops that dress single external line like  . Its often convenient to use Feyn. rules that directly give $i\mathcal{A}$.

Here Z is residue of single particle pole. If exact $_{\wedge}$ connected 2-pt function is



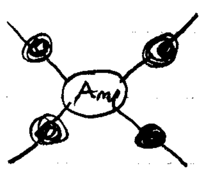
$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-i\Sigma)\frac{i}{p^2-m_0^2} + \cdots = \frac{i}{p^2-m_0^2-\Sigma(p^2)}$$

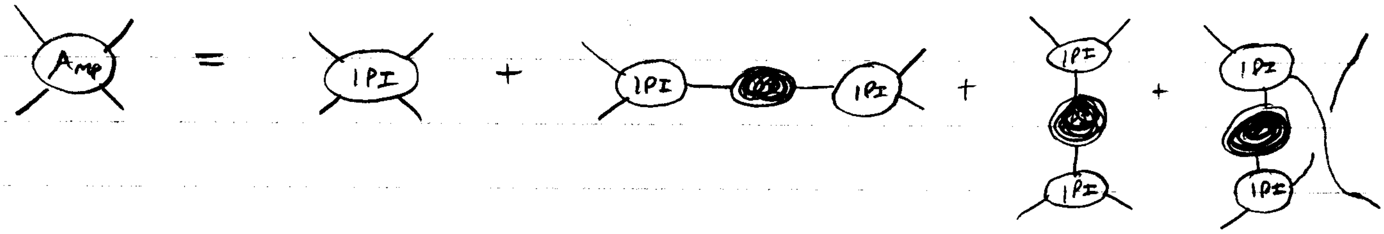
$$= \frac{iZ}{p^2-m^2} + \text{regular}$$

1PI = one-particle irreducible, graphs that do not fall apart if a single line is erased

- The 1PI graphs are the only ones we need to consider to renormalize the theory. In fact they are the basic building blocks of all other graphs

eg. 4-pt function





The generating function for 1PI graphs is called the effective action, $\Gamma[\varphi]$

Lets derive a relation between $\Gamma[\varphi]$ and $W[J]$

$$\Gamma[\varphi] = +\frac{1}{2}\int đ^d k\ \tilde{\varphi}(-k)\left(k^2-m^2-\Sigma(k^2)\right)\tilde{\varphi}(k)$$

$$+\sum_{n=3}^{\infty}\frac{1}{n!}\int đ^d k_1 \cdots đ^d k_n\ (2\pi)^d\ \delta^d(k_1+\dots+k_n)\ \underbrace{V_n(k_1,\dots,k_n)}\ \tilde{\varphi}(k_1)\cdots\tilde{\varphi}(k_n)$$

momentum space result for 1PI n-pt function

- can Fourier transform to position space if desired
- Tree graphs with this action give complete scattering

Consider $Z_\Gamma[J] = \int \mathcal{D}\varphi\ e^{i\Gamma[\varphi] + i\int d^dx\, J\varphi} = e^{iW_\Gamma[J]}$

$$W_\Gamma[J]\Big|_{\text{tree graphs}} = W[J]$$

To isolate tree graphs introduce $\hbar$, $\exp\left[\frac{i}{\hbar}\left(\Gamma[\varphi] + \int J\varphi\right)\right]$

each propagator gives $\hbar$
" source " $\hbar^{-1}$
" vertex " $\hbar^{-1}$

$$\hbar^{\overbrace{P}^{\#\text{ internal lines}}-\overbrace{E}-\overbrace{V}^{\#\text{ mom. conserving }\delta\text{-functions}}} = \hbar^{\overbrace{L}^{\#\text{ loops}}-\overbrace{1}^{1\text{ overall }\delta\text{-function}}}$$

so $W_\Gamma[J] = \sum_{L=0}^{\infty}\hbar^{L-1}\,W_\Gamma^{(L)}[J]$

and $\hbar\to 0$ isolates $L=0$, tree graphs

For $\hbar\to 0$ use stationary phase approx for path integral

$\Rightarrow$ $\dfrac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x)$ quantum eqtn of motion

let $\varphi_J(x)$ be the solution, then

$$Z_\Gamma[J] = \exp\left(\frac{i}{\hbar}\underbrace{\left(\Gamma[\varphi_J] + \int d^dx\, J\varphi_J\right)}_{\text{tree's}} + \mathcal{O}(\hbar^0)\right)$$

so $\boxed{W[J] = \Gamma[\varphi_J] + \int d^dx\, J(x)\,\varphi_J(x)}$ Legendre Transform so

$$\langle 0|\phi(x)|0\rangle_J = \frac{\delta W[J]}{\delta J(x)} = \frac{\delta \Gamma[\varphi]}{\delta J(x)} + \varphi_J(x) + \int d^d y\; J(y)\,\frac{\delta \varphi_J(y)}{\delta J(y)}$$

$$= \int d^d y \left[\cancel{\frac{\delta \Gamma[\varphi]}{\delta \varphi_J(y)}\,\frac{\delta \varphi(y)}{\delta J(x)} + J(y)\,\frac{\delta \varphi(y)}{\delta J(x)}} \right] + \varphi_J(x)$$

$= \varphi_J(x)$, so φ_J is also vev of field operator

Propagator $G(x,y) = -\dfrac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = -\dfrac{\delta \varphi_J(y)}{\delta J(x)} \xrightarrow{FT} \dfrac{i}{p^2-m^2-\Sigma(p^2)}$

$\Gamma^{(2)}(x,y) \equiv \dfrac{\delta^2 \Gamma[\varphi]}{\delta \varphi(x)\,\delta \varphi(y)} = -\dfrac{\delta J(y)}{\delta \varphi_J(x)} =$ inverse propagator $= G^{-1}$

$\Big\downarrow$ FT

$p^2 - m^2 - \Sigma(p^2)$

Higher Pt Functions $\quad \dfrac{\delta}{\delta J(x')} = \displaystyle\int dz'\, \frac{\delta \varphi(z')}{\delta J(x')}\,\frac{\delta}{\delta \varphi(z')} = -\int dz'\, G(x',z')\,\frac{\delta}{\delta \varphi(z')}$

$\delta/\delta J = -(\text{propagator}) * \delta/\delta\varphi$

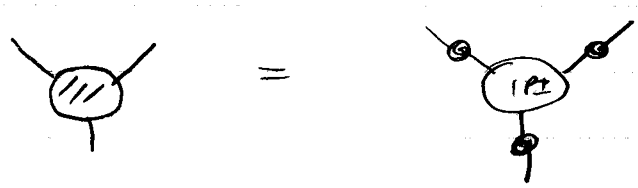
eg. $\displaystyle\int dx'\, \frac{\delta^2 W[J]}{\delta J(x)\,\delta J(x')}\,\frac{\delta^2 \Gamma[\varphi]}{\delta \varphi(x')\,\delta \varphi(y)} = \delta(x-y)$

$\dfrac{\delta}{\delta J(z)}$: $\quad 0 = \displaystyle\int dx'\, \frac{\delta^3 W[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)}\, G^{-1}(x',y) - \int dx'\,dz'\, G(x,x')\,G(z,z')\,\frac{\delta^3 \Gamma[\varphi]}{\delta \varphi(x')\,\delta \varphi(z')\,\delta \varphi(y)}$

so

$$\frac{\delta^3 W[J]}{(\delta J)^3} = \int dx'\,dy'\,dz'\; G\cdot G\cdot G\; \frac{\delta^3 \Gamma}{(\delta \varphi)^3}$$

ie



Chapter 2 The Renormalization Group

Goals

① Improve our understanding of renormalizable QFT (dim $\mathcal{O} \le 4 =$ spacetime dim.) by showing that it is (most often) the low energy limit of QFT with no restrictions on operator dimensions

② Understand why quantum fluctuations at short distances (large momenta Λ) only effect the value of a few parameters $m(\Lambda), g(\Lambda), \ldots$ in renormalizable QFT (or at low energies)

③ Explore and exploit the scheme dependence of coupling constants (renormalization schemes). Renormalization Group Equations (RGE)

④ Use RGE to avoid a breakdown of perturbation theory due to large logs, utilizing a smart definition of couplings (eg. $\alpha \ln(q^2/m_e^2)$, for $q^2 \gg m_e^2$ in QED)

Or equivalently, what is the most useful & physically reasonable definition "$\alpha(q^2)$" for a problem dominated by the scale q^2.

①, ② are Wilsonian R.G., ④ is Gell-Mann Low R.G.

In Lecture we'll build on material covered in QFT-II (eg. ① & ②, QED, $\overline{MS}$ → QCD). These Lecture Notes include background material as well.

Wilsonian Point of View

QFT should be regarded as an effective field theory valid in certain range of energies with a finite physical UV cutoff Λ_0 (imposing $p_E^2 \le \Lambda_0^2$)

[For $\mathcal{L}_{SM}^{\Lambda_0}$ this cutoff might be scale of quantum gravity or a heavy particle we have not seen.]

Take Λ_0 seriously.

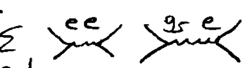
eg. non-renormalizable massless QED

$$\mathcal{L}_{QED}^{\Lambda_0} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \bar\Psi i \not{\partial} \Psi + \frac{g_5}{\Lambda_0} \bar\Psi \sigma^{\mu\nu} F_{\mu\nu} \Psi + \frac{g_6}{\Lambda_0^2} (\bar\Psi\Psi)^2 + \dots \qquad \infty \ \# \text{ of terms}$$

$g_{5,6}$ dimless, Λ_0 dim-1 constant

$$\sigma(e^+e^- \to e^+e^-) \sim \frac{\alpha^2}{E^2} + \frac{\alpha g_5^2}{\Lambda_0^2} + \dots$$

$\propto$  $+ \dots$

[$\sum_{pol}$  + h.c. vanishes by chirality]

For $E \ll \Lambda_0$ $\quad g_5, g_6, \dots$ are irrelevant! & we call operators with $d > 4$ irrelevant operators

mass scale of irrelevant operators defines what "low" means in ①

① says $\left(\mathcal{L}_{QED}^{\Lambda_0}\right)_{E \ll \Lambda_0} = \mathcal{L}_{QED} + \mathcal{O}\left(\frac{E}{\Lambda_0}\right)$

[This example was tree level, but ① is true with loops See 8.324 or lecture notes included at end of this chapter.]

We should write down <u>all</u> interactions consistent with symmetries of the theory.

- For $E \ll \Lambda_0$, $\mathcal{L}_{SM}^{\Lambda_0}$ will look like $\mathcal{L}_{SM}$.
- We can constrain Λ_0 with measurements

eg. our $\frac{g_5}{\Lambda_0}$ term in $\mathcal{L}_{QED}^{\Lambda_0}$ contributes a $\frac{4 g_5}{\Lambda_0}$

to electron magnetic moment. SM calculated value agrees to $10^{-10}\, e/2m_e$ (!), so $\Lambda_0/g_5 \gtrsim 8\times 10^{10}\, m_e = 4\times 10^{7}$ GeV

(naturalness $g_5 \sim g_6 \sim 1$)

[Note: in SM $\bar\psi \sigma^{\mu\nu} F_{\mu\nu} \psi$ is not consistent with $SU_L(2)$ gauge symmetry]

More "Wilson" later.

Teasing Physics out of UV divergences

Goals: • Regularization & Renormalization, <u>two different things</u>

- understand scheme dependence
- understand "μ" in dim. reg.

Regularization: cutoff on UV loop momenta (dim. reg., Λ, Pauli-Villars)

Renormalization: pick a <u>scheme</u> to give definite meaning to parameters in $\mathcal{L}$

eg. $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!}\lambda_0 \phi_0^4 = \mathcal{L}[\phi_0, m_0, \lambda_0]$

bare

$= \mathcal{L}[\phi, m, \lambda] + \frac{1}{2}\delta_Z (\partial_\mu \phi)^2 - \frac{1}{2}\delta m\, \phi^2 - \frac{\delta\lambda}{4!}\phi^4$

ren. field   counterterms

$\phi_0 = Z^{1/2}\phi$, $\delta Z = Z - 1$

$\delta m = m_0^2 Z - m^2$

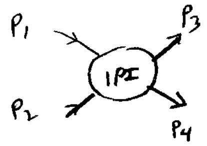
$\delta\lambda = \lambda_0 Z^2 - \lambda$

[δ's remove UV div., but also remove finite term, need way to specify these]

On-shell Ren. Scheme

 $\Big|_{p^2=m^2} = 0$ fixes δm

$\frac{d}{dp^2}$  $\Big|_{p^2=m^2} = 0$ fixes δZ

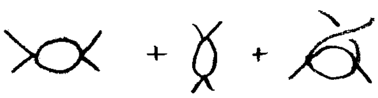


$\Big|_{s=t=u=\frac{4}{3}m^2} = -i\lambda$ fixes $\delta\lambda$

$s=(p_1+p_2)^2$
$t=(p_1-p_3)^2$
$u=(p_1-p_4)^2$

(can use dim. reg. or cutoff ; same result for observables $\sigma(E, m^2, \lambda)$.)

Recall



$$\overset{\text{cutoff}}{=} \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{m^2-x(1-x)s-i0}\right)+(s\to t)+(s\to u)-3\right]$$

Λ ← Euclidean cutoff

$$\overset{\text{dim.reg.}}{=} \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\frac{1}{\epsilon}+\ln\left(\frac{1}{m^2-x(1-x)s}\right)+\text{''}+\text{''}+B\right]$$

$d=4-2\epsilon$ dimensions

 $= -i\,\delta\lambda$

$$\delta\lambda = \frac{\lambda^2}{32\pi^2}\begin{cases}\left[3\ln\left(\frac{\Lambda^2}{m^2}\right)+A-3\right] & \text{cutoff}\\ \left[\frac{3}{\epsilon}+3\ln\left(\frac{1}{m^2}\right)+A+B\right] & \text{dim. reg.}\end{cases}$$

A, B are numbers

Sum is

$$A^{ren} = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{m^2}{m^2-x(1-x)s}\right)+(s\to t)+(s\to u)-A\right]$$

[$Z=1+O(\lambda^4)$ in ϕ^4, so irrelevant here]

Same for both regulators

Now: $\ln(1/m^2)$ might bother you.

In dim. reg. dimension of fields and couplings changes.

$\int d^dx\ (\partial_\mu\phi\,\partial^\mu\phi)$ $\quad [\phi]=1-\epsilon$

$\int d^dx\ \lambda\phi^4$ $\quad [\lambda]=2\epsilon$

write $\lambda=\mu^{2\epsilon}\lambda(\mu)$ $\quad$ (or $\lambda_0=\mu^{2\epsilon}\lambda(\mu)Z_\lambda$ later)

$\lambda(\mu)$ ← dimensionless.

Last Time

ϕ^4 On-Shell Renormalization Scheme

- defines mass m and coupling λ, and fixes field normalization

$$\text{1PI}\Big|_{p^2=m^2} = 0 = \left(\quad + \overset{\delta m}{\times} + \cdots \right)_{p^2=m^2} \qquad \text{fixes } \delta m$$

$$\frac{d}{dp^2}\,\text{1PI}\Big|_{p^2=m^2} = 0 \qquad \text{fixes } \delta Z \qquad \left(\text{@ 1-loop } Z=1,\ \delta Z = 0 \atop \lambda_0 = \lambda + \delta\lambda \right)$$

$$p_1, p_2 \to \text{1PI} \to p_3, p_4 \ \Big|_{s=t=u=\frac{4}{3}m^2} = -i\lambda = \left(\times + \underbrace{\quad + \overset{\delta\lambda}{\otimes} + \cdots}_{=0} \right)\Big|_{(\ldots)}$$

$$(s+t+u = 4m^2)$$

One-Loop $\phi\phi \to \phi\phi$

$$A^{ren} = \quad + \quad + \quad + \overset{\delta\lambda}{\otimes}$$

$$= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \left[\ln\left(\frac{m^2}{m^2 - x(1-x)s}\right) + (s\to t) + (s\to u) - A \right]$$

<u>same</u> for dim.reg & with a cutoff regulator [more in section]

Dim. Reg. Extend space time dimension to $d = 4-2\epsilon$ dimensions,

$$\int \frac{d^dk}{k^4} \sim \frac{dk\, k^{3-2\epsilon}}{k^4} \quad \text{so } \epsilon > 0 \text{ make it convergent in UV for } k\to\infty.$$

action $\int d^dx\, \mathcal{L}$ is dimensionless, look at mass dimension

$$[\partial_\mu\phi\,\partial^\mu\phi] = +d \qquad \text{so } [\phi] = 1-\epsilon$$

$$[\lambda\phi^4] = +d \qquad \text{so } [\lambda] = 2\epsilon$$

Introduce dimensionless coupling $\lambda(\mu)$:

$$\lambda = \mu^{2\epsilon}\lambda(\mu) \qquad \text{or} \qquad \lambda_0 = \mu^{2\epsilon}\lambda(\mu)\, Z_\lambda$$

in On-shell Scheme, this is part of regulator, <u>not in A^{ren}</u>

in $\overline{MS}$ (later) it will play a role for the scheme.

Note: $\delta\lambda = \dfrac{\lambda^2}{32\pi^2}\left[\dfrac{1}{\epsilon} + \ln\dfrac{1}{m^2} + \cdots\right]$

$\delta\lambda = \dfrac{\mu^{2\epsilon}\lambda^2(\mu)}{32\pi^2}\left[\dfrac{1}{\epsilon} + \ln\dfrac{\mu^2}{m^2} + \cdots\right]$

$\mu^{2\epsilon}$: dim $= 2\epsilon$

either way, same A^{ren}, $\delta\lambda$ is μ indep. here

<u>Suppose</u> $s, t, u \gg m^2$, then $A^{ren} \sim \lambda^2 \ln\left(\frac{m^2}{s}\right) + \cdots$
a large log which can potentially spoil λ-expansion

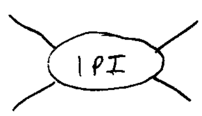
Its natural here to consider $m \to 0$, but both A^{ren} &
$\lambda = \lambda_0 + \dfrac{\lambda_0^2}{32\pi^2}\left[3 \ln\dfrac{\Lambda^2}{m^2} + \cdots\right]$ blow up.

So $\sigma(s, x, \lambda, m^2) = s^{D/2}\,\sigma[x, \lambda, m^2/s]$ does not have a good $m^2 \to 0$ limit

σ: observable dim-D; $x = t/s,\ u/s$

? Can we change λ defn to make $m \to 0$ OK ?

<u>Off Shell Momentum</u>
<u>Subtraction Scheme</u> "μ_R"



$\Big|_{s=t=u=-\mu_R^2} \equiv -i\lambda(\mu_R)$

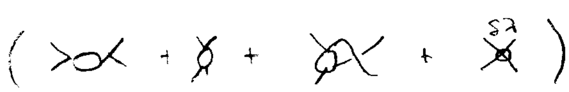
minus ensures $\lambda(\mu_R) \in \mathbb{R}$ for any μ_R

μ_R = arbitrary renormalization scale

(different μ_R = different schemes)

<u>Tree</u> $X = -i\lambda = -i\lambda(\mu_R)$

<u>One-Loop</u> set



$\Big|_{s,t,u=-\mu_R^2} = 0$

$\Rightarrow \quad \delta\lambda = \dfrac{\lambda(\mu_R)^2}{32\pi^2}\displaystyle\int_0^1 dx\left[3\ln\left(\dfrac{\Lambda^2}{m^2 + x(1-x)\mu_R^2}\right) + \cdots\right]$

good $m \to 0$ limit

$$A^{ren}\Big|_{m\to 0} = \frac{i\lambda^2(\mu_R)}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\mu_R^2}{-x(1-x)s}\right)+\cdots\right]$$

$$\sigma(s,x,\lambda(\mu_R),m^2,\mu_R^2) = s^{D/2}\,\sigma\left(x,\lambda(\mu_R),\frac{m^2}{s},\frac{\mu_R^2}{s}\right)$$

has good $m\to 0$ limit

Freedom encoded in μ_R solves our other problem. For $\mu_R \simeq s$ A^{ren} & σ have no large logs, just $\ln\left(\frac{\mu_R^2}{-s}\right)$ so perturbation theory, $\lambda \ln\frac{\mu_R^2}{-s} \ll \lambda^2$, is OK.

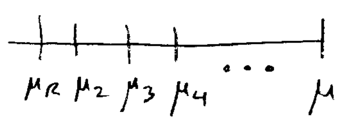
$\boxed{\text{Relate Schemes}}$
$$\lambda_0 Z^2 = (\lambda+\delta\lambda)^{OS} = (\lambda+\delta\lambda)^{\mu_R}$$

OS to μ_R:
$$\lambda(\mu_R) = \lambda + \delta\lambda^{OS} - \delta\lambda^{\mu_R}$$
$$= \lambda + \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left[3\ln\left(\frac{m^2+x(1-x)\mu_R^2}{m^2}\right)+\#\right] + O(\lambda^3)$$

if $\mu_R \gg m$ series may not converge

μ_R' to μ_R:
$$\lambda(\mu_R') = \lambda(\mu_R) + \frac{\lambda^2(\mu_R)}{32\pi^2}\int_0^1 dx\left[3\ln\left(\frac{m^2+x(1-x)\mu_R'^2}{m^2+x(1-x)\mu_R^2}\right)\right] + O(\lambda^3)$$

here $m\to 0$ o.k.

take small steps  then series converges

Differential Method derive a differential eqtn for this

Let
$$\lambda(\mu_R') = G\left(\lambda(\mu_R),\frac{\mu_R'}{\mu_R},\frac{m}{\mu_R}\right)$$
$$\mu_R'\frac{d}{d\mu_R'}\lambda(\mu_R') = \frac{\mu_R'}{\mu_R}\frac{d}{dz}G\left(\lambda(\mu_R),z,\frac{m}{\mu_R}\right)\Big|_{z=\mu_R'/\mu_R}$$

set $\mu_R'=\mu_R$:
$$\boxed{\mu_R\frac{d}{d\mu_R}\lambda(\mu_R) = \beta\left(\lambda(\mu_R),\frac{m}{\mu_R}\right)}$$
Callan - Symanzik Eqtn.

where
$$\beta \equiv \left[\frac{\partial}{\partial z}G\left(\lambda(\mu_R),z,\frac{m}{\mu_R}\right)\right]\Big|_{z=1}$$

no large logs for $m\to 0$

Compute β with perturbative result for G (take $m\to 0$ for simplicity)

$$\lambda(\mu_R') = \lambda(\mu_R) + \frac{3}{32\pi^2}\lambda^2(\mu_R)\ln\left(\frac{\mu_R'^2}{\mu_R^2}\right) + \mathcal{O}(\lambda^3(\mu_R))$$

$$\beta(m\to 0) = \frac{3}{16\pi^2}\lambda^2(\mu_R) + \cdots$$

β-function for ϕ^4 theory

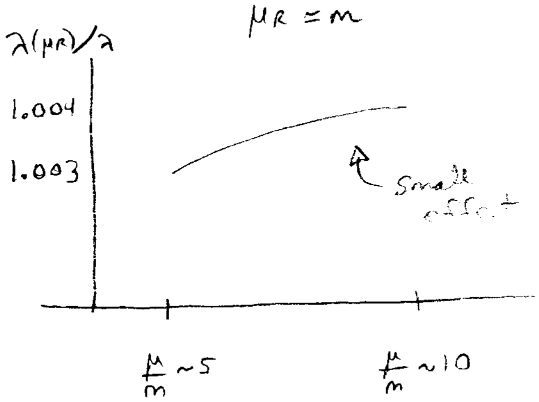
<u>Solution</u> $\lambda(\mu_R) = \dfrac{-16\pi^2}{3\ln(\mu_R/\Lambda)}$ $\quad\Lambda$ = integration constant

Thus $\dfrac{1}{\lambda(\mu_R)} = \dfrac{1}{\lambda(\mu_R')} - \dfrac{3}{16\pi^2}\ln\dfrac{\mu_R}{\mu_R'}$, $\quad \boxed{\lambda(\mu_R) = \dfrac{\lambda(\mu_R')}{1-\frac{3}{16\pi^2}\lambda(\mu_R')\ln\frac{\mu_R}{\mu_R'}}}$

This result sums the large logs between μ_R' & μ_R.

<u>Relation to OS-scheme</u>

Lets cheat a bit (we'll see how much momentarily) and state that $\lambda(\mu_R')\big|_{\mu_R'=m} = \lambda$. Then $\lambda(\mu_R) = \dfrac{\lambda}{1-\frac{3\lambda}{16\pi^2}\ln\mu_R/m}$



Expanding for $\lambda\ln\mu_R/m \ll 1$,

$$\lambda(\mu_R') = \lambda + \frac{3\lambda^2}{16\pi^2}\ln\frac{\mu_R'}{m} + \mathcal{O}(\lambda^3)$$

This agrees with our earlier relation between OS & μ_R scheme only if $\mu_R' \gg m$.

Note: here $\Lambda \simeq m\exp\left(\dfrac{16\pi^2}{3\lambda}\right)$ is enormous. $\left(\begin{matrix}\lambda = .1\\ \Lambda/m \sim 10^{228}\end{matrix}\right)$

Our coupling, $\lambda(\mu_R)$, blows up at $\mu_R\simeq\Lambda$, but since Λ is clearly larger than our Wilsonian cutoff Λ_0 (and Mplank) we shouldn't worry about it.

(and not cheat)

To do better with OS scheme we should keep m-dependent terms in β, which I'll discuss below for QED.

Avoiding large logs requires a <u>running coupling</u>.

The Logs are physical. Our integrals were cutoff at scale $\sim \mu$, so changing μ changes range of d.o.f. that are taken into account.

$\Rightarrow$ to do calculations at a given energy we should first get rid of d.o.f. at much higher energy.
Then coupling depends on scale you probe, $\mu \simeq s$.

<u>Extension</u>

$\Big|_{p^2 = -\mu_R^2} = 0$ gives $m(\mu_R)$

$\frac{d}{dp^2}$ 

$\Big|_{p^2 = -\mu_R^2} = 0$ gives $Z(\mu_R)$

$$\phi = Z^{-1/2}(\mu_R)\,\phi_0$$

$$\frac{Z^{1/2}(\mu_R')}{Z^{1/2}(\mu_R)} = G^{\phi}\left(\lambda(\mu_R), \frac{\mu_R'}{\mu_R}, \frac{m}{\mu_R}\right)$$

take $\mu_R' \, d/d\mu_R'$, set $\mu_R' = \mu_R$

$$\boxed{\mu_R \, d/d\mu_R \; Z^{1/2}(\mu_R) = \gamma^{\phi}\left(\lambda(\mu_R), \frac{m}{\mu_R}\right) Z^{1/2}(\mu_R)}$$

or $\quad \mu_R \, d/d\mu_R \, \ln Z^{1/2}(\mu_R) = \gamma^{\phi}\left(\lambda(\mu_R), \frac{m}{\mu_R}\right)$

where $\gamma^{\phi} = \left[\frac{\partial}{\partial z} G^{\phi}\left(\lambda(\mu_R), z, \frac{m}{\mu_R}\right)\right]\Big|$ is the <u>anomalous</u>

QED

[review, establish notation, offshell scheme & mass in β]

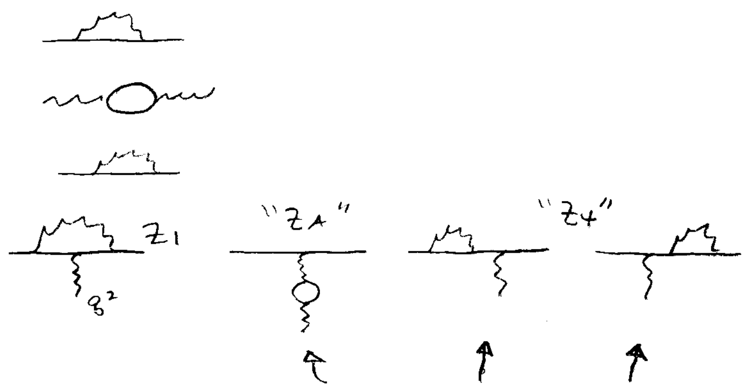
$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial\cdot A)^2 + \text{counterterms}$$

renormalized fields

covariant gauge fixing $\xi = 1$ is Feyn. gauge

Here bare

$$\psi_0 = Z_\psi^{1/2}\,\psi$$
$$A_0^\mu = Z_A^{1/2}\,A^\mu$$
$$m_0 = (m - \delta m)\,Z_\psi^{-1}$$
$$e_0 = Z_e\, e$$
$$\xi_0 = Z_A\,\xi$$



not truncated, a schematic for effect of 1PI subgraphs

$$\bar{\psi}_0\, e_0\, \not{A}_0\, \psi_0 = \underbrace{Z_\psi Z_A^{1/2} Z_e}_{Z_1}\; \bar{\psi}\, e\, \not{A}\, \psi$$

$$Z_e = \underbrace{Z_1 Z_\psi^{-1}}_{=1}\, Z_A^{-1/2} = Z_A^{-1/2}$$

(true for any scheme consistent with gauge symm.)

$= 1$ to all orders by Word Identity & (review is given on subsequent pages)

$$e_0 A_0 = e A$$

Calculate

$$= \underbrace{\frac{Z_A^{-1}}{1 - \Pi_0(q^2)}}\;\frac{(-i)}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - \frac{i}{q^4}\,\xi\, q^\mu q^\nu$$

ren. fields

$$\equiv \frac{1}{1 - \Pi(q^2)}$$

Renormalization schemes define "e", "m" and fix δm, Z_ψ, Z_A (regulator dependent div. terms and finite terms)

Aside/Review

Ward - Takahashi Identities

Gauge symmetry implies non-trivial relations between/on correlation functions.

Consider QED generating function

gauge fixing

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi \; \exp\left[i\int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar\psi(i\partial\!\!\!/ - m)\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + J^\mu A_\mu + \bar\eta\psi + \bar\psi\eta \right)\right]$$

correlators from A_μ: $-i\,\delta/\delta J^\mu$

ψ: $-i\,\delta/\delta\bar\eta$

$\bar\psi$: $+i\,\delta/\delta\eta$

eg. $$\langle 0|\, T\,\psi(x)\,\bar\psi(y)\,|0\rangle = \left(-i\frac{\delta}{\delta\bar\eta(x)}\right)\left(-i\frac{\delta}{\delta\eta(y)}\right) \frac{Z[J,\eta,\bar\eta]}{Z[0,0,0]}\Bigg|_{J=\eta=\bar\eta=0}$$

$$\frac{Z[J,\eta,\bar\eta]}{Z[0,0,0]} = e^{iW[J,\eta,\bar\eta]}$$

generating function for connected graphs

Physical Results are gauge invariant, $\delta Z[J,\eta,\bar\eta] = 0$

$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha$ $\qquad \delta A_\mu = 1/e\,\partial_\mu\alpha$

$\psi' = e^{-i\alpha}\psi$ $\qquad \delta\psi' = -i\alpha\psi$

$\bar\psi' = \bar\psi e^{i\alpha}$ $\qquad \delta\bar\psi' = i\alpha\bar\psi$

Change variable in path integral to $\{A',\psi',\bar\psi'\}$ and expand to first order in $\alpha(x)$

$$0 = \delta Z = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\; e^{i(\cdots)} \int d^4x \left(-\frac{1}{e\xi}(\partial_\mu A^\mu)\,\partial^2\alpha + \frac{1}{e}J^\mu\partial_\mu\alpha - i\alpha(\bar\eta\psi - \bar\psi\eta) \right)$$

Int by parts & pull out $\int d^4x\,\alpha(x)$ & mult by $(-e)$

$$\int [\mathcal{D}A\psi\bar\psi]\; e^{i(\cdots)} \left(\frac{1}{\xi}\partial^2\partial_\mu A^\mu + \partial^\mu J_\mu + ie(\bar\eta\psi - \bar\psi\eta) \right) = 0$$

$$\left[\frac{-i}{\xi}\partial^2\partial_\mu \frac{\delta}{\delta J^\mu} + \partial^\mu J_\mu + e\left(\bar\eta\frac{\delta}{\delta\bar\eta} - \eta\frac{\delta}{\delta\eta}\right)\right] Z[J,\eta,\bar\eta] = 0$$

$$\boxed{\frac{1}{\xi}\,\partial^2\partial^\mu\,\frac{\delta W}{\delta J^\mu} + \partial_\mu J^\mu + ie\left(\bar\eta\,\frac{\delta W}{\delta\bar\eta} - \eta\,\frac{\delta W}{\delta\eta}\right) = 0}$$

↑L3
↓L4

Problem Set: Implications for photon propagator

Word Identity for 1PI Graphs

$$\Gamma[\underbrace{A,\psi,\bar\psi}_{\text{classical sources}}] = W[J,\eta,\bar\eta] - \int d^4x\,\left(\bar\eta\psi + \bar\psi\eta + J^\mu A_\mu\right)$$

$$\frac{\delta\Gamma}{\delta A_\mu(x)} = -J^\mu(x)\,, \qquad \frac{\delta\Gamma}{\delta\psi(x)} = +\bar\eta(x)\,, \qquad \frac{\delta\Gamma}{\delta\bar\psi(x)} = -\eta(x)$$

$$\frac{\delta W}{\delta J_\mu(x)} = A^\mu(x)\,, \qquad \frac{\delta W}{\delta\bar\eta(x)} = -\psi(x)\,, \qquad \frac{\delta W}{\delta\eta(x)} = +\bar\psi(x)$$

so
$$\boxed{\frac{1}{\xi}\,\partial^2\partial^\mu A_\mu(x) - \partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)} + ie\left(\psi(x)\,\frac{\delta\Gamma}{\delta\psi(x)} - \bar\psi(x)\,\frac{\delta\Gamma}{\delta\bar\psi(x)}\right) = 0}$$

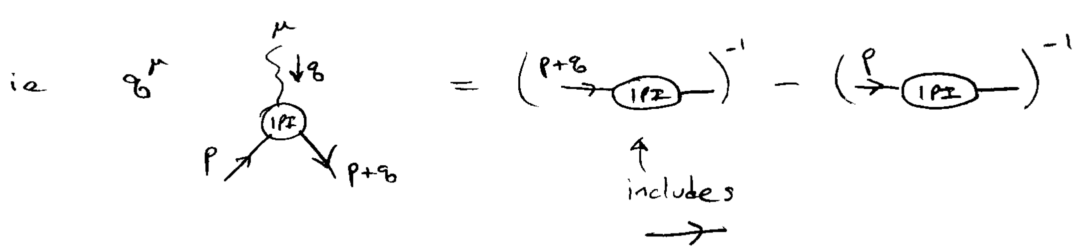
Simplest word identity for interactions

take $\dfrac{\delta}{\delta\bar\psi(x_1)}\dfrac{\delta}{\delta\psi(x_2)}$:

$$-\partial^x_\mu\,\frac{\delta^3\Gamma}{\delta\bar\psi(x_1)\,\delta\psi(x_2)\,\delta A_\mu(x)} = -ie\,\delta(x-x_2)\,\frac{\delta^2\Gamma}{\delta\bar\psi(x_1)\,\delta\psi(x_2)} + ie\,\delta(x-x_1)\,\frac{\delta^2\Gamma}{\delta\bar\psi(x_1)\,\delta\psi(x_2)}$$

Fourier Transform, pull out $\delta^4(p'-p-q)$

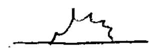
$$q^\mu\,\Gamma_\mu(p,q,p+q) = S_F^{-1}(p+q) - S_F^{-1}(p)$$

ie


$q^\mu \to 0$

$$\Gamma_\mu(p,0,p) = \frac{\partial S_F^{-1}(p)}{\partial p^\mu}$$

tree level : $\quad \gamma_\mu = \dfrac{\partial}{\partial p^\mu}(\not{p} - m) = \gamma_\mu \quad \checkmark$

one-loop: $\quad \gamma^\mu$  $= \dfrac{\partial}{\partial p^\mu}$ 

This implies $\quad Z_1 = Z_\psi \quad$ (on-shell scheme, or $\overline{MS}$ scheme, ...)

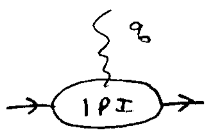
the renormalization constants in $\mathcal{L}_{QED}$ are related such that

$$\mathcal{L}^{ct} = (Z_\psi - 1)\,\overline{\psi}\,(i\not{\partial} - m)\,\psi - (Z_1 - 1)\,\overline{\psi}\, e \not{A}\, \psi \quad + \cdots$$

$$= (Z_\psi - 1)\,\overline{\psi}\,(i\not{D} - m)\,\psi \quad + \cdots$$

end
Aside

$\boxed{\text{On-Shell Scheme}}$ where $\alpha = \frac{1}{137}$

Want  $\Big|_{q^\mu = 0} = -ie\gamma^\mu$. Since $Z_e = Z_A^{-1/2}$ we can equivalently impose condition on 

 $\Big|_{\text{as } q^2 \to 0}$ $= \frac{(-i)}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - i\,\xi \frac{q_\mu q_\nu}{q^2}$

so no residue "Z_{LSZ}" in this scheme

<u>Want</u> $\Pi(q^2 = 0) = 0$.

$Z_A = \left[1 - \Pi_0(0)\right]^{-1}$ Imposing this we find

$$\Pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{m^2 - q^2 x(1-x)}{m^2}\right) + O(e^4)$$

$$Z_A = 1 - \frac{e^2(\mu)}{12\pi^2}\left[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right] + O(e^4)$$

<u>Comments</u> • $e_0 = Z_e e = Z_e \mu^\epsilon e(\mu)$ in dim. reg.
$d = 4 - 2\epsilon$

<u>Here</u> μ is just convenience for dimensionality in Z_A, but has no physical significance. $\mu\, d/d\mu\, Z_A = 0$

$\mu\, d/d\mu\, e = 0$, coupling here is constant

• $\lim_{m \to 0} \Pi(q^2)$ diverges $\Longleftrightarrow$ $\Pi(q^2)$ has large log, $\ln(q^2/m^2)$, for $q^2 \gg m^2$

The mass of ψ plays an important role in definition of on-shell scheme, and stops us from taking the massless limit.

$\boxed{\text{Offshell Momentum Subtraction ("}\mu_R\text{") scheme}}$

 $$\Big|_{q^2=-\mu_R^2} = \left(\frac{-i g^{\mu\nu}}{q^2} + q^\mu q^\nu \text{ term} \right)_{q^2 \to -\mu_R^2}$$

Now $\underset{\uparrow\ \mu_R\text{-scheme}}{Z_A^{-1}(\mu_R)} = 1 - \Pi_0(-\mu_R^2) = \underset{\uparrow}{Z_A^{-1}} \left(1 - \underset{\uparrow}{\Pi}(-\mu_R^2)\right)$ on-shell scheme

bare $e_0 = Z_A^{-1/2}(\mu_R)\, e(\mu_R) = Z_A^{-1/2}\, e$

$$e(\mu_R) = \left[\frac{Z_A(\mu_R)}{Z_A} \right]^{1/2} e$$

perturbative (mass dependent) relation between schemes

$e(\mu_R) = e + e^3[\cdots] + \cdots$

find $$Z_A(\mu_R) = 1 - \frac{e^2(\mu_R)}{12\pi^2}\left[\frac{1}{\epsilon} - \int_0^1 dx\, 6x(1-x) \ln\left(m^2 + \mu_R^2 x(1-x)\right) \right]$$

β-function

$$\beta = \mu_R \frac{d}{d\mu_R} e(\mu_R) = Z_A^{1/2} \underbrace{Z_A^{-1/2} \mu_R\, {}^{d}\!/\!{}_{d\mu_R} Z_A^{1/2}}_{\gamma^{(A)}} e_0 = e(\mu_R)\gamma^{(A)}$$

$$= \frac{e(\mu_R)^3}{2\pi^2} \int_0^1 dx \frac{x^2(1-x)^2 \mu_R^2}{m^2 + \mu_R^2 x(1-x)}$$ $\leftarrow$ have not taken $m \to 0$ here

(i) $\mu_R \gg m$ $\quad \beta = \dfrac{e(\mu_R)^3}{12\pi^2}$ $\quad$ usually called QED β-function

(ii) $\mu_R \ll m$ $\quad \beta = \dfrac{e(\mu_R)^3}{60\pi^2} \dfrac{\mu_R^2}{m^2}$

- Smooth through $\mu_R = m$, and connects smoothly to on-shell coupling $e(\mu_R = 0) = e$
- $m \to 0$ limit exists for $\Pi(q^2)$ in this scheme, get $\ln(\mu_R^2/q^2)$ rather than $\ln(m^2/q^2)$
- For $m \to \infty$ (Non. relativistic limit) get $\beta = 0$, $e(\mu_R) = e$ the on-shell "e" is charge we use in NR-Quantum Mechanics

- running coupling $e(\mu_R)$. Integrals are cutoff at scale $\sim \mu_R$ and changing μ_R changes range of d.o.f. that are taken into account in integrations.

→ to calculate at "q^2" we should get rid of d.o.f. at scales $\gg q^2$, and hence take $\mu_R^2 \simeq q^2$, which minimizes the log

[coupling depends on scale you probe]

Solutions: General $\quad \alpha(\mu_R) = \dfrac{\alpha}{1 - \frac{2\alpha}{3\pi} f(\mu_R/m)}$

↑ function with properties mentioned above

$\mu_R \gg m$ gives $\quad \alpha(\mu_R) = \dfrac{\alpha}{1 - \frac{2\alpha}{3\pi} \ln(\mu_R/m)}$

QED $\alpha(\mu_R)$ runs slowly $\qquad \alpha \simeq \frac{1}{137}$, $\alpha(\mu_R = m_W) \simeq \frac{1}{128}$

[Note: here rescaling function from pg 24 is

$$G^A\left(e(\mu_R), \frac{\mu_R'}{\mu_R}, \frac{m}{\mu_R}\right) = \left[\frac{Z_A(\mu_R')}{Z_A(\mu_R)}\right]^{1/2} = \left[\frac{1 - \pi(-\mu_R'^2)}{1 - \pi(-\mu_R^2)}\right]^{-1/2}$$

$$= 1 + \frac{e(\mu_R)^2}{4\pi^2} \int_0^1 dx \; x(1-x) \ln\left(\frac{m^2 + \mu_R'^2 x(1-x)}{m^2 + \mu_R^2 x(1-x)}\right) + \cdots$$

where $e(\mu_R') = G^A(e(\mu_R), \mu_R'/\mu_R, m/\mu_R)$]

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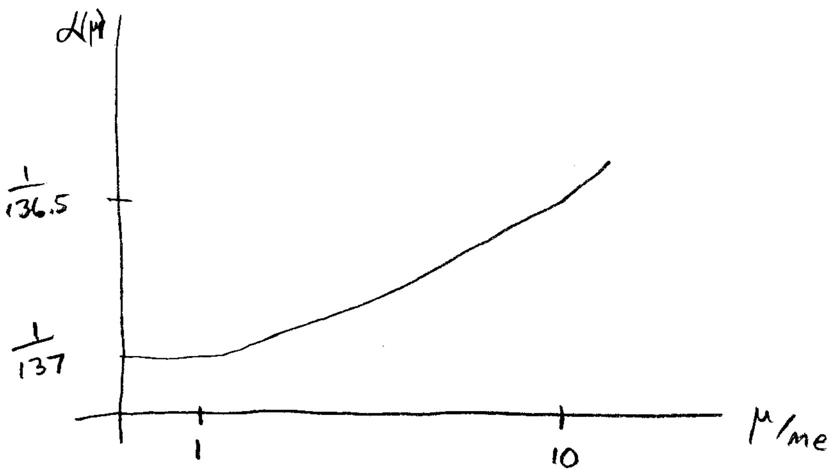
$$= 1 + \frac{e(\mu_R)^2}{4\pi^2} \int_0^1 dx \; x(1-x) \ln\left(\frac{m^2 + \mu_R'^2 x(1-x)}{m^2 + \mu_R^2 x(1-x)}\right) + \cdots$$

where $e(\mu_R') = G^A(e(\mu_R), \mu_R'/\mu_R, m/\mu_R)$]



$\boxed{\text{Minimal Subtraction Scheme}}$

- efficient method to get β, γ in dim. reg.
- use μ as sliding scale (taking place of μ_R)
- simple counterterms

QED: $\int d^dx \; \bar{\Psi}_0 \not{A}_0 \Psi_0 \, e_0$ $\qquad d = 4-2\epsilon$

dim: $-d$ $\quad \frac{3}{2}-\epsilon$ $\quad 1-\epsilon$ $\quad \hookrightarrow$ so $[e_0] = \epsilon$

$\boxed{e_0 = Z_e \mu^{\epsilon} e(\mu)}$ $\qquad [e(\mu)] = 0$

e_0, Z_e, $e(\mu)$ all depend on d

Laurent Series

$$\mu^{-\epsilon} e_0(d) = \underset{\substack{\uparrow \\ \text{analytic}}}{e(\mu,d)} \underbrace{\left(1 + \sum_{k=1}^{\infty} \frac{a_k(e(\mu,d))}{\epsilon^k}\right)}_{Z_e \text{ only has pole terms and no explicit } \ln\mu\text{'s}}$$

Differentiate $\mu\, d/d\mu$ & equate powers ϵ^k

$O(\epsilon^{-1})$ & $O(\epsilon^0)$ terms give $\qquad e = e(\mu,d)$

$$\boxed{\beta(e(\mu,d),d) \equiv \mu\, d/d\mu\; e(\mu,d) = -\epsilon e + e^2\, d/de\; a_1(e)}$$

Only simple pole term!

Only divergent part of graphs!!

Aside

Proof: Let $\dot{a}_k = d/de\; a_k$, then $\mu\, d/d\mu$ gives

$$-\epsilon\left[e + e\sum_k \frac{a_k(e)}{\epsilon^k}\right] = \beta(e,d)\left(1 + \sum_{k=1}^{\infty}\frac{a_k(e)}{\epsilon^k}\right) + e\sum_{k=1}^{\infty}\frac{\dot{a}_k(e)\,\beta(e,d)}{\epsilon^k} \qquad (*)$$

β analytic: LHS at most ϵ^1, so $\beta(e,d) = -\epsilon e + \beta(e)$

$O(\epsilon^0)$ terms: $\qquad \beta(e) = -e a_1(e) + e a_1(e) + e^2 \frac{d}{de} a_1(e)$

Higher Poles give: $\qquad e^2\, d/de\; a_{k+1} = \beta(e)\, d/de\,(e a_k)$ $\qquad$ recursion relation

coefficients $\frac{1}{\epsilon^{k>1}}$ at some order in pert theory are given by lower order information

$$\boxed{\begin{array}{l} d \to 4 \\ \mu \, d/d\mu \; e(\mu) = e^2 \, d/de \; e(\mu) , \quad \mu \frac{d}{d\mu} \alpha(\mu) = 4\alpha^2 \frac{d}{d\alpha} a_1(\alpha) \end{array}}$$

$$\alpha(\mu) = \frac{e^2(\mu)}{4\pi}$$

eg. $Z_A = 1 - \dfrac{e^2}{12\pi^2 \epsilon} + \cdots$, $Z_e = 1 + \dfrac{e^2}{24\pi^2 \epsilon} + \cdots$

$a_1 = \dfrac{e^2}{24\pi^2}$, $\beta = e^2 \dfrac{e}{12\pi^2} = \dfrac{e^3}{12\pi^2}$ ✓ <u>massless</u> β - function

(same as μ_R-scheme for $\mu_R \gg m$)

$\boxed{\overline{MS}\text{ -scheme}}$ $\mu^2_{MS} \equiv \mu^2_{\overline{MS}} \dfrac{e^{\gamma_E}}{4\pi}$ to remove common constants

typically $\dfrac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \dfrac{\mu^{2\epsilon}_{MS}}{s^\epsilon} = \dfrac{1}{16\pi^2}\left(\dfrac{1}{\epsilon} + \ln\left(\dfrac{\mu^2_{\overline{MS}}}{s}\right) + \mathcal{O}(\epsilon)\right)$

<u>Comments</u>

- In dim. reg. powers of dimensionful cutoff don't appear $(\Lambda, \Lambda^2, \ldots)$. Poles at $d=4$ correspond to log divergences
  $\frac{1}{\epsilon} + \ln(\mu^2/-s) + \cdots$

- $1/\epsilon$ coefficients give anom. dim. or β-function (any loop order) MS or $\overline{MS}$

- Threshold corrections from $\mu \sim m$ are included by
  i) only including light particles (mass $\le \mu$) in β-function
  ii) matching for $\mu \sim m$

[More on Homework]

We must also renormalize local products of operators.

e.g. $\mathcal{O}_0 = (\phi_0^2)(x)$, renormalized $\mathcal{O}(\mu) = Z^{\mathcal{O}}(\mu)\,\mathcal{O}_0$

$$\mu \frac{d}{d\mu} \mathcal{O}_0 = 0 \quad \Rightarrow \quad \mu \frac{d}{d\mu} \mathcal{O}(\mu) = \gamma^{\mathcal{O}}(\alpha(\mu))\,\mathcal{O}(\mu)$$

anom. dim. $\gamma^{\mathcal{O}} = (Z^{\mathcal{O}})^{-1} \mu\, {}^{d}/_{d\mu}\, Z^{\mathcal{O}}$

MS, $\overline{\text{MS}}$: $$Z^{\mathcal{O}} = 1 + \sum_{k=1}^{\infty} \frac{a_k^{\mathcal{O}}(e)}{\epsilon^k}$$

$$\gamma^{\mathcal{O}} = -e \frac{d}{de} a_1^{\mathcal{O}}$$

[more in section & on Homework]

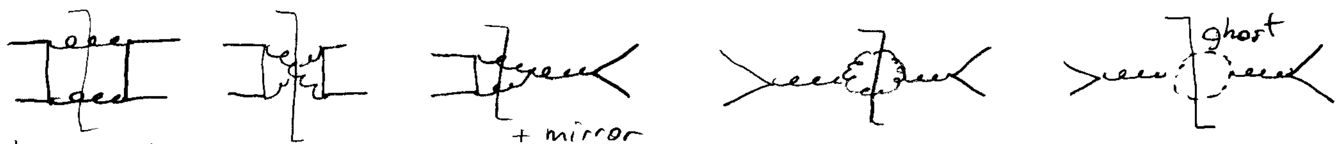
QCD, Renorm., β-function, Asymptotic Freedom

Faddeev-Popov Gauge Fixing gave ← cov. gauges

$$\mathcal{L}_{QCD} = -\frac{1}{4}\left(F^A_{\mu\nu}\right)^2 + \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{2\xi}\left(\partial^\mu A^A_\mu\right)^2 + \bar{c}^A\left(-\partial^\mu D^{AC}_\mu\right)c^C$$

ghosts $c, \bar{c}$ are anticommuting, Lorentz scalars, adjoint in color
- negative d.o.f. that cancel unphysical timelike & longitudinal polarization states of gauge bosons

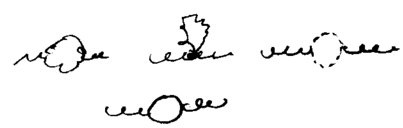
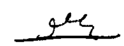
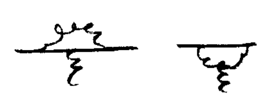
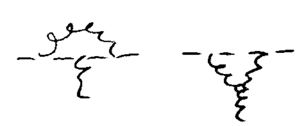
eg.



optical thm

& $q\bar{q} \to gg$ only should produce transverse gluons

Renormalization

		graphs
$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}\right)^2$	Z_A	
$- g_0 f^{ABC}\left(\partial_\mu A^A_{0\nu}\right) A^{B\mu}_0 A^{C\nu}_0$	$Z_g Z_A^{3/2} = Z_1$	
$-\frac{1}{4} g_0^2 \left(f^{EAB} A^A_{0\mu} A^B_{0\nu}\right)\left(f^{ECD} A^{C\mu}_0 A^{D\nu}_0\right)$	$Z_g^2 Z_A^2 = Z_1'$	← you draw them
$+ \bar{\psi}_0 i\not{\partial} \psi_0$	Z_ψ	
$- m_0 \bar{\psi}_0 \psi_0$	$\delta m \bar{\psi}\psi$	
$+ g_0 \bar{\psi}_0 \not{A}_0 \psi_0$	$Z_g Z_\psi Z_A^{1/2} = Z_1^F$	
$- \bar{c}^A_0 \partial^2 c^A_0$	Z_c	
$- g_0 \bar{c}^A_0 f^{ABC} \partial^\mu A^0_{B\mu} c^C_0$	$Z_g Z_c Z_A^{1/2} = Z_1^c$	
$- \frac{1}{2\xi_0}\left(\partial^\mu A^A_{0\mu}\right)^2$	1	

↑ longitudinal part of propagator not altered.

where $A_0 = Z_A^{1/2} A \qquad g_0 = Z_g\, g$

$\psi_0 = Z_\psi^{1/2}\psi \qquad M_0 = (m - \delta m) Z_\psi^{-1}$

$C_0 = Z_C^{1/2} C \qquad \xi_0 = Z_A\, \xi \leftarrow$ insures 1

Interactions have the same g_0

$$Z_g = \frac{g^0}{g} = \underset{\uparrow}{Z_1 Z_A^{-3/2}} = \underset{\uparrow}{Z_1'^{\,1/2} Z_A^{-1}} = \underset{\uparrow}{Z_1^F Z_A^{-1/2} Z_\psi^{-1}} = \underset{\uparrow}{Z_1^C Z_C^{-1} Z_A^{-1/2}}$$

4 different ways we could calculate Z_g and then β_{QCD}

Relations can be written

$$\frac{Z_1}{Z_A} = \frac{Z_1'}{Z_1} = \frac{Z_1^F}{Z_\psi} = \frac{Z_1^C}{Z_C}$$

<u>Slavnov - Taylor</u> <u>Identities</u>

these are QCD analogs of Word identity $Z_1 = Z_\psi$

$\boxed{\text{Better Order}}$

Gauge Symmetry in QCD $\Rightarrow$ S.T. Identities $\Rightarrow$ equality of g couplings

Analog in U(1) case: renormalized coupling $e = \frac{Z_\psi}{Z_1} Z_A^{1/2} e_0 = Z_A^{1/2} e_0$ does not depend on species of fermion

In QCD things are more complicated

- $Z_1^F \neq Z_\psi$, Z_A is gauge dependent

- but first two terms of $\beta(g) = b_0 g^3 + b_1 g^5 + \cdots$ are gauge independent. In $\overline{MS}$ full $\beta(g)$ is.

- A manageable derivation of S.T. identities requires studying BRST symmetry of gauge-fixed action [Reading]

$\overline{MS}$ $\quad \beta_{QCD} = \mu \frac{d}{d\mu} g(\mu) = g^2 \frac{d}{dg} a_1(g)$

$$Z_g = 1 + \sum_{k=1}^{\infty} \frac{a_k}{\epsilon^k}$$

Compute Z_1^F, Z_3, Z_2 to get a_1 [Peskin 16.5]

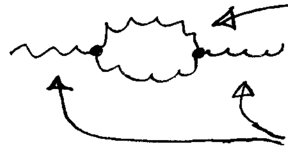
A simpler method

<u>Background Field Gauge</u>

Find a gauge where $g A^\mu$ is not renormalized

so $\quad Z_g = Z_A^{-1/2}$

Idea



need gauge fixing for propagators

would like gauge inv. for external lines.

Let $A_\mu \to A_\mu^a + Q_\mu^a$ ← Quantum field

↑ fixed background

QCD action $S[A+Q]$ is invariant under gauge trnsfm

$$A_\mu + Q_\mu \to A_\mu + Q_\mu + \frac{1}{g} \partial_\mu \alpha - i [A^\mu + Q^\mu, \alpha]$$

① Fix A_μ, $\quad Q_\mu \to Q_\mu + \frac{1}{g} [D_\mu^{A+Q}, \alpha]$

$$i D^{A+Q} \equiv i \partial + g(A+Q)$$

gauge trnsfm of quantum field

(2) let $A_\mu \to A_\mu + \frac{1}{g}[D_\mu^A, \alpha]$ ie $A_\mu \to U(A + \frac{i}{g}\partial)U^{-1}$

$Q_\mu \to Q_\mu - i[Q_\mu, \alpha]$ $Q_\mu \to U Q_\mu U^{-1}$

- gauge trnsfm of bkgnd
- Q behaves like adjoint matter field

Usual Generating Functions ($A=0$)

$$Z[J] = \int dQ \det\left[\frac{\delta G^a}{\delta \alpha^b}\right] \exp i \int d^4x \left[\mathcal{L}(Q) - \frac{1}{2\xi}(G^a)^2 + J_\mu^a Q^{\mu a}\right]$$

for Green's functions, $G^a = \partial_\mu A^{\mu a}(x)$, $\delta G^a(x)/\delta\alpha^b(x') = \underbrace{\partial^\mu D_\mu^{ab}}_{\text{ghost kinetic term}} \delta^4(x-x')$

$W[J] = -i \ln Z[J]$ for conn. G. fn's

$\Gamma[\bar{Q}] = W[J] - \int d^4x\, J_\mu^a \bar{Q}_\mu^a$; $\bar{Q}_\mu^a = \frac{\delta W}{\delta J_\mu^a}$

for 1PI G. fn's

Now Consider

$$Z[J,A] = \int dQ \exp i\left[S[Q+A] + J_\mu^a Q^{\mu a} + g.f. + \text{ghosts}\right]$$

Let $G^a = (D_\mu^A Q^\mu)^a = \partial_\mu Q^{\mu a} + g f^{abc} A_\mu^b Q^{\mu c}$

$\mathcal{L}_{g.f.} = -\frac{1}{2\xi}(G^a)^2$ fixes gauge in (1)

but still invariant under (2)

(1). $\frac{\delta Q_\mu}{\delta \alpha} = \frac{1}{g} D_\mu^{A+Q}$, $\frac{\delta G}{\delta \alpha} = \frac{1}{g} D_\mu^A D^{\mu\, A+Q}$

$\mathcal{L}_{ghost} = \bar{c}\left(-D_\mu^A D^{\mu\, A+Q}\right) c$ also inv. under (2) under ② we let ghost $c \to u c u^{-1}$

And $S[Q+A] + \mathcal{L}_{g.f.} + \mathcal{L}_{ghost}$ is inv. under ②

So $Z[J,A] = Z[uJu^{-1}, \; uAu^{-1} + u\frac{i}{g}\partial u^{-1}]$

(make change of var $Q \to uQu^{-1}$ in path integral)

$Z[J,A] = e^{iW[J,A]}$

$\Gamma[\tilde{Q}, A] = W[J,A] - \int d^4x \, J_\mu^a \tilde{Q}^{\mu a}$

$\tilde{Q} = \frac{\delta W}{\delta J}$

And $\Gamma[\tilde{Q}, A] = \Gamma[u\tilde{Q}u^{-1}, \; uAu^{-1} + u\frac{i}{g}\partial u^{-1}]$

Use gauge inv. $\Gamma[0, A]$ to compute β-function

- $\tilde{Q} = 0$ so only A external fields
- $\int dQ$ so only Q internal lines.

Lets check that $\Gamma[0, A]$ is equal to our usual effective action $\Gamma[\bar{Q} = A]$, but in a strange gauge

Shift $Q \to Q - A$, $S[Q+A] \to S[Q]$ and

$Z[J, A] = Z[J]_A \; e^{-i\int d^4x \, J^\mu_a A^{\mu a}}$

where we have an A-dependent gauge: $\mathcal{L}_{g.f} = \frac{-1}{2\xi}\left(\partial_\mu Q^{\mu a} - \partial_\mu A_\mu^a + g f^{abc} A_\mu^b Q^{\mu c}\right)^2$

& correct ghost term $\bar{c}(-D^{\mu A} D_\mu^Q) c$

$W[J, A] = W[J]_A - \int d^4x \, J^\mu_a A^{\mu a}$

$\tilde{Q} = \frac{\delta W[J,A]}{\delta J} = \frac{\delta W[J]_A}{\delta J} - A = \bar{Q} - A$

$$\Gamma[\tilde{Q}, A] = \left(W[J]_A - \int d^4x\, J^\mu_a A^{\mu a} \right) - \int d^4x\, J^\mu_a \tilde{Q}^{\mu a}$$

$$= \Gamma[\underbrace{\tilde{Q} + A}_{\bar{Q}}]_A$$

$\Gamma[0, A] = \Gamma[A]_A$, so bkgnd field action with b.g. field on external lines is usual eff. action in particular gauge

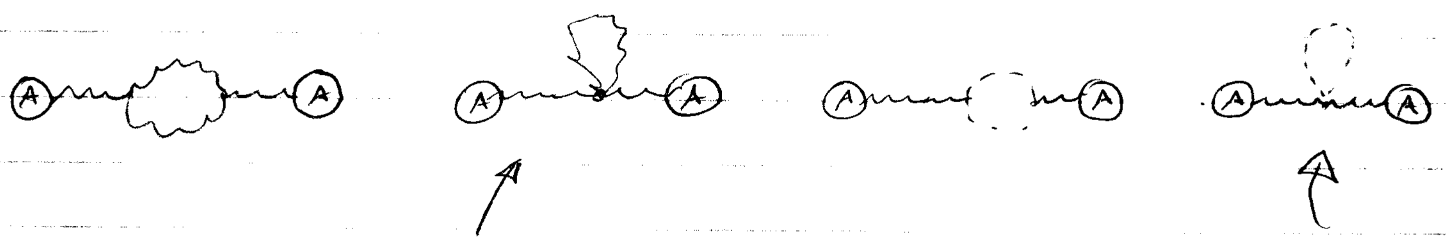
$\Gamma[0,A]$: divergences must preserve gauge inv.

so $(F^a_{\mu\nu})^2$ must be multiplicatively renormalized

$$(F^a_{\mu\nu})_0 = Z_A^{1/2} \left[\partial_\mu A_\nu - \partial_\nu A_\mu + g \underbrace{Z_g Z_A^{1/2}}_{1} f^{abc} A^b_\mu A^c_\nu \right]$$

as desired.

So Compute

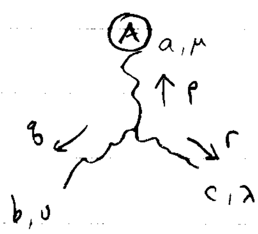


$\sim \int \frac{d^dk}{k^2}$ $\qquad$  $\qquad$ $\int \frac{d^dk}{k^2}$

no $1/\epsilon$ pole

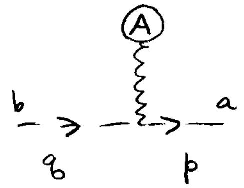
Just 3 graphs!

Feyn. Rules



$$= g f_{abc} \left[g_{\mu\lambda} \left(p - r - \frac{1}{\xi} q \right)_\nu + g_{\nu\lambda} (r - q)_\mu + g_{\mu\nu} \left(q - p + \frac{1}{\xi} r \right)_\lambda \right]$$

$\xi = 1$ is bkgnd field Feyn. Gauge



$$= -g\, f^{acb}\,(p+q)_\mu$$

$$= \frac{i g^2 C_A}{16\pi^2}\,\delta^{ab}\,[g_{\mu\nu}k^2 - k_\mu k_\nu]\,\frac{10}{3\epsilon}$$

$$= \frac{i g^2 C_A}{16\pi^2}\,\delta^{ab}\,[g_{\mu\nu}k^2 - k_\mu k_\nu]\,\frac{1}{3\epsilon}$$

$$Z_A = 1 + \frac{g^2}{16\pi^2\epsilon}\left(\frac{11}{3}C_A - \frac{4}{3}n_f T_F\right) \qquad C_A = 3 \text{ for } SU(3)$$

$$Z_g = 1 - \frac{g^2/2}{16\pi^2\epsilon}\left(\frac{11}{3}C_A - \frac{4}{3}n_f T_F\right)$$

from  n_f flavors of quarks

same as QED, but $tr[T^A T^B] = T_F\,\delta^{AB}$, $T_F = \frac{1}{2}$

$$\beta_{QCD}(g) = g^2\frac{d}{dg}a_1 = \frac{-g^3}{16\pi^2}\underbrace{\left[\frac{11}{3}C_A - \frac{4}{3}n_f T_F\right]}_{\equiv\ \beta_0}$$

QED: $n_f = 1,\ C_A = 0,\ T_F = 1$ is our old friend $\beta_{QED} = \frac{g^3}{12\pi^2}$
$(\beta_0 = -4/3)$

QCD: $n_f \le 6 < \frac{33}{2}$ so $\beta_{QCD} < 0$!!!

<u>Solution</u>

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)}{2\pi}\beta_0 \ln\left(\frac{\mu}{\mu_0}\right)}$$

$$\alpha_s = \frac{g^2}{4\pi}$$

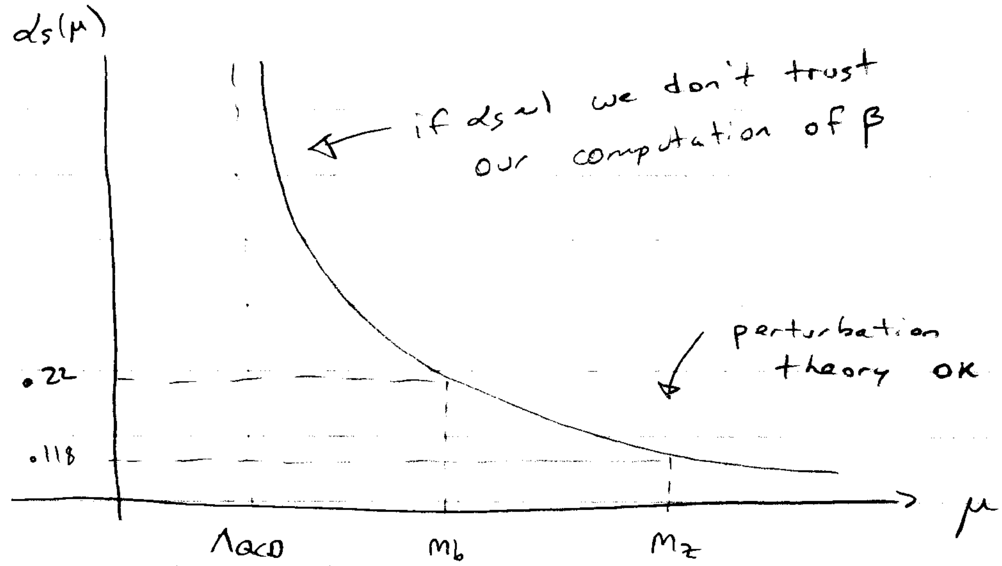
$$\mu\frac{d}{d\mu}\alpha_s = -\frac{\alpha_s^2(\mu)}{2\pi}\beta_0$$

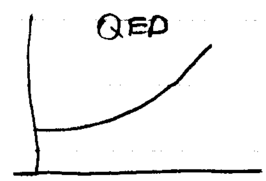
boundary condition is value $\alpha_s(\mu_0)$

<u>or</u>

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \ln(\mu/\Lambda_{QCD})}$$

Λ_{QCD} = dimension-1 constant of integration





- <u>Asymptotic Freedom</u> as $\mu \to \infty$ $\alpha_s \to 0$

  Free quarks at large energy

  ['68 DIS expt $e^- p \to e^- X$ observed weaker int. at larger E]

- We trade dimensionless parameter α_s for dimensionful parameter Λ_{QCD}. <u>Dimensional transmutation</u>

if we consider massless $\mathcal{L}_{QCD}$ then its scale invariant, but this symmetry is broken by Λ_{QCD}

data: $\alpha_s(m_Z) \simeq .118$, $\Lambda_{QCD} \sim 250\,\text{MeV}$ [exact value depends on n_f; order in pert. theory]

for $p \sim \Lambda_{QCD}$ interaction are obviously not perturbative

- At small μ (i.e. long distances as in the lab) quarks are <u>confined</u> into color singlet hadrons

(more later)

qqq baryons $\qquad$ $\bar{q}q$ mesons

In nature $m_u, m_d \ll \Lambda_{QCD}$ so consider $m_q \to 0$.
Only dimensionful parameter is Λ_{QCD} so

$$M_{proton} \propto \Lambda_{QCD}, \qquad (\text{size})^{-1} \propto \Lambda_{QCD} \qquad \text{etc.}$$

- Strong dependence on μ means R.G. is crucial to determine interaction strength

e.g. α_s is twice as big for b-physics as for Z-physics

Heuristic Explanation of Asymptotic Freedom

QED

Consider vacuum fluctuations that gave rise to $e^2(\mu)$ in QED as a dielectric medium.

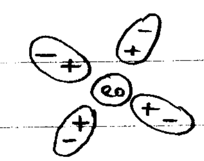
$$e^2(k) = \frac{e_0^2}{1 + b_0 e_0^2 \ln \Lambda^2/k^2} \simeq \frac{e_0^2}{\epsilon(k)}$$

Λ = cutoff

e_0^2 = bare charge $e_0(\Lambda)$

here $b_0 = -\frac{\beta_0}{16\pi^2} > 0$ so $\epsilon > 1$ Fermions screen charge

and $e^2(k)$ is smaller at long distance $\sim k^{-1}$



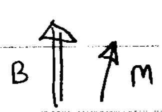
QCD $b_0 < 0$ so $\epsilon < 1$, gluon vacuum fluctuations antiscreen color charge. Why?

[U(1) language OK at lowest order]

$\mu\epsilon = 1$ in the vacuum, so ask why does vacuum screen color magnetic charge, $\mu > 1$. Easier [$\vec{B}$ confines particles to orbits, acts as IR cutoff]

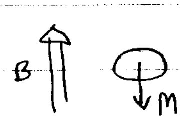
↑ avoids dielectric breakdown from pair prod. too.

Recall



Magnetic field causes intrinsic magnetic moments to line up with $\vec{B}$ giving $\mu > 1$

Paramagnetism $\quad \vec{m} = \frac{\mu - 1}{4\pi\mu}\vec{B} = \frac{\chi_m}{4\pi}\vec{B}$



A current loop develops a magnetic moment to oppose applied $\vec{B}$, giving $\mu < 1$

Diamagnetism

Magnetic Susceptibility $\quad \chi(k) = \frac{\mu - 1}{\mu} = 1 - \frac{1}{\mu} = 1 - \epsilon = -b_0 e_0^2 \ln\left(\frac{\Lambda^2}{k^2}\right)$

Energy density $\quad U = -\frac{1}{2}\chi B^2 = b_0 e_0^2 \ln\left(\frac{\Lambda}{k}\right) B^2$

So compute vacuum energy density for free bosons or fermions with arbitrary spin in a B-field.
Find term $\propto \ln\Lambda\, B^2$ ~~and~~ read off b_0.

<u>Diamagnetic Term</u> In B-field the continuous free particle spectrum $\rightarrow$ discrete Landau levels. Can think of particles executing quantized circular motion. [$\psi(\vec{x})$ does in some gauge]

Take massless charge e particle in $\vec{B} = B\hat{z}$, $\vec{A} = Bx\hat{y}$ ← Landau gauge

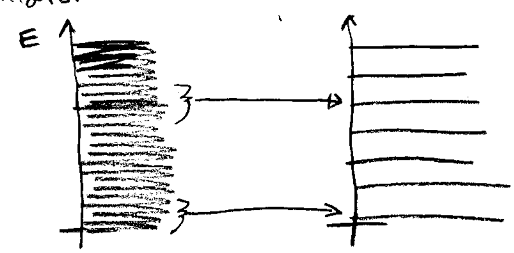
$$H^2 = (\bar{p} - e\bar{A})^2 = p_z^2 + p_x^2 + \underbrace{(p_y - eBx)^2}_{\text{shifted oscillator}};$$

$[H, p_z] = 0$, $[H, p_y] = 0$; [recall Non Rel. Solution]

$$E^2 = p_z^2 + 2eB\left(n + \tfrac{1}{2}\right)$$

degeneracy /unit area $g_n = \dfrac{eB}{2\pi}$

no states disappear



Vacuum energy /unit volume

$$U = \sum_{n=0}^{\infty} \int \frac{dp_z}{2\pi}\left[p_z^2 + (2n+1)eB\right]^{1/2} \frac{eB}{2\pi}$$

As $B \to 0$ we get free cts spectrum. Need $O(B^2)$ term.

Relate Sum to integral

$$\int_{-\frac{1}{2}\epsilon}^{(N+\frac{1}{2})\epsilon} dx\, F(x) = \sum_{n=0}^{N} \int_{(n-\frac{1}{2})\epsilon}^{(n+\frac{1}{2})\epsilon} dx\, F(x) = \sum_{n=0}^{N} \int_{(n-\frac{1}{2})\epsilon}^{(n+\frac{1}{2})\epsilon} dx \left[F(n\epsilon) + (x - n\epsilon)F'(n\epsilon) + \frac{1}{2}(x-n\epsilon)^2 F''(n\epsilon) + \cdots\right]$$

$$= \sum_{n=0}^{N}\left[\epsilon F(n\epsilon) + \frac{\epsilon^3}{24}F''(n\epsilon) + \cdots\right]$$

Invert:

$$\sum_{n=0}^{\infty} \epsilon F(n\epsilon) = \int_0^{\infty} dx\, F(x) - \frac{\epsilon^2}{24}\int_0^{\infty} dx\, F''(x) + \cdots$$

$\epsilon = eB \quad , \quad (n+\tfrac{1}{2})\, eB \to x$

$$F(x) = \left[p_z^2 + 2x \right]^{1/2}$$
$$F'(x) = \left[p_z^2 + 2x \right]^{-1/2}$$
$$F''(x) = -\left[p_z^2 + 2x \right]^{-3/2}$$

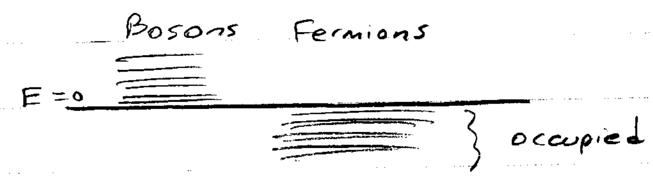
$2x = \vec{q}_\perp^{\,2} \,, \quad dx = \dfrac{d\vec{q}_\perp^{\,2}}{2} = \dfrac{d^2 q_\perp}{(2\pi)}$ so $\displaystyle\sum_n \int \frac{dp_z}{2\pi} \frac{eB}{2\pi} \to \int \frac{d^3p}{(2\pi)^3}$

$$U = \int d^3p \left[\vec{p}_z^{\,2} + \vec{q}_\perp^{\,2} \right]^{1/2} + \frac{(eB)^2}{24} \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\vec{p}^{\,2})^{3/2}} \qquad d^3p = 4\pi p^2 dp$$

↑ free particle

$$= U(B=0) + \frac{e^2B^2}{48\pi^2} \ln \Lambda * \begin{cases} +1 & \text{bosons} \\ -1 & \text{fermions} \end{cases}$$

since sign of fermion vacuum energy is opposite to bosons



thus $\quad b_0^{\text{diamag.}} = \dfrac{1}{48\pi^2} \begin{cases} +1 & \text{boson} \\ -1 & \text{fermion} \end{cases}$

for <u>each</u> spin state

<u>Paramagnetic Term</u>

$$H^2 = (\bar{p} - e\bar{A})^2 - 2eBS_z$$

↑ $g = 2$ for <u>massless</u> elementary particle of any spin

$$E^2 = E_{\text{Landau}}^2 - 2eBS_z$$
$$E = E_{\text{Landau}} - \frac{eBS_z}{E_{\text{Landau}}} - \frac{1}{2} \frac{e^2 S_z^2 B^2}{E_{\text{Landau}}^3}$$

so

$$U^{\text{para}} = -\frac{1}{2} e^2 S_z^2 B^2 \int \frac{d^3p}{(2\pi)^3 (\vec{p}^{\,2})^{3/2}} = -\frac{e^2B^2 \ln\Lambda}{4\pi^2} S_z^2$$

$$b_0^{\text{para}} = -\frac{S_z^2}{4\pi^2} \begin{cases} +1 & \text{boson} \\ -1 & \text{fermion} \end{cases}$$

$$b_0 = \frac{-1}{16\pi^2}\left[4 S_z^2 - \frac{1}{3}\right] \begin{cases} +1 & \text{boson} \\ -1 & \text{fermion} \end{cases}$$ for each state

Fermions: $S_z = \pm \frac{1}{2}$, 2 helicity states, charge Q

$$b_0 = 2\left(\frac{-Q^2}{16\pi^2}\right)\left(1-\frac{1}{3}\right)(-1) = \frac{Q^2}{12\pi^2}$$ QED result ✓

Vectors: $S_z = \pm 1$, 2 states, charge Q

$$b_0 = 2\left(-\frac{Q^2}{16\pi^2}\right)\left(4-\frac{1}{3}\right)(+1) = \frac{-1}{16\pi^2}\frac{22Q^2}{3} < 0$$

compare $$b_0^{QCD} = -\frac{1}{16\pi^2}\left(11 - \frac{2}{3}n_f\right)$$

to get proper factor we need to improve our analysis so that we consider quarks and gluons as carrying effective U(1) charges from the SU(3) algebra [which can be done to give 11, but goes beyond the point of our analysis, we already computed b_0 in QCD directly]

Hence

Asymptotic freedom is consequence of large magnetic moment of spin = 1 charged particles making vacuum paramagnetic

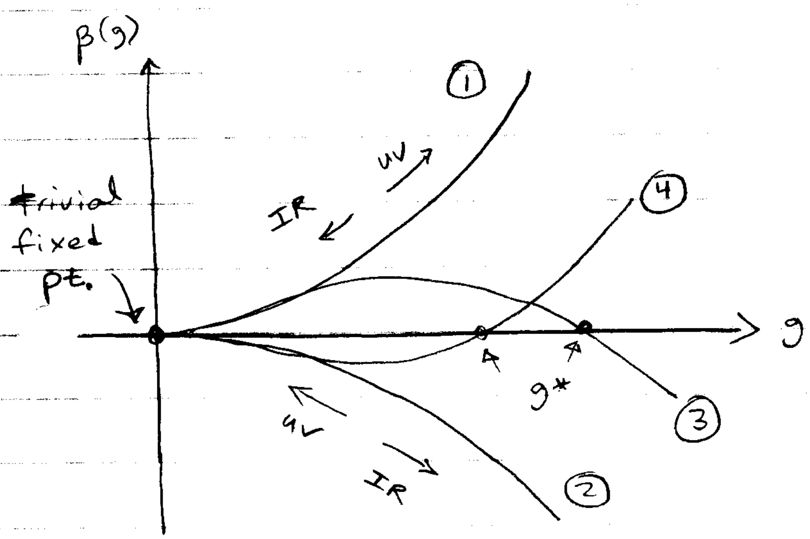
Mag. Moment of Fermions make vacuum diamagnetic because their zero-pt fluctuations have neg. energy

Asymptotic Behavior & Fixed Points

coupling g, $\mu \frac{d}{d\mu} g = \beta(g)$ so

$$\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{\beta(g)} = \ln\left(\frac{\mu}{\mu_0}\right).$$

As $\mu \to 0$ (IR flow)
or $\mu \to \infty$ (UV flow)
either $g(\mu)$ goes to g^* where $\beta(g^*) = 0$ or
g goes to ∞



g^* = fixed points

Case ① $g(\mu)$ is driven away from $g=0$ as $\mu \to \infty$
$\beta(g) > 0$ Looks like QED & ϕ^4 theory (at small g)
If behavior persits than $\int^{\infty} \frac{dg}{\beta(g)} < \infty$ and
g diverges at a finite scale $\mu = M$, $\quad M = \mu_0 \exp \int_{g(\mu_0)}^{\infty} \frac{dg}{\beta(g)}$
leading to unphysical effects

Best viewed as Effective Field theories with $p \ll M$
(where new ops. and d.o.f. are relevant at $p \sim M$)
eg QED $M = e^{647} M_e$ enormous (Weak Int. enter first)

② is like QCD, $\beta < 0$ at small g. Large energy behaviour is under control!
[Well defined QFT without "UV completion"]

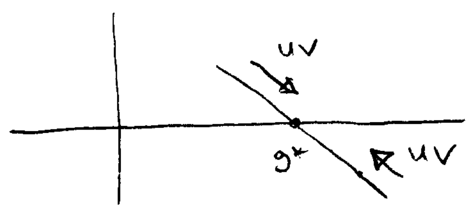
We flow towards trivial fixed point as $\mu \to \infty$

$$\int_{g(\mu_0)}^{g} \frac{dg'}{\beta(g')} \to +\infty \quad \text{for} \quad g \to 0$$

Fixed points at intermediate couplings

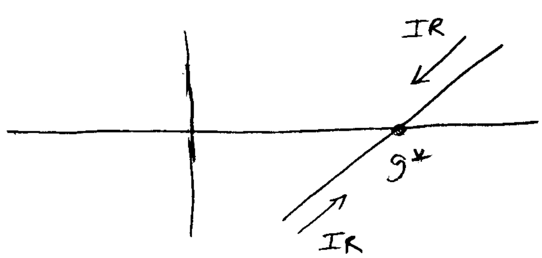
(Note: Same direction of flow for $\beta > 0$, $\beta < 0$ as in ①, ②)

③



$g(\mu \to \infty) = g_*$
UV stable fixed point
here $\beta'(g^*) < 0$

④



$g(\mu \to 0) = g_*$
IR stable fixed point
$\beta'(g^*) > 0$

Existence of these fixed points & slope of β at the fixed point are scheme independent.
Anomalous dimensions at fixed point $\gamma(g^*)$ are scheme independent.

As $E \to 0$, $L \to \infty$, mass sets scale of length and correlation function $G^{(n)} \sim L^{-D} L^{-n\gamma(g^*)} f\left[(T-T_c) L^{2(1-\gamma_m(g^*))}\right]$

For critical phenomena: corr. length $\xi \sim (T-T_c)^{-2(1-\gamma_m)}$
All 2nd order phase transitions with scalar order parameter behave the same way (as a renorm. scalar field theory).

Lets come back to our statement that [with some qualifications] "non-renormalizable" theories with dim $\mathcal{O} > 4$ <u>flow</u> to renormalizable theories at low energy

Recall our eg. at tree level, $\frac{g_5}{\Lambda_0} \bar{\Psi} \sigma^{\mu\nu} F_{\mu\nu} \Psi$ gave

$\sigma \sim \frac{\alpha^2}{E^2} + \frac{\alpha\, g_5^2}{\Lambda_0^2} + \cdots$ in massless QED.

In dim. reg. with say $\overline{MS}$, we get no powers of a cutoff in loop computations. Say we compute a dimless quantity Π with one g_5 insertion

only depends on external low E scales

$$\Pi = \underbrace{\frac{g_5 E}{\Lambda_0}}_{\text{"tree"}} + \underbrace{\frac{g_5}{\Lambda_0} \int d^dp \cdots d^dp_n \frac{(\cdots)}{(\cdots)}}_{\text{"loops"}} \sim E/\Lambda_0 \text{ too!}$$

naive dimensional analysis that says we can drop g_5 for $E \ll \Lambda_0$ carries through.

Though correct this doesn't really explain what's going on, nor what the assumptions are. To do so we consider the <u>Wilsonian R.G.</u> with a hard cutoff, and show that R.G. flow is related to removing high energy modes, and that ops with dim > 4 are suppressed.

Take Scalar Field theory with physical <u>Euclidean</u> cutoff Λ_0

$$S(\phi_0, \Lambda_0) = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi_0)^2 + \underbrace{g_2^0 \phi_0^2 + g_4^0 \phi_0^4 + g_6^0 \phi_0^6 + g_{6'}^0 (\partial^\mu \phi_0)^2 \phi_0^2 + \cdots}_{\sum_i g_i^0 \mathcal{O}_i} \right]$$

which we'll use to describe physics at $p \leq E_0 \ll \Lambda_0$

et $Z[J,\Lambda_0] = \int\limits_{|p|<\Lambda_0} D\phi_0 \; \exp\left[- S_0[\phi_0, \Lambda_0] - \int J \phi_0 \right]$

where $\int\limits_{|p|<\Lambda_0} D\phi_0 = \int \prod\limits_{|p|<\Lambda_0} d\phi_0(p)$

$J(p) = J(p)\, \Theta(\epsilon_0^2 - p^2)$ source for small momenta

Let $\phi_0(p) = \phi_1 + \chi = \phi_1(p)\, \Theta(\Lambda_1 - |p|) + \chi(p)\, \Theta(\Lambda_1 < |p| < \Lambda_0)$

where $\Lambda_1 = b\Lambda_0 < \Lambda_0 \qquad (b<1)$

<u>Remove χ modes</u>

χ propagator is $\propto \dfrac{\Theta(\Lambda_1 < |k| < \Lambda_0)}{k^2}$

Integrate over χ to get $\qquad J\phi_0 = J\phi_1$

$Z[J,\Lambda_0] = \int\limits_{p<\Lambda_1} D\phi_1 \; \exp\left[- S_1(\phi_1, \Lambda_1) + \int J\phi_1 \right]$

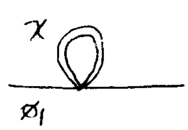
What is S_1 ? In fact

$$S_1[\phi_1, \Lambda_1] = \int d^4x \left[\frac{(\partial^\mu \phi_1)^2}{2} + \sum_i g_i^{(1)}(\Lambda_1, \Lambda_0, \vec{g}^{(0)})\, O_i(\phi_1) \right]$$

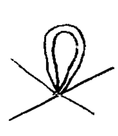
Its easy to imagine proving this by working to all orders in pert. theory.

Expand $\phi_0^4 = \phi_1^4 + 4\phi_1^3\chi + 6\phi_1^2\chi^2 + 4\phi_1\chi^3 + \chi^4$, etc

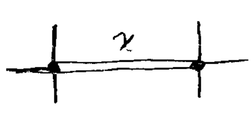
$\phi_1^2\chi^2$:



$= g_4^{(0)} \phi_1^2 \int_{\Lambda_1}^{\Lambda_0} \frac{d^dk}{k^2} = \phi_1^2 \underbrace{g_4^{(0)} f(\Lambda_0, \Lambda_1)}_{\text{term in } g_2^{(1)}}$



term $g_6^{(0)} f(\Lambda_0, \Lambda_1)$ in $g_4^{(1)}$



term $(g_4^{(0)})^2 f(\Lambda_0, \Lambda_1)$ in $g_6^{(1)}$

[so $g_6^{(1)}$ is generated even if initially $g_6^{(0)} = 0$]

We need small steps (b close to 1) to guarantee locality of S_1.

Repeat $S_0 \to S_1 \to S_2 \to S_3 \to \dots \to S_n$
until $\Lambda_n \simeq E_0$ and we only have low energy modes

Consider dimensionless $\lambda_i(\Lambda) = \Lambda^{-\Delta_i} g_i(\Lambda)$
then the above process gives

$$\lambda_i(\Lambda') = G_i\left(\lambda_j(\Lambda), \frac{\Lambda'}{\Lambda}\right)$$

no other dimensionful parameters, IR, UV safe

$\Lambda' = \Lambda_K$
$\Lambda = \Lambda_{K-1}$

take $\Lambda' \, d/d\Lambda'$, then set $\Lambda' = \Lambda$

$$\boxed{\Lambda \frac{d}{d\Lambda} \lambda_i(\Lambda) = \beta_i(\vec{\lambda}(\Lambda))}$$

where
$$\beta_i = \left[\frac{\partial}{\partial z} G_i(\vec{\lambda}(\Lambda), z)\right]_{z=1}$$

Wilsonian R.G.E. (compare to earlier β-fn defn)

Our Goal: Think of set of local interactions as ∞-dim space parameterized by the couplings. We want to show that for $\Lambda \ll \Lambda_0$ we flow to a <u>stable</u> subspace parameterized by only renormalizable couplings (and independent of Λ_0 & initial conditions $g_i^{(0)}$)

Proof in Polchinski handout, we'll study simplified example
eg. $\lambda_4 = g_4$, $\lambda_6 = \Lambda^2 g_6$

$$\left.\begin{aligned} \Lambda \frac{d}{d\Lambda} \lambda_4 &= \beta_4(\lambda_4, \lambda_6) \\ \Lambda \, d/d\Lambda \, \lambda_6 &= 2\lambda_6 + \beta_6(\lambda_4, \lambda_6) \end{aligned}\right] \quad (*)$$

Let $\bar{\lambda}_i$ solve $(*)$ and consider small perturbation

$$\lambda_i = \bar{\lambda}_i + \epsilon_i$$

evaluated at $\bar{\lambda}_i$

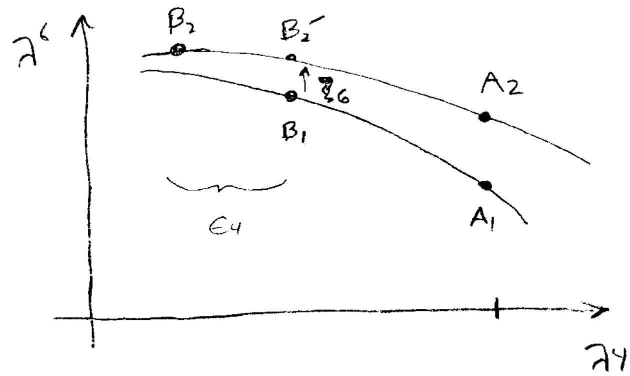
then $$\Lambda \frac{d\epsilon_4}{d\Lambda} = \overline{\frac{\partial \beta_4}{\partial \lambda_4}}\, \epsilon_4 + \overline{\frac{\partial \beta_4}{\partial \lambda_6}}\, \epsilon_6$$ ← linearized in ϵ_i

$$\Lambda \frac{d\epsilon_6}{d\Lambda} = 2\epsilon_6 + \overline{\frac{\partial \beta_6}{\partial \lambda_4}}\, \epsilon_4 + \overline{\frac{\partial \beta_6}{\partial \lambda_6}}\, \epsilon_6$$

damping term

Flow $A_2 \to B_2$
$A_1 \to B_1$

Converging, but perhaps B_2 gets there first (don't use $\| B_2 - B_1 \|$ for convergence)



Use distance $\| B_2' - B_1 \|$ by
$$\xi_6 = \epsilon_6 - \frac{(d\bar{\lambda}_6/d\Lambda)}{\underbrace{(d\bar{\lambda}_4/d\Lambda)}_{\text{slope}}}\, \epsilon_4$$
↖ subtract off tangent vector along trajectory

... get
$$\Lambda\, d/d\Lambda\ \xi_6 = 2\xi_6 + \left\{ \overline{\frac{\partial \beta_6}{\partial \lambda_6}} + \overline{\frac{\partial \beta_4}{\partial \lambda_4}} - \Lambda \frac{d}{d\Lambda} \ln \bar{\beta}_4 \right\} \xi_6$$

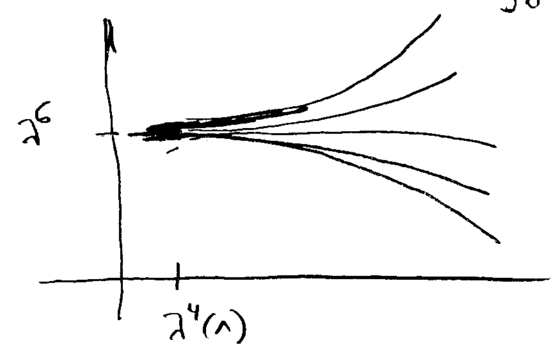
[Hint: to derive it note that $\Lambda \frac{d\bar{\lambda}_i}{d\Lambda}$ obey linearized diff eqtns analogous to those for ϵ_i]

Solution
$$\xi_6(\Lambda) = \xi_6(\Lambda_0) \left(\frac{\Lambda^2}{\Lambda_0^2}\right) \frac{\bar{\beta}_4(\Lambda_0)}{\bar{\beta}_4(\Lambda)} \exp\left(\int_{\Lambda_0}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \left(\overline{\frac{\partial \beta_6}{\partial \lambda_6}} + \overline{\frac{\partial \beta_4}{\partial \lambda_4}} \right) \right)$$

If couplings are such that $\frac{\bar{\beta}_4(\Lambda_0)}{\bar{\beta}_4(\Lambda)}$ and integrand are slowly varying (eg. if they were all pert.) then
not beating the quadratic

$$\xi_6(\Lambda) \to 0 \quad \text{for} \quad \Lambda \ll \Lambda_0$$

We converge to trajectory where value of $\lambda^4(\Lambda)$ determines λ^6, independent of Λ^0 & initial conditions



So action only depends on renormalizable couplings (Q.E.D.)

[We don't get zero for λ^6 in infrared, because recall that λ_4 induces λ^6 when integrating out modes.]

Advantages of Wilsonian RGE

- exact correspondence with modes is clear
- we get proof of renormalizability without explicitly dealing with subdivergences or overlapping div.

Disadvantages

- always have non-renorm. operators
- cutoff destroys symmetries: manifest gauge inv, chiral symmetry, Lorentz Inv, ...

Luckily the physical implications of the RGE are the same whether its Wilsonian or a Gell-Mann-Low RGE.

Ch 3. Spontaneous Symmetry Breaking (SSB)

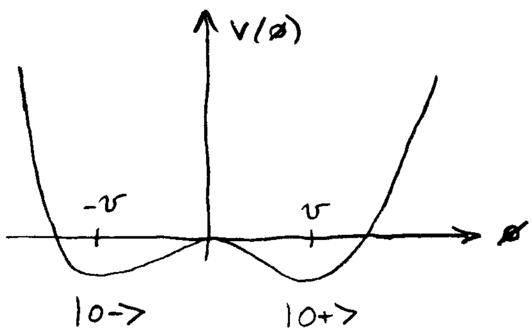
<u>SSB</u> A conserved charge Q, $\frac{dQ}{dt} = 0$, which is symmetry of the dynamics (& hence the Lagrangian), but <u>not</u> a symmetry of the vacuum.

$\partial_\mu J^\mu = 0$, $[Q, H] = 0$ still but $e^{i\alpha Q}|0\rangle \neq |0\rangle$ ie $Q|0\rangle \neq 0$

$\boxed{\text{Eg 1}}$ $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$$

Discrete
$\mathbb{Z}_2$ symmetry $\phi \to -\phi$,

negative mass term $m^2 = -\mu^2$



minima of potential are at
$$\phi = \phi_0 = \pm v = \pm\sqrt{\frac{6}{\lambda}}\,\mu$$
not $\phi = 0$

$\langle 0+|\phi|0+\rangle = v$, $\langle 0-|\phi|0-\rangle = -v$

v = vacuum expectation value (vev)

For QFT in ∞ Volume the tunneling probability is zero, and we will sit in $|0+\rangle$ or $|0-\rangle$.

[See Weinberg Ch 19.1]

Pick $|0\rangle = |0+\rangle$.

Look at field fluctuations about minima
$$\phi(x) = v + \rho(x)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\mu^2(v^2 + \cancel{2v\rho} + \rho^2) - \frac{\lambda}{4!}(v^4 + \cancel{4v^3\rho} + 6v^2\rho^2 + 4v\rho^3 + \rho^4)$$

$$= \frac{1}{2}(\partial_\mu\rho)^2 - \frac{1}{2}(2\mu^2)\rho^2 - \sqrt{\frac{\lambda}{6}}\,\mu\rho^3 - \frac{\lambda}{4!}\rho^4 + \text{constant}$$

- $m_\rho = \sqrt{2}\,\mu > 0$
- ρ^3 and ρ^4 interactions
- hidden $\mathbb{Z}_2$: $\phi = v + \rho \to -\phi = -v - \rho$ is $\rho \to -\rho - 2v$

  "spontaneously broken"

<u>Renormalizability?</u>

Original $\mathcal{L}$ needs $\phi_0 = Z_\phi^{1/2}\phi$, $\mu_0\phi_0 = Z_\mu^{1/2}\mu\phi$, $\lambda_0 = Z_\lambda \lambda$

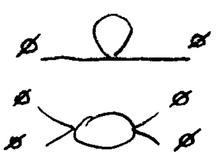
New $\mathcal{L}$ has ρ^3 term (Z_{ρ^3} ?) . Quantum corrections could also induce linear ρ term  $\to \rho$

(Z_ρ ?)

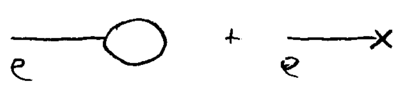
Z_ϕ, Z_μ, Z_λ are enough to renormalize shifted theory.

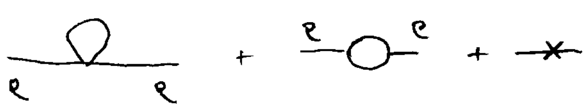
$$\mathcal{L} = Z_\phi \tfrac{1}{2}(\partial_\mu \rho)^2 + \rho\mu^2 v\,(Z_\mu - Z_\lambda Z_\phi) - \tfrac{1}{2}\rho^2\mu^2(3Z_\lambda Z_\phi^2 - Z_\mu)$$

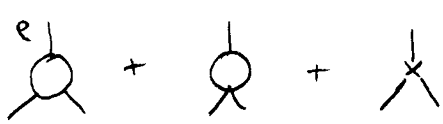
$$- \frac{Z_\lambda Z_\phi^2}{6}\lambda v \rho^3 - \frac{Z_\lambda Z_\phi^2}{4!}\lambda \rho^4$$

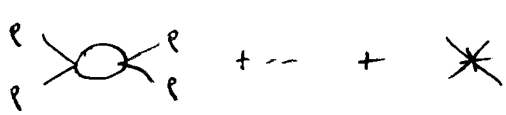
<u>1-loop</u>  gives Z_μ ($Z_\phi \propto \lambda^2$ two-loops)

+ ... get Z_λ

then  = finite

 = finite

 = finite

 = finite

Often its convenient to replace our renormalization condition on the mass $\left(\text{—(1PI)—} \big|_{p^2=m^2} = 0 \right)$ by

(1PI)—— $= 0$. This then ensures that

$$v^{\text{full}} \equiv \langle 0|\phi|0\rangle = v^{\text{tree}} = \sqrt{\frac{6}{\lambda}}\,\mu$$

without finite loop corrections.

Goldstone's Theorem For every spontaneously broken continuous global symmetry there is a massless particle.

$\boxed{\text{eg } 2}$ O(N) model $\quad \mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi}\cdot\partial^\mu\vec{\phi} - \lambda(\vec{\phi}\cdot\vec{\phi} - v^2)^2$

$$\vec{\phi} = (\phi_1, \ldots, \phi_N)$$

Symmetry group $G = O(N)$ has $\frac{N(N-1)}{2}$ cts. symmetries (eg $O(3)$, 3 rotations)

$$\delta\phi_i = i\,\epsilon^a\, T^a_{ij}\,\phi_j$$

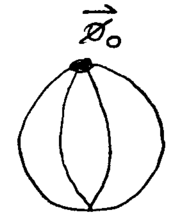
↑ rotation matrix

Symmetry current $\quad J^a_\mu = \pi_{\mu i}\, i T_a^{ij}\,\phi_j = i\,\partial_\mu\vec{\phi}^T\cdot T^a\cdot\vec{\phi}$

$\partial_\mu J^\mu = 0$

Minimize potential: $\vec{\phi}^2 = v^2$. Rather than 2 vacua we have vacuum manifold $\phi_1^2 + \phi_2^2 + \cdots + \phi_N^2 = v^2$ ie S^{N-1}. Each point on S^{N-1} is allowed vacuum. Vacua are equivalent, related by O(N) symmetry

Symmetry is "broken" by choice of vacuum around which we quantize the theory



e.g. $\vec{\phi}_0 = \underbrace{(0, 0, \ldots, 0}, v)$ standard vacuum

$\vec{\phi}_0$ leaves $H_0 = O(N-1)$ as unbroken symmetry

Say $G \xrightarrow{\vec{\phi}_0} H_0$, $G/H_0 = S^{N-1}$

$\langle 0 | \phi_N | 0 \rangle = v$

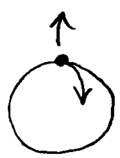
$\langle 0 | \phi_{i \neq N} | 0 \rangle = 0$

Fluctuations about $\vec{\phi}_0$: $\vec{\phi} = (\phi_1, \ldots, \phi_{N-1}, v + \rho)$
$= (\vec{\phi}_T, v+\rho)$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \vec{\phi}_T)^2 + \frac{1}{2} (\partial_\mu \rho)^2 - \lambda (\vec{\phi}_T \cdot \vec{\phi}_T + 2v\rho + \rho^2)^2$$

no quadratic term for $\vec{\phi}_T \rightarrow$ $\vec{\phi}_T$ are <u>massless</u>

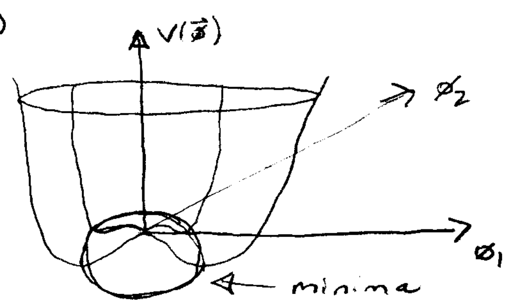
ρ is massive $m_\rho^2 = 8\lambda v^2$

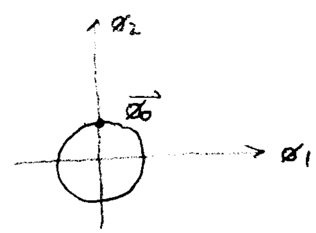


- "radial modes" cost energy and are massive
  They are ones taking us off G/H_0
- "Goldstone" modes move us along G/H_0
  and are massless [tangent = along locally]

$\dim G - \dim H_0 = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1$ massless modes

e.g. $O(2)$





Charges

$$\langle 0 | [Q^a, \phi_i] | 0 \rangle = T^a_{ij} \langle 0 | \phi_j | 0 \rangle = T^a_{iN} \, v$$

- broken symmetry : $Q^a |0\rangle \neq 0$, $T^a_{ij} \phi_0^j \neq 0$
  generator "a"
- unbroken symmetry : $Q^b |0\rangle = 0$, $T^b_{ij} \phi_0^j = 0$
  generator "b"

SSB in Effective Action

Taking $J \to 0$

$$\langle 0 | \phi(x) | 0 \rangle = \left. \frac{\delta W[J]}{\delta J(x)} \right|_{J=0} = \phi_{cl}(x)$$

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x) = 0$$ has solution $\phi_{cl}(x)$, extremum of eff. action

Lets make a derivative expansion of $\Gamma[\phi]$

$$\Gamma[\phi] = \int d^4x \left[- \mathcal{U}(\phi) + \frac{1}{2} Z(\phi)(\partial^\mu \phi)^2 + \cdots \right]$$

Specialize to ϕ_{cl} independent of x^μ (solutions to $\frac{\delta \Gamma}{\delta \phi} = 0$ that depend on x^μ are solitons)

which is appropriate for Lorentz Invariant vacuum state

$$\Gamma[\phi_{cl}] = - \mathcal{V}_4 \, \mathcal{U}(\phi_{cl}) = - \mathcal{V}_4 \, V_{eff}(\phi_{cl})$$

↑ "effective potential"

$$\left. \frac{\partial V_{eff}(\phi)}{\partial \phi} \right|_{\phi = \phi_{cl}} = 0$$ is extremum condition, just like a classic potential, but V_{eff} includes quantum corrections

Proof of Goldstone's Thm

Symmetries of action are also symmetries of $V \equiv V_{eff}$

$$\delta_a V(\phi) = \frac{\partial V}{\partial \phi_i} \delta_a \phi_i \overset{\text{cts symm.}}{=} \frac{\partial V}{\partial \phi_i} T^a_{ij} \phi_j = 0$$

take $\frac{\partial}{\partial \phi_K}$:
$$\underbrace{\frac{\partial V}{\partial \phi_i}}_{0} T^a_{iK} + \frac{\partial^2 V(\phi)}{\partial \phi_i \partial \phi_K} T^a_{ij} \phi_j = 0$$

vacuum $\phi = \phi_0$
$$0 + \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_K}\right|_{\phi=\phi_0} T^a_{ij} \phi_{0j} = 0$$

Recall: Inverse Propagator $\Delta^{-1}_{ij}(k) = \left.\frac{\delta^2 \Gamma[\phi]}{\delta \phi_i(k)\, \delta \phi_j(-k)}\right|_{\phi=\phi_0}$

Quadratic term in eff. ptnl. is <u>mass</u> <u>matrix</u>
$$M_{ij} = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\phi=\phi_0}$$

So $\quad M_{Ki} \left(T^a_{ij} \phi_{0j} \right) = 0$

For every broken generator a, $T^a_{ij} \phi_{0j} \neq 0$, and we have eigenvector of mass matrix with eigenvalue zero. A massless Goldstone mode for each broken gen.

1-Loop Effective Potential

— Setup perturbation theory

$$\mathcal{L} = \underbrace{\mathcal{L}_1}_{\text{renormalized parameters}} + \delta\mathcal{L} \leftarrow \text{counterterms}$$

$\phi_{cl}[J] = \langle 0|\phi|0\rangle_J$

Let $\left.\dfrac{\delta \mathcal{L}_1}{\delta\phi}\right|_{\phi=\phi_{cl}} = -J_1(x)$ [classical field eqtn for $\mathcal{L}_1$]

Full source $J(x) = J_1(x) + \delta J(x)$ where δJ is fixed by demanding $\langle\phi\rangle_J = \phi_{cl}$

$$e^{iW[J]} = \int \mathcal{D}\phi \; e^{i\int d^4x \, [(\mathcal{L}_1 + J_1\phi) + (\delta\mathcal{L} + \delta J\phi)]}$$

Expand about ϕ_{cl}: $\phi = \phi_{cl} + \eta$

$$\int d^4x \, (\mathcal{L}_1 + J_1\phi) = \int d^4x \, (\mathcal{L}_1(\phi_{cl}) + J_1\phi_{cl}) + \int d^4x \, \eta(x) \underbrace{\left(\frac{\delta\mathcal{L}_1}{\delta\phi} + J_1 \right)}_{\to 0}$$

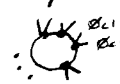
$$+ \frac{1}{2}\int d^4x\, d^4y \; \eta(x)\eta(y) \frac{\delta^2\mathcal{L}_1}{\delta\phi(x)\delta\phi(y)} + \underbrace{\frac{1}{3!}\int d^4x\, d^4y\, d^4z \; \eta(x)\eta(y)\eta(z) \frac{\delta^3\mathcal{L}_1}{\delta\phi\,\delta\phi\,\delta\phi}}_{\text{perturbation}}$$

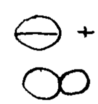
$+ \cdots$

Quadratic Path Integral:

$$\int \mathcal{D}\phi \; e^{i\int d^4x\,(\mathcal{L}_1 + J_1\phi)} = \underbrace{e^{i\int d^4x\,(\mathcal{L}_1(\phi_{cl}) + J_1\phi_{cl})}}_{\text{tree-level}} \underbrace{\left(\det\left(\frac{-\delta^2\mathcal{L}_1}{\delta\phi\,\delta\phi} \right) \right)^{-1/2}}_{\text{One-loop}} \Big(1 + \underbrace{\cdots}_{\text{2-loop corrections}} \Big)$$

η-loop in ϕ_{cl} bkgnd

$= \sum_i$ 



include $e^{i\int d^4x\,(\delta\mathcal{L}_1 + \delta J\phi)}$: $\quad * \; e^{i\int d^4x\,(\delta\mathcal{L}_1(\phi_{cl}) + \delta J\phi_{cl})} \quad * \quad [\,1 + \cdots\,]$

← all we need at 1-loop is tree level counterterms

to 1-loop

$$W[J] = \int d^4x\,(\mathcal{L}_1(\phi_{cl}) + J_1\phi_{cl}) + \frac{i}{2}\ln\det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi\,\delta\phi}(\phi_{cl}) \right) + \int d^4x\,(\delta\mathcal{L}_1(\phi_{cl}) + \delta J\phi_{cl})$$

$$\Gamma[\phi_{cl}] = \int d^4x\; \mathcal{L}_1(\phi_{cl}) + \frac{i}{2}\ln\det\left(-\frac{\delta^2\mathcal{L}_1(\phi_{cl})}{\delta\phi\,\delta\phi} \right) + \int d^4x\; \delta\mathcal{L}_1(\phi_{cl})$$

eg. O(N) model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}\mu^2 \vec{\phi}^2 - \lambda(\vec{\phi}^2)^2 \qquad (\mu = 2\sqrt{\lambda}\, v)$$

$$= \frac{1}{2}\mu^2 \vec{\phi}_{cl}^{\,2} - \lambda \vec{\phi}_{cl}^{\,4} + (\mu^2 - 4\lambda\vec{\phi}_{cl}^{\,2})\vec{\phi}_{cl}\cdot\vec{\eta}$$
$$+ \frac{1}{2}(\partial_\mu \vec{\eta})^2 + \frac{1}{2}\mu^2\vec{\eta}^{\,2} - \lambda\left[2\vec{\phi}_{cl}^{\,2}\vec{\eta}^{\,2} + 4(\vec{\phi}_{cl}\cdot\vec{\eta})^2\right] + \mathcal{O}(\eta^3)$$

$$\frac{\delta^2 \mathcal{L}}{\delta\phi^i \delta\phi^j} = -\partial^2 \delta^{ij} + \mu^2\delta^{ij} - \lambda\left[4\vec{\phi}_{cl}^{\,2}\delta^{ij} + 8\phi_{cl}^i \phi_{cl}^j\right]$$

$$\det\left(-\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\right) = \left[\det(\partial^2 + (4\lambda\vec{\phi}_{cl}^{\,2} - \mu^2))\right]^{N-1} \det(\partial^2 + (12\lambda\vec{\phi}_{cl}^{\,2} - \mu^2))$$

for $\phi_{cl}^i = \delta^{iN} v$

Need:

$$\left[\begin{aligned} \det A &= \prod_i a_i \\ &= \exp\left[\sum_i \ln a_i\right] \\ &= \exp\left[\mathrm{Tr}\ln B\right] \end{aligned}\right.$$

$$\ln\det(\partial^2 + m^2) = \mathrm{Tr}\ln(\partial^2 + m^2)$$
$$= \sum_k \ln(-k^2 + m^2)$$
$$= \mathcal{V}_4 \int đ^4 k \, \ln(-k^2 + m^2)$$

regulate

$$= \mathcal{V}_4\, i \int đ^4 k_E \ln(k_E^2 + m^2) = \left(\mathcal{V}_4\, \mu_{ms}^{2\epsilon}\right)\left(-i\frac{\partial}{\partial\alpha}\right)\int đ^d k_E\, (k_E^2 + m^2)^{-\alpha}\Big|_{\alpha=0}$$

a standard dim-reg. integral

$$= \mathcal{V}_4\, (-i)\, \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} (m^2)^{d/2} \mu_{ms}^{2\epsilon}$$

counterterms

$$\text{So} \quad V_{eff}(\phi_{cl}) = -\Gamma[\phi_{cl}]/\mathcal{V}_4$$
$$= -\frac{1}{2}\mu^2\vec{\phi}_{cl}^{\,2} + \lambda\vec{\phi}_{cl}^{\,4} + \frac{1}{2}\delta_\mu \vec{\phi}_{cl}^{\,2} + \delta_\lambda \vec{\phi}_{cl}^{\,4}$$
$$- \frac{\Gamma(-d/2)}{2(4\pi)^{d/2}} \mu_{ms}^{2\epsilon}\left[(N-1)(4\lambda\vec{\phi}_{cl}^{\,2} - \mu^2)^{d/2} + (12\lambda\vec{\phi}_{cl}^{\,2} - \mu^2)^{d/2}\right]$$
$$+ (2\text{-loops}) + \cdots$$

Expand in $\overline{ms}$, cancel $1/\epsilon$'s:

$$V_{eff}^{\overline{ms}}(\phi_{cl}) = -\frac{1}{2}\mu^2\phi_{cl}^2 + \lambda\phi_{cl}^4 + \frac{1}{64\pi^2}\Big[(N-1)(4\lambda\phi_{cl}^2 - \mu^2)^2\left[\ln\left(\frac{4\lambda\phi_{cl}^2 - \mu^2}{\mu_{\overline{ms}}^2}\right)\right.$$
$$\left.\left. - \frac{3}{2}\right] + (12\lambda\phi_{cl}^2 - \mu^2)^2\left[\ln\left(\frac{12\lambda\phi_{cl}^2 - \mu^2}{\mu_{\overline{ms}}^2}\right) - \frac{3}{2}\right]\right]$$

$\lambda = \lambda(\mu_{\overline{ms}})$

$m = m(\mu_{\overline{ms}})$

Note: • function of $\vec{\phi}_{cl}^{\,2}$, so $O(N)$ symmetric

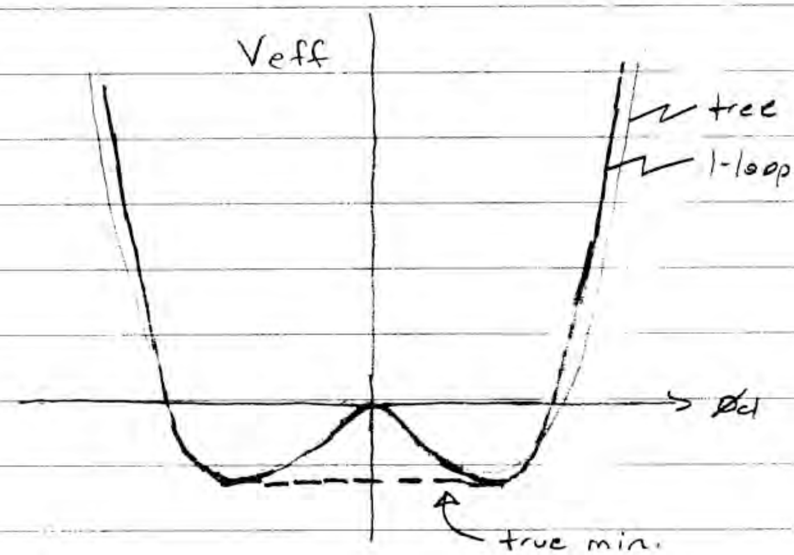
• $\left.\dfrac{\partial V_{eff}}{\partial \phi_{cl}}\right|_{\vec{\phi}_{cl}=\phi_0\delta_{iN}} = 0$ determines $\phi_0 = v$

corrects tree level relation

$$v^2 = \mu^2/4\lambda$$

• pole mass $m_{pole} = \mu$ + corrections $\leftrightarrow$ $\overline{MS}$ parameters

eg. $N=1$



Final Discussion:

Our calculation is only valid for $|\vec{\phi}_{cl}| \geq |\text{minimum}|$.

2 issues • $V_{eff}(\phi_c)$ has $\ln(4\lambda\vec{\phi}_c^{\,2}-\mu^2-i0)$ & $\ln(12\lambda\vec{\phi}_c^{\,2}-\mu-i0)$ and has imaginary terms for $\vec{\phi}_c^{\,2} < \mu^2/4\lambda$

• $V_{eff}(\phi_c)$ is the minimum of an energy density $\dfrac{\langle 0|H|0\rangle}{v_3}$ for all states satisfying $\langle 0|\vec{\phi}|0\rangle = \vec{\phi}_{cl}$ (Weinberg 16.3) and can be proven to be convex

This is violated between our minima.

It turns out our calculation is not valid in this region, one must simultaneously account for the multiple minima when evaluating the $\int \mathcal{D}\phi$. The imaginary part indicates we can decay to a lower energy state, giving a flat region btwn minima.

eg. $N=1$ $\quad |\lambda\rangle = \sqrt{\lambda}\,|0+\rangle + \sqrt{1-\lambda}\,|0-\rangle$, $\quad \langle 0+|0-\rangle = 0$ (∞-volume)

$\langle\lambda|\lambda\rangle = 1$, $\quad 0 \leq \lambda \leq 1$

if $\langle 0\pm|H|0\pm\rangle = E_0$, $\quad \langle\lambda|H|\lambda\rangle = E_0$

and $\langle\lambda|\phi|\lambda\rangle = \lambda\langle 0+|\phi|0+\rangle + (1-\lambda)\langle 0-|\phi|0-\rangle = \lambda v + (1-\lambda)(-v)$

$= \phi_{cl}$ that interpolates between two minima

Another Proof of Goldstone's Thm

Symmetry $\Rightarrow$ conserved current $\partial^\mu j_\mu(x) = 0$

charge $Q = \int d^3x\, j^0(x)$

$[Q, \phi_n] = T_{nm}\,\phi_m$

- don't refer to vacuum, so still true when spont. broken

Given conserved $j_\mu(x)$ and $\epsilon > 0$, lets prove that if every state but vacuum has $p^2 > \epsilon$, then $\langle 0|\delta\phi|0\rangle = 0$.

Consider

$$\langle 0|j_\mu(x)\,\phi(0)|0\rangle = \sum_n \langle 0|\, j_\mu(x)\,|n\rangle\langle n|\,\phi(0)\,|0\rangle$$

$$= \sum_n \langle 0|\, e^{i\hat{P}\cdot x} j_\mu(0)\, e^{-i\hat{P}\cdot x}\,|n\rangle\langle n|\,\phi(0)\,|0\rangle$$

$$= \sum_n e^{-iP_n\cdot x}\,\langle 0|\, j_\mu(0)\,|n\rangle\,\langle n|\,\phi(0)\,|0\rangle$$

$$= \int d^4k\; e^{-ik\cdot x}\;\underbrace{\sum_n \delta^4(k-P_n)\,\langle 0|\,j_\mu(0)|n\rangle\langle n|\phi(0)|0\rangle}_{\equiv\ \frac{i}{(2\pi)^3}\,\theta(k^0)\,k^\mu\,\rho(k^2)}$$

$$\langle 0|\,\phi(0)\,j_\mu(x)\,|0\rangle = \int d^4k\; e^{ik\cdot x}\;\underbrace{\sum_n \delta^4(k-P_n)\,\langle 0|\phi(0)|n\rangle\langle n|\,j_\mu(0)|0\rangle}_{\frac{i}{(2\pi)^3}\,\theta(k^0)\,k^\mu\,\tilde{\rho}(k^2)}$$

Vacuum doesn't contribute $\langle 0|j_\mu|0\rangle = 0$, so no state with $P_n^2 < \epsilon$ contributes, $\rho(k^2<\epsilon) = 0$

$\tilde{\rho}(k^2<\epsilon) = 0$

Act with ∂^μ

$$0 = \int d^4k\; e^{-ik\cdot x}\; k^2\,\rho(k^2)\;\frac{\theta(k^0)}{(2\pi)^3}$$

so $0 = k^2\rho(k^2)$ and $\rho(k^2) = 0$ for any k^2

$0 = k^2\tilde{\rho}(k^2)$ $\qquad \tilde{\rho}(k^2) = 0$

so in particular

$$\int d^3x \quad \langle 0 | [j_0(\vec{x},t), \phi(\vec{y},t)] | 0 \rangle = \langle 0 | \delta\phi | 0 \rangle = 0$$

Λ

Q.E.D.

[might need regulator]

So $\langle 0 | \delta\phi | 0 \rangle \neq 0$ and $\partial^\mu j_\mu = 0 \Rightarrow \exists$ physical state with $p^2 = 0$ and nonzero $\rho, \tilde{\rho}$.

G-Thm

Lets construct ρ & $\tilde{\rho}$

$$\langle 0 | [j_\mu(x), \phi(0)] | 0 \rangle = \frac{i}{(2\pi)^3} \int d^4k \; \theta(k^0) \, k^\mu \left(e^{-ik\cdot x} \rho(k^2) - e^{ik\cdot x} \tilde{\rho}(k^2) \right)$$

$$= \frac{-\partial}{\partial x^\mu} \int d\mu^2 \left[\rho(\mu^2) \, \Delta_+(x,\mu^2) + \tilde{\rho}(\mu^2) \, \Delta_+(-x,\mu^2) \right]$$

where $\Delta_+(x,\mu^2) \equiv \int \frac{d^4p}{(2\pi)^3} \, \theta(p^0) \, \delta(p^2-\mu^2) \, e^{-ip\cdot x}$

Spacelike $x^2 < 0$ $\quad \Delta_+(x) = \Delta_+(-x)$, $\quad [j_\mu(x), \phi(0)] = 0$

so $\rho_n(\mu^2) = -\tilde{\rho}_n(\mu^2)$

Also $\left. \frac{\partial}{\partial x^0} \Delta_+(x,\mu^2) \right|_{t=0} = \frac{-i}{2} \, \delta^3(\vec{x})$

So $\langle 0 | [j_0(\vec{x},0), \phi(0)] | 0 \rangle = i \, \delta^3(\vec{x}) \int d\mu^2 \, \rho_n(\mu^2)$

integrate d^3x

$$\langle 0 | [Q^a, \phi_i(0)] | 0 \rangle = i \int d\mu^2 \, \rho^a_i(\mu^2)$$

↑ put back indices

Solution: $\rho^a_i(\mu^2) = -i \, \delta(\mu^2) \, T^a_{ij} \langle 0 | \phi_j | 0 \rangle$

↑ single particle states of zero mass

Call the Goldstone States $|\pi^a(p)\rangle$, $p^2=0$

$$\frac{i\,\theta(k^0)\,k^\mu\,e_i^a(k^2)}{(2\pi)^3} = \sum_n \delta^4(k-P_n)\,\langle 0|\,j_\mu^a\,|n\rangle\langle n|\,\phi_i\,|0\rangle$$
$$= \sum_b \int \frac{d^3p_\pi^b}{2E_\pi^b}\,\delta^4(k-P_\pi^b)\,\langle 0|\,j_\mu^a\,|\pi_b\rangle\langle \pi_b|\,\phi_i\,|0\rangle$$

$$\frac{\theta(k^0)\,\delta(k^2)\,k^\mu\langle 0|\,\delta^a\phi_i\,|0\rangle}{(2\pi)^3} = \frac{\delta(k^2)\,\theta(k^0)}{(2\pi)^3}\sum_b \langle 0|\,j_\mu^a(0)\,|\pi_b(k)\rangle\langle \pi_b(k)|\,\phi_i\,|0\rangle \qquad \boxed{*}$$

Comments broken gen. "a"

① $e_i^a \neq 0$ so $\langle \pi_b(k)|\,\phi^i(0)\,|0\rangle = z_b^i \neq 0$

and $|\pi\rangle$ is Lorentz scalar

Lorentz Inv.

Also $\langle 0|\,j_\mu^a(x)\,|\pi^b(p)\rangle = -i\,f\,\underbrace{\delta^{ab}}_{}\,p_\mu\,e^{-ip\cdot x} \neq 0$

pick states so its diagonal

so $|\pi\rangle$ has same internal quantum numbers and parity as $\dot{j}_0^a$ $[\,P_0 \xrightarrow{P} P_0\,]$

[if $z=0$ or $f=0$ then $\langle 0|\,\delta\phi\,|0\rangle = 0$]

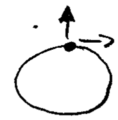
② This Proof is non-perturbative. Massless particles do not have to correspond to elementary fields in $\mathcal{L}$.
The formula $\boxed{*}$ holds even when ϕ is composite field,
eg. $\phi(x) = \bar{\psi}\,\gamma_5\,\psi(x)$

③ Often at low energy, massless goldstones are the only relevant excitations.
Full Symmetry Group G $\rightarrow$ constrains their interactions!

④ $\boxed{*}$ says "$v = F\,z$"

eg. O(N) model

had $\vec{\phi} = (\phi_1, \ldots, \phi_{N-1}, \rho + v)$

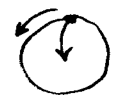


instead use $\vec{\phi} = (\rho + v) \exp\left[\frac{i}{v} X^a \pi^a\right] \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

[here $Z=1$, $f=v$]

X^a: N-1 broken gen.

π^a: polar coord. fields



At low energy $E \ll m_\rho$ can ignore radial modes and

$$\mathcal{L} = \frac{v^2}{2}\left[\left(\partial_\mu e^{i/v\, X^a \pi^a}\right)\left(\partial^\mu e^{-i/v\, X^a \pi^a}\right)\right]_{NN} + \cdots$$

$$= \frac{1}{2}(\partial_\mu \pi^a)^2 + \mathcal{O}\left(\frac{1}{v^2}\partial^2 \pi^4\right) + \cdots$$

- Non-linear realization of the symmetry [more later]
  (transfm of π^a)
- <u>Derivative</u> <u>couplings</u>, zero momentum π^a is just motion along vacuum manifold & costs no energy
- Broken currents produce Goldstone's from vacuum
  $$J_\mu^a = i\,\partial_\mu \vec{\phi}^\dagger \cdot T^a \cdot \vec{\phi} \;\propto\; v\,\partial_\mu \pi^b \left[X^b, T^a\right]_{NN} + \cdots$$
- We can <u>construct</u> low energy interactions of Goldstone modes using an effective Lagrangian and knowledge of symmetry breaking pattern

Chiral Symmetry in QCD

$$\mathcal{L}_{QCD} = \bar{\psi}^i (i\not{\partial} - m_i)\psi^i \qquad + \text{ glue}$$

$m_u = 3$ MeV

$m_d = 6$ "

$m_s = 100$ " $\leftarrow$ $\Lambda_{QCD} \sim 250$ MeV , $\Lambda_\chi \sim 1$ GeV

$m_c = 1.4$ GeV

$m_b = 4.7$ "

$m_t = 172$ "

$m_u \neq m_d$, but $m_{u,d} \ll \Lambda$

Let's neglect $m_{u,d}$ (later m_s too)

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

Introduce $1 = P_L + P_R$ where $P_{L,R} = \dfrac{1 \mp \gamma_5}{2}$, $\gamma_5^2 = 1$

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\, i\not{\partial} P_L \psi + \bar{\psi}\, i\not{\partial} P_R \psi \\ &= \bar{\psi}_L i\not{\partial} \psi_L + \bar{\psi}_R i\not{\partial} \psi_R\end{aligned}$$

Transform $\psi_L \to U_L \psi_L$ $\qquad U_L$ 2×2 unitary

$\psi_R \to U_R \psi_R$ $\qquad U_R$ " "

$U(2)_L \times U(2)_R$ symmetry

Two $U(1)$'s : $U(1)_V$ $\quad \psi_L \to e^{i\theta}\psi_L,\ \psi_R \to e^{i\theta}\psi_R$ $\quad$ Baryon #

$U(1)_A$ $\quad \psi_L \to e^{-i\theta}\psi_L,\ \psi_R \to e^{i\theta}\psi_R$ $\quad$ Anomalous (more later)

So consider

$G = SU(2)_L \times SU(2)_R$, $U_{L,R}$ 2×2 unitary, $\det U = 1$

$$U_L = e^{i\alpha_L^a \sigma^a/2}$$

QCD dynamics induces a non-zero expectation value for the operator $\bar{\psi}\psi$ (scalar)

$\langle 0 | \bar{\psi}\psi | 0 \rangle \neq 0$, $\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$

H = Group that leave vacuum invariant has $U_L = U_R$
ie $\psi \to U \psi$ which is SU(2) vector = <u>isospin</u>

$SU(2)_L \times SU(2)_R \to SU(2)$ vector
3 gen 3 gen $\to$ 3 gen 6 - 3 = 3 goldstone bosons.
states $|\pi^a\rangle$

3 broken generators "$SU(2)_A$"

$$e^{i\alpha_A^a \sigma^a/2} P_R + e^{-i\alpha_A^a \sigma^a/2} P_L$$

(doesn't close to form group)

$$\delta_a \psi = i\left(\frac{\sigma^a}{2} P_R - \frac{\sigma^a}{2} P_L\right)\psi = \frac{i\sigma^a}{2} \gamma_5 \psi$$

Noether Current $J_5^{\mu a} = \bar{\psi} \gamma^\mu \gamma_5 \frac{\sigma^a}{2} \psi$

$$\langle 0 | J_5^{\mu a}(0) | \pi^b(p_\pi) \rangle = -i F_\pi p_\pi^\mu \delta^{ab}$$ [From Goldstone Theorem]

[Here $\phi = \bar{\psi}\sigma^a\gamma_5\psi$, check with ⊛]

Under parity

$$\underbrace{\langle 0 | P}_{\langle 0|} \underbrace{P^{-1} j_5^{ia}(0) P}_{j_5^{ia}(0)} \underbrace{P^{-1} | \pi^b(\vec{p}_\pi) \rangle}_{\text{must be } -|\pi^b(-\vec{p}_\pi)\rangle} = -i F_\pi p_\pi^i$$

So state is a <u>pseudoscalar</u>

$|\pi^b\rangle$ is an iso triplet under SU(2) vector

From $\pi^- \to \mu^- \bar{\nu}_\mu$ $F_\pi = \frac{130 \text{ MeV}}{\sqrt{2}}$ [Homework]

Do we have massless isotriplet pseudoscalar hadrons ?

Close : pions $m_\pi^\pm = 139.57$ MeV, $m_\pi^0 = 134.98$ MeV } isotriplet

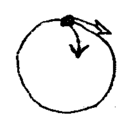
Small $m_u, m_d \neq 0$ cause explicit breaking of the symmetry, and make $m_\pi \neq 0$.

The simplest way to see how this works is using a Chiral Lagrangian [multi-commutators in Weinberg]

$G = SU(2)_L \times SU(2)_R \to H = SU(2)_V$ and we want goldstone fields to parameterize coset G/H

[compare $O(N)/O(N-1) = S^{N-1}$ from before

$$\vec{\Phi} = v \exp\left[i/v \, X^a \pi^a \right] \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

]

[more in section]

We'll use use elements $g_L g_R^{-1}$ to parameterize G/H

$$\bar{g}_R^{\,j} \, g_L^{\,k}(x) \sim v \, \Sigma_{kj}(x)$$ local orientation of condensate

$$\text{where} \quad \Sigma(x) = \exp\left[\frac{2i}{f} M(x) \right], \quad \langle 0 | \Sigma | 0 \rangle = \mathbb{1}$$

$$M = \frac{\pi^a \sigma^a}{\sqrt{2}} = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}$$

$SU(2)$ triplet, $M^\dagger = M$, $M \to V M V^\dagger$ adjoint rep

- The exponential representation for fields is not mandatory, different choices correspond to different coordinates (related by field redefn's)

See Coleman, Wess, Zumino Phys Rev 177 (1969) 2239

$G = SU(2)_L \times SU(2)_R \qquad \Sigma \to L \Sigma R^+$

$\quad L^\epsilon \qquad R^\epsilon$

Construct Lagrangian from lowest dim operators

$tr\, \Sigma\Sigma^+ = 3$ no good

$\left(\frac{\partial^\mu}{\Lambda_\chi}, \frac{M_\pi}{\Lambda_\chi}\right)$ expansion

$$\mathcal{L}_\chi = \frac{f^2}{8}\, tr\left(\partial^\mu \Sigma\, \partial_\mu \Sigma^+\right) + \ldots$$

$\Lambda_\chi \simeq 1\,GeV$ is chiral symm. breaking scale

gives standard normalization for $\pi^\pm$, π^0 kinetic term

$\partial^\mu \pi^+ \partial_\mu \pi^- + \frac{1}{2}(\partial^\mu \pi^0)^2$

$\boxed{\text{Mass term}}$ $\quad \mathcal{L}_{QCD} = -\bar{\Psi}_L M_q \Psi_R - \bar{\Psi}_R M_q \Psi_L$

$$M_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

We want term in $\mathcal{L}_\chi$ that breaks symmetry in same way

<u>Spurion Analysis</u> pretend $M_q \to L M_q R^+$

(think of it as v.e.v. of scalar field if you like)

then $\mathcal{L}_{QCD}$ is invariant. Construct invariant operator in $\mathcal{L}_\chi$.

$$\boxed{\mathcal{L}_\chi^{mass} = v_0\, tr\left(M_q^+ \Sigma + M_q \Sigma^+\right)} \qquad [\text{Hermitian}]$$

now set back to real world

To compute meson masses expand to quadratic order

$$M^2 = \begin{pmatrix} \frac{1}{2}\pi^{0\,2} + \pi^+\pi^- & -- \\ -- & \pi^+\pi^- + \frac{1}{2}\pi^{0\,2} \end{pmatrix}$$

$$\Sigma = 1 + \frac{2iM}{f} - \frac{2}{f^2} M^2 + \ldots$$

$$\mathcal{L} = -\frac{4 v_0}{f^2} (m_u + m_d) \left(\frac{1}{2} \pi^{0 2} + \pi^+ \pi^- \right)$$

So $\quad M_{\pi^\pm}^2 = M_{\pi^0}^2 = \frac{4 v_0}{f^2} (m_u + m_d)$ $\qquad$ [and $M_{\pi^0} \neq M_{\pi^\pm}$ due to e.m.]

then $\quad m_{u,d} \neq 0 \Rightarrow M_\pi \neq 0$ as desired,

plus meson mass-squared is linear in quark mass.

Extend to $SU(3)_L \times SU(3)_R \to SU(3)_V$ $\quad$ 8 g. bosons.

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2\eta}{\sqrt{6}} \end{pmatrix} = \frac{\pi^a \lambda^a}{\sqrt{2}}$$

different basis π^a

$\pi^0 = \pi^3, \quad \pi^+ = \frac{\pi^1 + i\pi^2}{\sqrt{2}}$

From mass term we find Gell-Mann-Okubo formula

$m_u = m_d \neq m_s$ $\qquad m_\eta^2 = \frac{4}{3} m_K^2 - \frac{1}{3} m_\pi^2 \simeq (566 \text{ MeV})^2$

$m_\eta^{expt} = 549 \text{ MeV}$

and $\quad \frac{m_u + m_d}{2 m_s} = \frac{m_\pi^2}{2m_K^2 - m_\pi^2} \simeq \frac{1}{26}$ $\qquad$ Strange & Kaon heavier.

<u>Couple Currents</u> $\qquad s(x),\ p(x),\ \ell^\mu(x)$

$$\mathcal{L}_{QCD} = \bar{\psi} i \not{\partial} \psi - \bar{\psi}_L (s + ip) \psi_R - \bar{\psi}_R (s - ip) \psi_L$$
$$- \bar{\psi} \gamma_\mu P_L \ell^\mu \psi$$

in path integral

iso-singlets & triplets $\qquad \ell^\mu = \ell^\mu_0 + \ell^\mu_a \sigma_a$

Inv. under

$\psi_L \to L(x) \psi_L \qquad (s + ip) \to L(x) (s + ip) R^\dagger(x)$

$\psi_R \to R(x) \psi_R \qquad \ell_\mu \to L(x) \ell_\mu L^\dagger(x) + i \partial_\mu L(x) L^\dagger(x)$

impose
Same Symmetry in $\mathcal{L}_\chi$ gives $(\chi = s + ip)$

$$\mathcal{L}_\chi = \frac{f^2}{8} \, tr\left(D_\mu \Sigma D^\mu \Sigma^+\right) + v_0 \, tr\left[\chi^+ \Sigma + \chi \Sigma^+\right]$$

$D_\mu \Sigma = \partial_\mu \Sigma + i l_\mu \Sigma$ left-handed cov. deriv.

generalizes mass-analysis where $\chi = s = M_q$.

Examples of response to sources

① $\langle 0 | \bar\Psi^j_R \Psi^k_L + \bar\Psi^j_L \Psi^k_R | 0 \rangle = -v \, \delta^{jk}$

$$-v \, \delta^j{}_j = \langle 0 | \bar\Psi \Psi | 0 \rangle = \langle 0 | -\frac{\delta \mathcal{L}}{\delta s} | 0 \rangle = -v_0 \langle tr(\Sigma + \Sigma^+) \rangle$$
$$= -4 v_0$$

$$v = 2 v_0$$

② $$J_{L\mu}^a = -\frac{\delta \mathcal{L}}{\delta l_\mu^a(x)} = \bar\Psi \gamma_\mu P_L \sigma^a \Psi$$

chiral $$J_{L\mu}^a = -\frac{i f^2}{4} \, tr\left(\sigma^a \Sigma \partial_\mu \Sigma^+\right)$$

$$\langle 0 | J_{L\mu}^a | \pi^-(p_\pi) \rangle = \ldots = -i f \, p_\pi^\mu$$

so $f = f_\pi \equiv \frac{F_\pi}{\sqrt{2}}$

③ W-boson Weak Interactions

$$l^\mu = -\frac{g_2}{\sqrt{2}} V_{ud} \, W^\mu_+ \, T^+ + h.c. \qquad T^+ = \frac{\sigma_1 + i\sigma_2}{2}$$

$$T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{L}^{int} = -\frac{g_2}{\sqrt{2}} V_{ud} \left[W_\mu^+ \, \bar u \gamma_\mu P_L d + W_\mu^- \, \bar d \gamma_\mu P_L u \right]$$

if V_{ud} = real

$$\mathcal{L}^\chi = \frac{f^2}{8} \, tr\left[\partial_\mu \Sigma \partial^\mu \Sigma^+ - \frac{i g_2}{\sqrt{2}} V_{ud} W^\mu_+ \left(T^+ \Sigma \partial^\mu \Sigma^+ - (\partial^\mu \Sigma)(\Sigma^+ T^+) \right) + \ldots \right]$$

W^μ_- terms

$$\Sigma = \exp\left(\frac{2i\pi}{f}\right) = 1 + \frac{2i\pi}{f} + \cdots$$

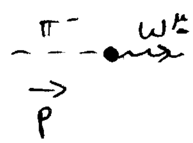
$$\mathcal{L}^{\chi}_{\text{linear } W} = \frac{f^2}{8}\left(\frac{-ig_2}{\sqrt{2}} V_{ud}\, W^{\mu}_{+}\right)\left(-\frac{4i}{f}\partial^\mu \pi^- + \underset{\text{more pions}}{\cdots}\right) + \underset{W^\mu_- \text{ term}}{\cdots}$$

$$= -\frac{g_2 V_{ud} f}{2\sqrt{2}}\left(W_\mu^+ \partial^\mu \pi^- + W_\mu^- \partial^\mu \pi^+\right)$$

With quarks 

$\langle 0 | \bar{u}\gamma_\mu P_L d | \pi^- \rangle = -i f_\pi p_\mu$

With $\mathcal{L}_\chi$ 

where 

$\propto f\, p_\mu$

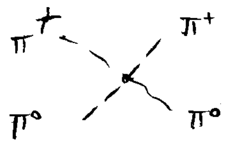
Same answer

But can compute results with more pions with $\underline{\mathcal{L}_\chi}$

SU(2) $\pi^- \to \pi^0 e^- \bar{\nu}$

no new parameters!

SU(3) $K^- \to \pi^0 \pi^0 e^- \bar{\nu}$ etc.

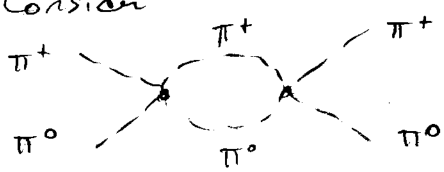
<u>Loops</u> 

$\sim p^2/f^2$

from $\mathcal{L} = \frac{f^2}{8} tr(\partial_\mu \Sigma)(\partial_\mu \Sigma^+)$

$\Rightarrow \frac{1}{6f^2} tr\left([\pi, \partial^\mu \pi][\pi, \partial^\mu \pi]\right)$

Consider



take $m_\pi = 0$

$$\sim \int d^d l \; \frac{(p^+ - l)^2}{f^2} \frac{1}{l^2} \frac{1}{(l - p_+ - p_0)^2} \frac{(l - p_+')^2}{f^2}$$

preserves chiral symm.

$$\sim \frac{p^4}{(4\pi)^2 f^4} = \frac{p^2}{f^2}\left(\frac{p^2}{(4\pi f)^2}\right)$$

loop suppressed

To renormalize these loops we need more terms

$$\mathcal{L} = L_1 \left[\mathrm{tr}(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger)\right]^2 + L_2\, \mathrm{tr}(\partial_\mu \Sigma \partial_\nu \Sigma^\dagger)\, \mathrm{tr}(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger)$$

$$\Big[\text{in } SU(2) \quad 2\,\mathrm{tr}(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger \partial_\nu \Sigma \partial^\nu \Sigma^\dagger) = \left(\mathrm{tr}(\partial^\mu \Sigma \partial_\mu \Sigma)\right)^2 \Big]$$

$L_1 \simeq f^2 \dfrac{\hat{L}_1}{\Lambda_\chi^2}$ physical cutoff Λ_χ

$\hat{L}_1 \sim O(1)$

$\sim \dfrac{p^4}{f^2 \Lambda_\chi^2}$

c.t. & loops same size $\Rightarrow$ $\Lambda_\chi \simeq 4\pi f > 1\,\mathrm{GeV}$

Chiral Expansion $\dfrac{p^2}{\Lambda_\chi^2}$, $\dfrac{m_\pi^2}{\Lambda_\chi^2}$

Higgs Mechanism

Symmetry Hidden, not "broken"

Spont. Symm. Breaking in Local Gauge theory

$\Rightarrow$ Massless Gauge bosons "eat" Goldstone bosons and become massive.

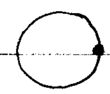
eg. U(1) gauge theory with complex scalar, $iD^\mu = i\partial^\mu + eA^\mu$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)(D_\mu\phi)^* - \lambda\left(\phi^*\phi - \frac{v^2}{2}\right)^2$$

U(1) symmetry $\quad \phi \to e^{i\alpha(x)}\phi(x), \quad A^\mu \to A^\mu + \frac{1}{e}\partial^\mu\alpha$

v.e.v. $\quad \langle 0|\phi|0\rangle = v/\sqrt{2}$ real, breaks U(1) circle

$\rightarrow$ massless goldstone



Let $\phi = \frac{(v+\rho)}{\sqrt{2}}\, e^{i\sigma/v}$ $\qquad$ ρ, σ are real fields

σ = variation on vac. manifold

$$\langle\rho\rangle = 0, \quad \langle\sigma\rangle = 0$$

$$(\partial_\mu - ieA^\mu)\phi = \frac{e^{i\sigma/v}}{\sqrt{2}}\left\{\partial_\mu\rho + (v+\rho)\frac{i\partial_\mu\sigma}{v} - ieA_\mu(v+\rho)\right\}$$

$$|D_\mu\phi|^2 = \frac{1}{2}\partial^\mu\rho\,\partial_\mu\rho + \frac{1}{2}(v+\rho)^2\left(eA_\mu - \frac{\partial_\mu\sigma}{v}\right)^2$$

$$\text{Potential} = -\frac{\lambda}{4}\left(\rho^2 + 2v\rho\right)^2$$

$$\mathcal{L}^{\text{Quadratic}} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}v^2\left(eA_\mu - \frac{\partial_\mu\sigma}{v}\right)^2 \underbrace{- \lambda v^2\rho^2}_{\text{massive } \rho} - \frac{1}{4}(F^{\mu\nu})^2$$

massless σ

massive A^μ $\qquad \mathcal{L}^{\text{mass}} = \frac{1}{2}M_A^2\, A^\mu A_\mu, \qquad M_A = ev$

cross-term $\quad \mathcal{L}_{cross} = -M_A A_\mu \partial^\mu \sigma \quad , \quad$  $= M_A k^\mu$

 $+$  $= i M_A^2 \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)$

Goldstone supplies $\frac{1}{k^2}$ pole to make vac. pol. amplitude transverse.
[Full discussion requires Fadeev-Popov for Spont. Broken theory to get gauge prop. - later]

Goldstone does not behave as independent physical particle.
To see this use

Unitary Gauge : $\quad \sigma(x) = 0$

ie choose $\alpha(x) = -\frac{\sigma(x)}{v}$, so new field $\phi'(x)$ is real for all x

and $e A'_\mu = e A_\mu - \frac{\partial_\mu \sigma}{v}$

say $\phi(x)$ is orthogonal to Goldstone direction

$$\mathcal{L}^{\text{Unit Gauge}}_{\text{Quad}} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{e^2 v^2}{2} A_\mu A^\mu - \lambda v^2 \rho^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

so A^μ has mass & σ has disappeared

d.o.f. unchanged $\quad$ 4 before: ϕ, A_μ^{massless} $\quad$ 4 after: ρ, A_μ^{massive}

A^μ still couples to ρ

$$\mathcal{L}^{\text{Unit. Gauge}}_{\text{Rest}} = \frac{e^2}{2}\left(2 v \rho + \rho^2\right) A_\mu A^\mu - \lambda \left(v \rho^3 + \frac{\rho^4}{4} \right)$$

<u>Non-Abelian Case</u> $\qquad \phi_i \to (1 + i\alpha^a(x)\, t^a)_{ij}\, \phi_j$

(in general)

its useful to pick a rep. where ϕ_i are real

then $\quad t^a_{ij} = i T^a_{ij}$ is purely imaginary

$t^\dagger = t \quad \Rightarrow \quad T^a_{ij} = -T^a_{ji}$ antisymm. & real

eg complex doublet $= \frac{1}{\sqrt{2}} \begin{pmatrix} -i\phi_1 - \phi_2 \\ \phi_4 + i\phi_3 \end{pmatrix}$ in $SU(2)$

4 real fields

$T^a = -\frac{i\sigma^a}{2}$, $\quad T^1 \begin{pmatrix} 0 \\ \phi_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -i\phi_4 \\ 0 \end{pmatrix}$ takes ϕ_4 to ϕ_1

& T^1_{14} is real etc.

Now $\quad D_\mu \phi = (\partial_\mu - i g A_\mu^a t^a)\phi = (\partial_\mu + g A_\mu^a T^a)\phi$

$$\frac{1}{2}(D_\mu\phi)^2 = \frac{1}{2}(\partial_\mu \phi_i)^2 + g A_\mu^a (\partial_\mu \phi_i T^a_{ij} \phi_j) + \frac{g^2}{2} A_\mu^a A^{\mu b} (T^a\phi)_i (T^b\phi)_i$$

suppose $\langle \phi^i \rangle = \phi_0^i$ for some i's , $\quad \vec{\phi}_0 \neq 0$

get $\quad \mathcal{L}_{mass} = \frac{1}{2} M^2_{ab} A_\mu^a A^{\mu b}$ where mass matrix

$$M^2_{ab} = g^2 (T^a \phi_0)_i (T^b \phi_0)_i$$

gives positive mass^2 to gauge bosons

[in any basis $\; M^2_{aa} = g^2 (T^a\phi_0)^2 \geq 0$ for fixed a

so in diagonal basis masses^2 are positive]

<u>But</u> generators that leave vacuum invariant (T^a in H)

leave corresponding gauge bosons massless

$T^a \vec{\phi}_0 = 0$

here $\mathcal{L}_{mix} = g A^a_\mu \partial_\mu \vec{\phi} \cdot (T^a \vec{\phi}_0)$

only terms $\parallel T^a \vec{\phi}_0$ (in $O(N)$, along sphere)

ie Goldstone bosons

Unitary Gauge

rotate to $\tilde{\phi}^i(x) = U^{ij}(x)\, \phi^j(x)$ orthogonal to Goldstones

so $0 = \sum_{i,j} \tilde{\phi}^i(x)\, (T^a)_{ij}\, \phi_0^j$ for each broken T^a generator

#("a") conditions which remove all Goldstone bosons

and $\mathcal{L}_{mix} = 0$

Higgs Mechanism for Weak Interactions

want $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$

bosons: $g A^a_\mu$, $g' B_\mu$ $\qquad$ $e A_\mu$ photon

generators $t^a = \frac{\sigma^a}{2}$, Y $\qquad$ Q

Any other choice can be rotated to this by global gauge transformation

use complex doublet $\phi = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, with vev $\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$

with $Y_H = \frac{1}{2}$

$$\phi \rightarrow e^{i\alpha^a(x) t^a}\, e^{i\alpha(x) Y}\, \phi$$

$\alpha^{1,2} = 0$, $\alpha^3 = \alpha$ gives $e^{i\alpha Q}\phi$, $Q = t^3 + Y \overset{\text{on } \phi}{=} \frac{\sigma^3}{2} + \frac{\mathbb{1}}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

h_1^+ is e.m. charged, h_2^0 is e.m. neutral

$e^{i\alpha Q}$ leaves ϕ_0 invariant, $U(1)_{em}$ unbroken

$$D_\mu \phi = \left(\partial_\mu - igA_\mu^a t^a - \frac{ig'}{2} B_\mu\right)\phi$$

$$\mathcal{L}_{mass} = \frac{1}{2}\,(0\ v)\left(gA_\mu^a t^a + \frac{g'}{2}B_\mu\right)\left(gA^{b\mu}t^b + \frac{g'}{2}B^\mu\right)\begin{pmatrix}0\\ v\end{pmatrix}$$

$$= \frac{v^2}{8}\left[\,g^2(A_\mu^1)^2 + g^2(A_\mu^2)^2 + (gA_\mu^3 - g'B^\mu)^2\,\right]$$

Three massive vector bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \mp iA_\mu^2\right) \qquad \text{mass} \quad M_W = \frac{gv}{2}$$

$$Z_\mu^0 = \frac{(gA_\mu^3 - g'B_\mu)}{\sqrt{g^2+g'^2}} = \cos\theta_W\, A_\mu^3 - \sin\theta_W\, B_\mu$$

mass $M_Z = \frac{v}{2}\sqrt{g^2+g'^2}$

θ_W = <u>weak mixing angle</u>

One massless boson (orthog. comb.)

$$A_\mu = \frac{g'A_\mu^3 + gB_\mu}{\sqrt{g^2+g'^2}} = \sin\theta_W\, A_\mu^3 + \cos\theta_W\, B_\mu$$

$M_A = 0$

$$\frac{M_W}{M_Z} = \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$$ given by gauge couplings!

<u>Coupling to Fermions</u> $\quad \bar\psi\, i\not{D}\,\psi \quad$ Invert $A_\mu^3, B_\mu \to Z_\mu, A_\mu$

$$iD_\mu = i\partial_\mu + \frac{g}{\sqrt{2}}\left(W_\mu^+ T^+ + W_\mu^- T^-\right) + g\, Z_\mu \frac{(T^3 - \sin^2\theta_W\, Q)}{\cos\theta_W}$$

$$+\, e\, A_\mu\, Q$$

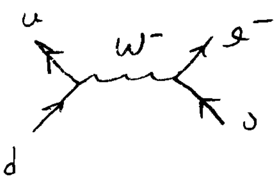
$$T^\pm \equiv \frac{1}{2}(\sigma^1 \pm i\sigma^2) = \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

where $e = g\sin\theta_W = \dfrac{gg'}{\sqrt{g^2+g'^2}}$

transition between components changes charge

<u>Three parameters</u> $\quad g, g', v \;\longrightarrow\; e,\ \theta_W,\ M_W$

flavor changing interaction

- From 

with $\langle W^{\mu +} W^{\nu -} \rangle = \dfrac{-i g^{\mu\nu}}{p^2 - m_W^2}$

justify later

one gets for $p^2 \ll M_W^2$

$$\frac{g^2}{2 m_W^2} \left(\bar{u}_L \gamma_\mu d_L \right) \left(\bar{e}_L \gamma^\mu \nu_L \right)$$

Fermi constant $4 \dfrac{G_F}{\sqrt{2}} = \dfrac{g^2}{2 m_W^2}$

- $\sin^2 \theta_W = 1 - \dfrac{M_W^2}{M_Z^2} = 0.224$ ← tested to 1% level

- v.e.v. $v = \dfrac{2 M_W}{g} = \dfrac{1}{(\sqrt{2} G_F)^{1/2}} = 246 \text{ GeV}$

Yukawa couplings gave fermion masses

$\boxed{\text{Higgs Boson}}$

SU(2) valued

general $\phi(x) = U(x) \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$, real $h(x)$, $\langle h \rangle = 0$

<u>Unitary Gauge</u>, $U(x) = 1$, 3 g. bosons are eaten to make W, Z massive, one remains, $h(x)$

Consider

$$\mathcal{L}_{Higgs} = |D_\mu \phi|^2 - \lambda \left(\phi^\dagger \phi - \frac{\mu^2}{2\lambda} \right)^2$$

$$\langle \phi \rangle = \frac{v}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{\mu^2}{\lambda} \right)^{1/2}$$

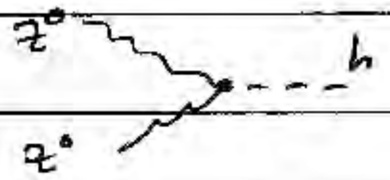
v was already a parameter, so this adds just one more.

$\mathcal{L}^{\text{Unitary-Gauge}}_{\text{Higgs}} = \frac{1}{2}|D_\mu h|^2 - \frac{1}{2} m_h^2 h^2 - \sqrt{\frac{\lambda}{2}}\, m_h h^3 - \frac{\lambda}{4} h^4$

Higgs Self-Couplings

- one more parameter λ or $m_h = \sqrt{2\lambda}\, v$
- No linear gauge boson terms in this Gauge, $\mathcal{L}_{mix} = 0$
- Higgs couples proportional to mass

$\frac{1}{2}|D_\mu h|^2 = \frac{1}{2}(\partial_\mu h)^2 + \left[m_W^2 W^{\mu+} W_\mu^- + \frac{1}{2} m_Z^2 Z^\mu Z_\mu\right]\left(1+\frac{h}{v}\right)$



$= \frac{2 i m_Z^2}{v}\, g^{\mu\nu}$ from "$v+h$"

Fermions: $\mathcal{L}_{mass} = -m_f \bar{f} f \left(1+\frac{h}{v}\right)$

top couples stronger to Higgs than Bottom, etc

Quantization of Spont. Broken Gauge Theory

Abelian Example $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi)$ $\quad U(1)$

$\phi = \frac{\phi^1 + i\phi^2}{\sqrt{2}}$, $\delta\phi^1 = -\alpha(x)\,\phi^2(x)$ $\quad D_\mu = \partial_\mu + ieA_\mu$

$\delta\phi^2 = +\alpha(x)\,\phi^1(x)$ $\quad \delta A_\mu = -\frac{1}{e}\partial_\mu\alpha$

break $U(1)$

$\langle\phi^1\rangle = v$, $\phi^1 = v + h(x)$, $\phi^2 = \varphi$, $M_A = ev$ as before

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu h - eA_\mu\varphi)^2 + \frac{1}{2}(\partial_\mu\varphi + eA_\mu(v+h))^2 - V$$

still invariant under exact local symmetry

$\delta h = -\alpha\varphi$, $\delta\varphi = \alpha(v+h)$, $\delta A_\mu = -\frac{1}{e}\partial_\mu\alpha$

Quantization: should **only** integrate over gauge indep. vars "x", the other vars "y" are redundant

Recall Fadeev-Popov Procedure let $z = (x,y)$, $S = \int d^4x\,\mathcal{L}$

$S = \int d^4x\,\mathcal{L} + J^{\mu a}A_{\mu a}$

$$Z[J^a] = e^{iW[J]} = \int dx\, e^{iS} = \int dz\, e^{iS}\,\delta(y)$$

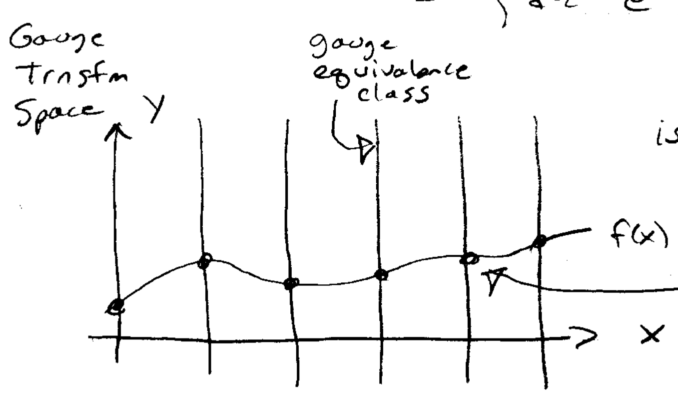
$$= \int dz\, e^{iS}\,\delta(y - f(x))$$

fix gauge dep. vars on $y = f(x)$

$$= \int dz\, e^{iS}\,\det\left(\frac{\delta G}{\delta y}\right)\delta[G(z)]$$

surface $G(z) = 0$

is function of A^μ, h, φ ie both "x" & "y"



representative of equiv. class

physical vars

simplest way to determine "y" is to use inf. gauge transformation

$$\frac{\delta G}{\delta y} = \frac{\delta G}{\delta(-\alpha^b/g)}$$ ← pick convenient norm

eg. general cov. gauge $G^a = \partial^\mu A_\mu{}^a$, $A_\mu{}^a \to A_\mu{}^a - \frac{1}{g} D_\mu^{ab}\alpha^b$

$$\frac{\delta G^a(x)}{\delta\left(-\frac{1}{g}\alpha^b(x')\right)} = \partial^\mu D_\mu^{ab}\ \delta^4(x-x')$$

Use $\delta[G(z) - \omega(x')]$ and integrate $\int d\omega \exp\left(\frac{-i}{2\xi}\int d^4x'\, \omega(x')^2\right)$ to get

$$\mathcal{L}_{FP} = \mathcal{L} - \frac{1}{2\xi} G^2 + \bar{c}\, \frac{\delta G}{\delta \alpha}\, c$$

from det.
(can rescale c to convenient norm)

For Spont. Broken U(1), use $\underline{R_\xi}$ $\underline{\text{gauge}}$

$$G = \partial_\mu A^\mu - \xi e v \varphi$$

$$\mathcal{L}_{g.fix} = \frac{-1}{2\xi} G^2 = \underbrace{-\frac{(\partial_\mu A^\mu)^2}{2\xi}}_{\text{Gen. Lorentz Gauge}} + \underbrace{\partial_\mu A^\mu e v \varphi}_{\text{cancels}} - \frac{\xi}{2} \underbrace{e^2 v^2 \varphi^2}_{\text{mass term}}$$

Gen. Lorentz Gauge — cancels $\mathcal{L}_{mix} = ev(\partial_\mu \varphi)A^\mu$ term in $|D_\mu \phi|^2$ — mass term for Goldstone

$$m_\varphi^2 = \xi (ev)^2 = \xi m_A^2$$

, fictitious particle with gauge dependent mass

$$\frac{\delta G}{\delta \alpha} = -\frac{1}{e}\partial^2 - \xi e v (v + h)$$

$$\mathcal{L}_{ghost} = \bar{c}\left[-\partial^2 - \xi m_A^2 (1 + h/v)\right] c$$

higgs couples U(1) ghost

<u>Propagators</u>	Higgs h	Goldstone φ	Ghost c
$\overset{k}{- - \!\!\!\to - -}$	$\dfrac{i}{k^2 - m_h^2}$	$\dfrac{i}{k^2 - \xi m_A^2}$	$\dfrac{i}{k^2 - \xi m_A^2}$

↑ from $V(\phi)$

gauge field

$$\mathcal{L}^A_{Quad} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{m_A^2}{2} A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Invert

$$\overset{\mu \quad k \quad \nu}{\sim\!\sim\!\sim} = \frac{-i}{(k^2-M_A^2)}\left(g^{\mu\nu} - \frac{k^\mu k^\nu\,(1-\xi)}{(k^2-\xi M_A^2)}\right)$$

$$= \frac{-i}{(k^2-M_A^2)}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{M_A^2}\right) \quad \frac{-i}{(k^2-\xi M_A^2)}\left(\frac{k^\mu k^\nu}{M_A^2}\right)$$

Transverse A^μ massive has ~~mass~~ M_A

Physical Higgs h " " M_h

Unphysical A^μ, φ, c " " $\sqrt{\xi}\,M_A$

Massive pol. sum (on-shell) $= \sum_{\epsilon\cdot k=0} \epsilon^\mu \epsilon^{\nu *} = -\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{M_A^2}\right)$

Examples

(1) $\xi = 0$ Lorentz — Goldstone massless $\dashrightarrow = i/k^2$

like in Global case — Gauge transverse $\sim\!\sim\!\sim = \frac{-i}{(k^2-M_A^2)}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$

(2) $\xi = 1$ Feynman - t'Hooft $\dashrightarrow = \frac{i}{k^2-M_A^2}$

good for pert. calculations. $\sim\!\sim\!\sim = \frac{-i g^{\mu\nu}}{k^2-M_A^2}$ same mass

(3) $\xi = \infty$ Unitary Gauge $\dashrightarrow = 0$ no Goldstone

physical intermediate states. $\sim\!\sim\!\sim = \frac{-i}{(k^2-M_A^2)}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{M_A^2}\right)$

doesn't fall off at large k, so not manifestly renormalizable. For any finite ξ $\sim\!\sim\!\sim \sim 1/k^2$ (unitary & renormalizable since $\underline{\xi}$ independent)

$SU(2)_L \times U(1)_Y$ Quantization

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + |D_\mu \phi^i|^2 - V(\phi)$$

very similar, but have
- indices i, a
- unbroken gen.

$$D_\mu \phi_i = \partial_\mu \phi_i + g A_\mu^a T^a_{ij} \phi_j + g' B_\mu T^4_{ij} \phi_j$$

for <u>real</u> <u>rep.</u> $\phi_1, \phi_2, \phi_3, \phi_4$ constructed from

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\phi^1 - \phi^2 \\ \phi^4 + i\phi^3 \end{pmatrix}$$

with $T^a \equiv -i\frac{\sigma^a}{2}$

$T^4 \equiv -iY = -\frac{i}{2}\mathbb{1}$

let $A^4_\mu = B_\mu$

V.e.V. $\langle \phi^4 \rangle = v$, $\phi_4 = v + \chi_4$, $\phi_{1,2,3} = \chi_{1,2,3}$

$$\mathcal{L}^{Quad} = -\frac{1}{4}(F_{\mu\nu}^a)^2\Big|^{Quad} + \frac{1}{2}(\partial_\mu \chi^i)^2 + \partial^\mu \chi_i A_\mu^a (\overset{\text{appropriate } g \text{ or } g'}{g} T^a_{i4} v) \leftarrow \mathcal{L}_{mix}$$
$$+ \frac{1}{2}(M_A^2)^{ab} A_\mu^a A_\nu^b - \frac{1}{2}(m_h^2)^{ij} \chi_i \chi_j$$

where $(m_h^2)_{ij} = \frac{\partial^2 V(\phi)}{\partial \phi_i \partial \phi_j}\Big|_{\text{V.e.V}} = \begin{cases} 0 & i \text{ or } j = 1,2,3 \quad \text{massless goldstones} \\ m_h^2 & i = j = 4 \quad \text{"Higgs"} \end{cases}$

Let $F^a{}_i \equiv \begin{cases} g\, T^a{}_{i4}\, v & a = 1,2,3 \\ g'\, T^4{}_{i4}\, v & a = 4 \end{cases}$ $\quad F^a{}_4 = 0$ not goldstone direction

In general $F^a{}_i = g(T^a \vec{\phi}_0)_i$ for $\vec{\phi}_0$ vacuum

Recall $M^2{}_{ab} = (g T^a \vec{\phi}_0) \cdot (g T^a \vec{\phi}_0) = F^a{}_i F^b{}_i$

$$\boxed{\text{gauge } (m_A^2)_{ab} = (F F^T)^{ab}}$$

To kill: $\mathcal{L}_{mix} \equiv -\partial^\mu A_\mu^a \chi_i F^a{}_i$

let $G^a \equiv \partial_\mu A^{a\mu} - \xi F^a{}_i \chi_i$

$$-\frac{1}{2\xi}(G^a)^2 = -\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \underbrace{\partial_\mu A^{a\mu}\chi_i F^a{}_i}_{\text{cancels } \mathcal{L}_{mix}} \underbrace{-\frac{1}{2}\xi \; F^a{}_i F^a{}_j \chi_i \chi_j}_{\text{Goldstone Mass-term}}$$

$i \neq 4,\ j \neq 4$

$$\boxed{(m_G^2)_{ij} = \xi \; F^a{}_i F^a{}_j = \xi\,(F^T F)_{ij}}$$

<u>Ghosts</u> $\quad \delta A_\mu{}^a = \frac{1}{g}(D_\mu \alpha)^a$

$$\delta \phi_i = -\alpha^a T^a{}_{ij}\phi_j$$

$$\frac{\delta G^a}{\delta \alpha^b} = \frac{1}{g}\left[(\partial_\mu D^\mu)^{ab} + \xi g \; F^a{}_i T^b{}_{ij}(v+\chi)_j\right]$$

$$\mathcal{L}_{ghost} = \bar{c}^a\left[-(\partial_\mu D^\mu)^{ab} - \xi\,(\underbrace{F^a{}_i F^b{}_i}) - \xi g F^a{}_i T^b{}_{ij}\chi_j\right]c^b$$

$a = 1,2,3,4$

<u>4 ghosts</u>

$$\boxed{(m^2{}_A)^{ab} = (FF^T)^{ab}}$$

Same as gauge bosons

So simply replace

Abelian $\quad e^2 v^2 \quad$ by $\quad (FF^T)^{ab}$

or $\quad (F^T F)^{ij}$

<u>Propagators</u>

Higgs h	Goldstones	Ghosts
$\left(\frac{i}{k^2 - m_h^2}\right)_{ij}$	$\left(\frac{i}{k^2 - \xi m_G^2}\right)_{ij}$	$\left(\frac{i}{k^2 - \xi m_A^2}\right)_{ab}$
1 higgs	3 goldstone	4 ghosts

$$a \sim\sim\sim^{\nu}_{b} = \left[\underbrace{\frac{-i}{k^2 - m_A^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2}\right)}_{\text{physical}} \underbrace{-\frac{i}{k^2 - \xi m_A^2}\frac{k^\mu k^\nu}{m_A^2}}_{\text{timelike}}\right]_{ab}$$

<u>What is $F^a{}_i$?</u>

$$F^a{}_i = \begin{cases} g\, T^a{}_{i4}\, v \\ g'\, T^4{}_{i4}\, v \end{cases}$$

$$F = \frac{v}{2} \begin{array}{c} \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \\ 0 & 0 & -g' \end{pmatrix} \\ i = \quad 1 \quad 2 \quad 3 \end{array} \begin{array}{c} a= \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \quad ; \quad F^a{}_4 = 0$$

~~eg.~~ use $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\phi^1 - \phi^2 \\ \phi^4 + i\phi^3 \end{pmatrix}$ $\qquad T^a = \frac{-i\sigma^a}{2}$, $T^4 = \frac{-i}{2}\mathbb{1}$

$$g' T^4 \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{v}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -ig' \end{pmatrix} \qquad \text{so} \quad T^4_{34} = -\frac{1}{2}$$

etc

Gluons & Ghosts

~~need~~

$$F F^T = \frac{v^2}{4} \begin{pmatrix} g^2 & 0 & & \\ 0 & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix} \begin{array}{l} \Big\} \text{ mass } M_W^2 = \frac{v^2 g^2}{4} \\ \Big\} \text{ det} = 0 \quad A^\mu \text{ massless} \end{array}$$

eigenvector $Z^0_\mu = \frac{(g A^3_\mu - g' B_\mu)}{\sqrt{g^2 + g'^2}}$, eigenvalue $\frac{(g^2+g'^2)v^2}{4} = M_Z^2$

- ghosts have ξ * same ~~mass~~ as phys. gauge bosons

Goldstones

$$\xi F^T F = \xi \frac{v^2}{4} \begin{pmatrix} g^2 & 0 & 0 \\ 0 & g^2 & 0 \\ 0 & 0 & g^2+g'^2 \end{pmatrix} \begin{array}{l} \Big\} \text{ two of mass } M_W^2 \xi \\ \Big\} \text{ one of mass } M_Z^2 \xi \end{array}$$

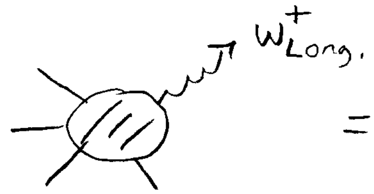
In diagonal basis same propagators as Abelian, with appropriate masses.

Goldstone Boson Equivalence Thm

Goldstone's were eaten by long. W's

A shame since their couplings were so simple. Is their a situation where they appear?

Yes, high energy



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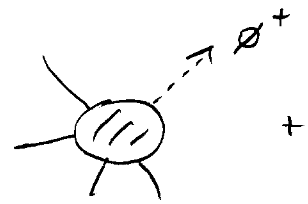


$+ \mathcal{O}\left(\frac{M_W^2}{E^2}\right)$

3 Pol. $\quad \epsilon^\mu k_\mu = 0 \qquad\qquad k^\mu = (E_k, 0, 0, k)$

norm $\quad \epsilon^2 = -1 \qquad\qquad \epsilon_L{}^\mu = \left(\frac{k}{m}, 0, 0, \frac{E_k}{m}\right) = \frac{k^\mu}{m} + \mathcal{O}\left(\frac{m}{E_k}\right)$

Proof

In Lorentz Gauge $\xi = 0$ (so $G = \partial_\mu A^{\mu a}$ & $\mathcal{L}_{mix} \neq 0$)

$0 = k^\mu \Big($  $\Big) = k^\mu \Big($  $+$  $\Big)$

Ward Identity

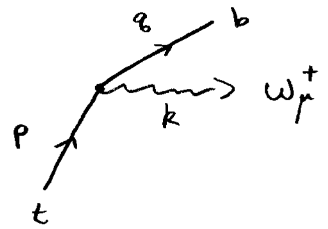
$$= k^\mu \Gamma_\mu(k) + k^\mu (iF k_\mu) \frac{i}{k^2} \Gamma(k) .$$

$$k^\mu \Gamma_\mu(k) = F\,\Gamma(k) = M_A\,\Gamma(k)$$

$$\epsilon_L^\mu \Gamma_\mu(k) = \Gamma(k) \qquad \#$$

SU(2) × U(1) Phenomenology (mostly gauge bosons)

Example Top Quark Decay



take $m_b = 0$, $V_{tb} = 1$

might guess $\Gamma_t \sim \frac{g^2 m_t}{4\pi}$, in fact $\Gamma_t \sim \frac{g^2 m_t^3}{4\pi m_W^2}$

due to longitudinal enhancement

$\boxed{\text{direct calc}}$

$$iA = \frac{ig}{\sqrt{2}} \bar{u}_b(q)\gamma^\mu P_L u_t(p) \epsilon^*_\mu(k)$$

$$\frac{1}{2}\sum_{\text{spins}} |A|^2 = \frac{g^2}{2}\left(q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu} q\cdot p\right) \sum_{\substack{\text{phys}\\ \text{pol.}}} \underbrace{\epsilon^*_\mu(k)\epsilon_\nu(k)}_{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}}$$

needed

$$= \frac{1}{2} g^2 \left(q\cdot p + \frac{2\, k\cdot p\; k\cdot q}{m_W^2}\right)$$

Kinematics: $q = p - k$

$q^2 = 0 = m_t^2 - 2p\cdot k + m_W^2$, $\quad 2p\cdot k = m_t^2 + m_W^2$

$p^2 = m_t^2 = m_W^2 + 2k\cdot q$ $\quad 2q\cdot p = 2q\cdot k = m_t^2 - m_W^2$

$k^2 = m_W^2 = m_t^2 - 2q\cdot p$

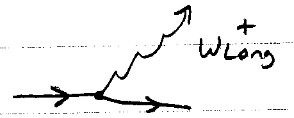
phase space

$$\Gamma = \frac{|\vec{P}_W|}{8\pi m_t^2} \frac{1}{2}\sum_{\text{spin}} |A|^2$$

$$= \frac{1}{16\pi}\frac{(m_t^2 - m_W^2)}{m_t^3}\left[\frac{g^2}{4}\frac{m_t^4}{m_W^2}\left(1 - \frac{m_W^2}{m_t^2}\right)\left(1 + \frac{2m_W^2}{m_t^2}\right)\right]$$

$$= \frac{g^2 m_t^3}{64\pi m_W^2}\left(1 - \frac{m_W^2}{m_t^2}\right)^2\left(1 + \frac{2m_W^2}{m_t^2}\right)$$

If $M_W \ll M_t$ use equiv. thm. (result should be same)



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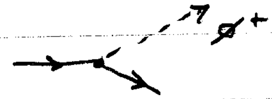


$+ \; \mathcal{O}\left(M_W^2 / M_t^2 \right)$

Top couples via Yukawa coupling λ_t

$$\mathcal{L} = -\lambda_t \, \epsilon^{ab} \, \overline{Q}_{La} \, (\phi^+)_b \, t_R \qquad + h.c.$$

doublets

$$= \lambda_t \, \overline{b}_L \, \phi^+ \, t_R$$

+1 charged Higgs, $\quad \phi^+ = \frac{1}{\sqrt{2}} (\phi^1 + i\phi^2)$

$$iA = i\lambda_t \, \overline{u}_b(q) \, P_R \, u_t(p)$$

$$\frac{1}{2} \sum_{pol} |A|^2 = \lambda_t^2 \, q \cdot p = \lambda_t^2 \frac{M_t^2}{2} + \cdots$$

$$\Gamma = \frac{1}{16\pi M_t} \left[\lambda_t^2 \frac{M_t^2}{2} \right] + \cdots = \frac{\lambda_t^2 M_t}{32\pi} + \cdots$$

convert λ_t to masses: $\quad M_t = \frac{\lambda_t v}{\sqrt{2}}$, $\quad v^2 = \frac{1}{\sqrt{2} G_F} = \frac{4 M_W^2}{g^2}$

$$\Gamma = \frac{M_t^3}{16\pi v^2} = \frac{g^2 M_t^3}{64\pi M_W^2} = \text{leading term}$$

Often top plays an important role in electroweak physics due to its large Y_t (its large mass M_t)

Z - Bosons & Neutral Currents

$$\mathcal{L} = \sum_i \overline{\Psi}_i \, i \not{D} \, \Psi_i \quad , \quad iD^\mu = i\partial^\mu + \frac{g}{\cos\theta_W} Z^\mu \left(T^3 - \sin^2\theta_W \, Q \right)$$

for Z coupling to fermions

2 common ways to write "neutral" currents

(1) $\mathcal{L} = g Z_\mu \sum J^\mu_i$ — V-A & V+A currents

$i = e_R, e_L, \nu_L, u_L, u_R, d_L, d_R$

eg. $J^\mu_{eL} = \frac{1}{\cos\theta_W} \bar{e}_L \gamma^\mu \left(-\frac{1}{2} + \sin^2\theta_W\right) e_L$ — $\gamma^\mu P_L$

$J^\mu_{eR} = \frac{1}{\cos\theta_W} \bar{e}_R \gamma^\mu \sin^2\theta_W \, e_R$ — $\gamma^\mu P_R$

(2) or use V, A basis

$$\mathcal{L} = \frac{g}{2\cos\theta_W} Z_\mu \sum_{f = e, \nu, u, d} \bar{\Psi}_f \left(g_V^{(f)} \gamma^\mu - g_A^{(f)} \gamma^\mu \gamma_5\right) \Psi_f$$

eg. $g_V^{(e)} = -\frac{1}{2} + 2\sin^2\theta_W$, $g_A^{(e)} = -\frac{1}{2}$

another common notation is

$$\mathcal{L} = e Z_\mu \sum_f \bar{\Psi}_f \left(v_f \gamma^\mu - a_f \gamma^\mu \gamma_5\right) \Psi_f$$

where $v_f = \frac{g_V^{(f)}}{2\sin\theta_W \cos\theta_W}$, $a_f = \frac{g_A^{(f)}}{2\sin\theta_W \cos\theta_W}$

What can we calculate?

<u>Z-width</u> Z-decay to massless fermions (not $M_t > M_Z$)

$$\Gamma_Z = \sum_f \frac{N_c^{(f)} g^2 M_Z}{48\pi \cos^2\theta_W} \left[g_V^{(f)2} + g_A^{(f)2} \right]$$

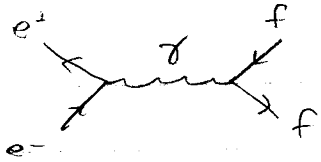
where $N_c = 3$ quarks

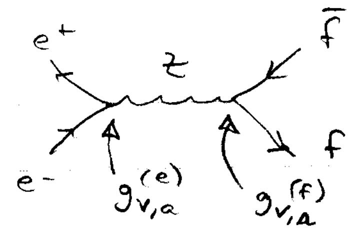
$N_c = 1$ leptons

$\Gamma_Z^{expt} = 2.495$ GeV

↑ how do we measure this if $Z \to \nu\bar{\nu}$ & we don't see final state neutrino's?

$e^+e^- \to f\bar{f}$

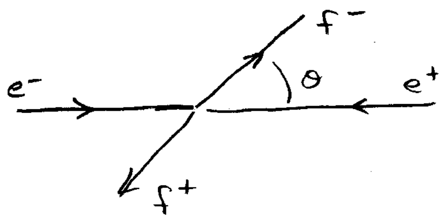




example $\quad f = \mu, \tau \quad , \quad s = E_{cm}^2$ $\qquad$ ( ) ( )* give $|M|^2$

$$\frac{d\sigma^{(0)}}{d\Omega} = \frac{\alpha^2}{4s}\Big\{ 1 + 2 g_v^{(e)} g_v^{(f)} \chi + \left(g_v^{(e)2} + g_a^{(e)2}\right)\left(g_v^{(f)2} + g_a^{(f)2}\right)\chi^2 (1+\cos^2\theta) + \left[4 g_a^{(e)} g_a^{(f)} \chi + 8 g_v^{(e)} g_a^{(e)} g_v^{(f)} g_a^{(f)} \chi^2\right]\cos\theta \Big\}$$

where



& $\quad \chi(s) = \dfrac{1}{(2 s_W c_W)^2}\,\dfrac{s}{s - M_Z^2}$

s_W: $\sin\theta_W$, c_W: $\cos\theta_W$

Note

$\cos\theta$ term. Z violates parity. Measure forward-backward asymmetry $\quad A_{FB} = \dfrac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}$

$$\sigma_F = \int_0^1 d\cos\theta \, \frac{d\sigma}{d\Omega}(2\pi) \quad , \quad \sigma_B = \int_{-1}^{0} d\cos\theta \, \frac{d\sigma}{d\Omega}(2\pi)$$

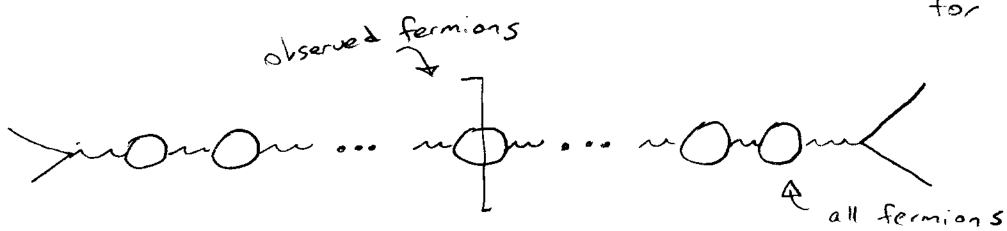
- What if $s \sim M_Z^2$? Then the fact that Z is a resonance rather than a stable particle becomes important.

Breit-Wigner $\quad \chi(s) = \dfrac{1}{(2 s_W c_W)^2}\,\dfrac{s}{s - M_Z^2 + i M_Z \Gamma_Z}$

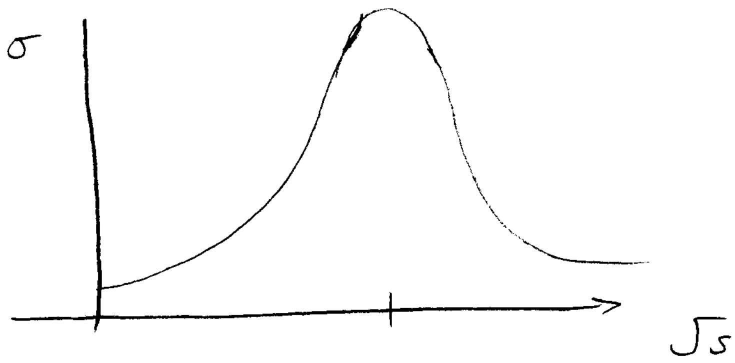
Γ_Z: total width

replace $\quad \chi \to \mathrm{Re}\,\chi \quad , \quad \chi^2 \to |\chi|^2$ above $\quad$ (goes back for $s \gg M_Z^2$)

Think of



measure total width from line-shape in
$e^+e^- \to$ hadrons or $e^+e^- \to$ leptons



shape sensitive to particles we don't see in final state

$M_Z = 91$ GeV

For $s \sim M_Z^2$, $\mathrm{Re}\,\chi = 0$ &
$$A_{FB} = 3\,\frac{g_v^{(f)} g_a^{(f)}}{g_v^{(f)2} + g_a^{(f)2}}\;\frac{g_v^{(e)} g_a^{(e)}}{g_v^{(e)2} + g_a^{(e)2}}$$

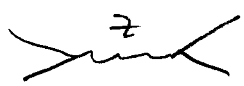
<u># ν's</u>

$$\Gamma_Z = N_\nu \Gamma_{\nu\nu} + \Gamma_{\substack{chg\\leptons}} + \Gamma_{hadrons}$$

measurement gave $N_\nu = 2.99 \pm 0.05$

<u>Low Energy Z</u>

$m_f^2 \ll M_Z^2$



integrate out Z just like W

$$\mathcal{L} = -\frac{1}{2}\,\frac{g^2}{4C_W^2}\sum_{f,f'} \bar{f}\left(g_v^{(f)}\gamma^\mu - g_a^{(f)}\gamma^\mu\gamma_5\right)f\;\frac{1}{M_Z^2}$$
$$* \;\bar{f}'\left(g_v^{(f')}\gamma^\mu - g_a^{(f')}\gamma^\mu\gamma_5\right)f'$$

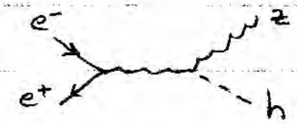
$$= -\rho\,\frac{G_F}{\sqrt{2}}\sum_{f,f'} \bar{f}\left(g_v^{(f)}\gamma^\mu - g_a^{(f)}\gamma^\mu\gamma_5\right)f\;\bar{f}'\left(g_v^{(f')}\gamma^\mu - g_a^{(f')}\gamma^\mu\gamma_5\right)f'$$

where $\rho = \dfrac{1}{\cos^2\theta_W}\,\dfrac{M_W^2}{M_Z^2}$ $\Rightarrow$ governs strength of neutral current relative to charged W-exchange

SM, tree level $\underline{\rho = 1}$

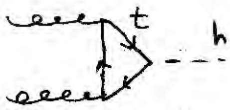
<u>Higgs</u> recall, couples like masses $m \to m\left(1 + \frac{h^{\circ}}{v}\right)$

<u>Production:</u>

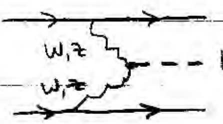
e^+e^- 

"Higgsstrahlung" LEP

$m_h > 114$ GeV @ 95% CL (Standard Model)

$p\overset{(-)}{p}$ 

"gluon fusion" dominant

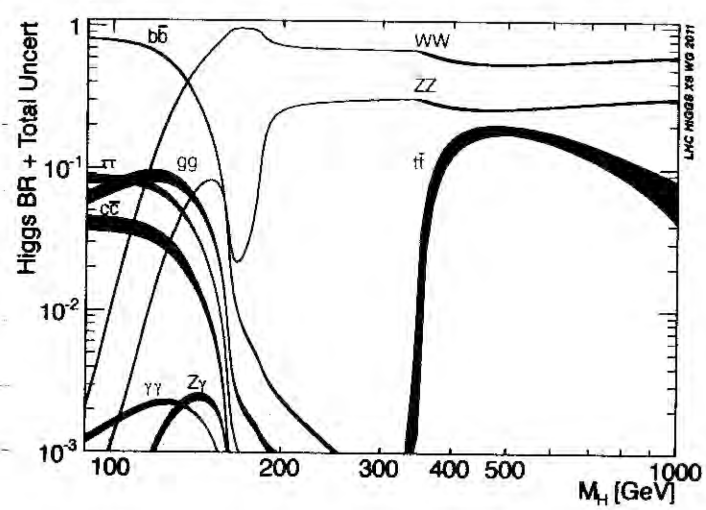


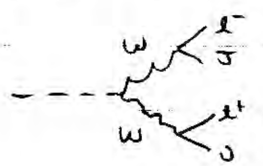
"Vector boson fusion", $\frac{1}{10}^{th}$, but can have smaller backgrounds

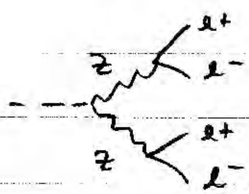
"Higgsstrahlung", contribute at Tevatron very small at LHC ($\bar{q}$ in initial state)

<u>Decay:</u>

dominant channel depends on value of m_H



• heavy $H \to WW$ 

$H \to ZZ$ 

• light $H \to b\bar{b}$ 

almost impossible due to large QCD bkgnd

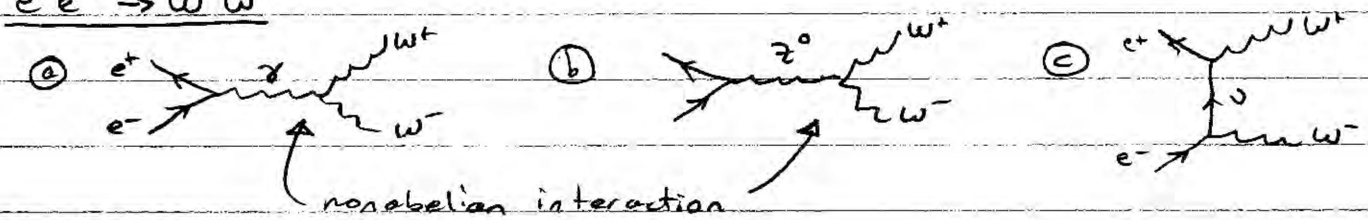
$H \to \tau\tau$ 

possible in vector boson fusion

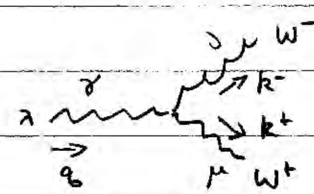
$H \to \gamma\gamma$ (Hmwk) tiny Br, but excellent γ resolution

More on naturalness, theoretical bounds on M_H, ... later

<u>$e^+e^- \to W^+W^-$</u>



eg.



$$= ie\left[g^{\mu\nu}(k_- - k_+)^\lambda + g^{\nu\lambda}(-q-k_-)^\mu + g^{\lambda\mu}(q+k_+)^\nu\right]$$

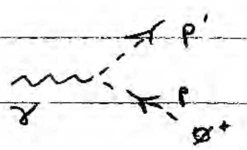
might guess $\dfrac{d\sigma}{d\cos\theta}(e^+e^-\to W^+W^-) \sim \underbrace{\dfrac{\alpha^2}{s}}_{\text{scalar result}} \Big|\underbrace{\epsilon(k^+)\cdot\epsilon(k^-)}_{s/m_W^2 \text{ for longitudinal}}\Big|^2 \sim s^0$

but this violates unitarity: $\dfrac{d\sigma}{d\cos\theta} \lesssim \dfrac{1}{s}$ for large s

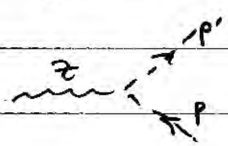
In fact each of a, b, c gives a cross-section $\dfrac{d\sigma}{d\cos\theta} \sim s^0$, but these terms <u>cancel</u> in sum of 3 graphs

From goldstone boson equiv. thm we know that this must happen since $\dfrac{d\sigma}{d\cos\theta}(e^+e^-\to\phi^+\phi^-) \sim \dfrac{1}{s}$

Scalar Calculation just needs $|\partial_\mu \phi|^2$



$= ie(p+p')^\mu$,



$= \dfrac{ie(p+p')^\mu}{\cos\theta_W \sin\theta_W}\left(\frac{1}{2} - \sin^2\theta_W\right)$

[Detailed Calc. in Peskin]

One-Loop Corrections to Weak Interactions

The parameters of the theory prior to spont. symmetry breaking are sufficient to renormalize it after spont. symmetry breaking (see eg Peskin Ch 11)

Parameters : M_{higgs} and one of

$\{g, g', \lambda\}$	couplings
$\{m_W, m_Z, v\}$	mass scales
$\{m_W, m_Z, e\}$	"on-shell" choice
$\{G_F, m_Z, e\}$	best measured
$\vdots$	

<u>Tree Level</u> :
$$M_W = \frac{gv}{2}, \quad M_Z = \sqrt{g^2+g'^2}\,\frac{v}{2}, \quad e = \frac{gg'}{\sqrt{g^2+g'^2}}$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{1}{2v^2}$$

- One relation amongst 4 observables. Same is true when we add radiative corrections Δr

$$\left(1 - \frac{m_W^2}{m_Z^2}\right)\frac{m_W^2}{m_Z^2} = \frac{\pi\alpha}{\sqrt{2}\,G_F\,m_Z^2}\,\frac{1}{(1-\Delta r)}$$

- Radiative corrections also shift $g_V^{(f)}$ and $g_A^{(f)}$. (& hence low energy Z physics)

$$g_V^f = \sqrt{\rho_f}\,\left(T_3^{(f)} - 2K_f\,S_W^2\,Q^{(f)}\right)$$

$$g_A^f = \sqrt{\rho_f}\;T_3^{(f)}$$

$$\rho_f = 1 + \Delta\rho + (\Delta\rho)^f_{\text{non-univ.}}$$
$$K_f = 1 + \underbrace{\Delta K}_{\text{universal term}} + \underbrace{(\Delta K)^f_{\text{non-univ}}}_{\text{process dependent}}$$

$\boxed{\text{On-Shell Scheme}}$

Define α, m_Z, m_W with on-shell renormalization conditions

$$\left[\quad \left.\text{(1PI)}\right|_{q=0} = -ie\gamma^\mu, \quad \mathrm{Re}\,\Pi_{WW}(q^2 = m_W^2) = 0, \ldots \right]$$

then define $\{g, g', v\}$ via (renormalized)

$$e^2 = \frac{g^2 g'^2}{g^2 + g'^2}, \quad m_W^2 = \frac{g^2 v^2}{4}, \quad m_Z^2 = \frac{(g^2+g'^2)v^2}{4}$$

And define on-shell weak mixing angle as

$$s_W^2 \equiv 1 - \frac{m_W^2}{m_Z^2} \quad \text{(to all orders)}$$

data $s_W = \sin\theta_W = 0.2231$, G_F from $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

Here

$$m_W = \frac{(\pi\alpha/\sqrt{2}\,G_F)^{1/2}}{s_W\,(1-\Delta r)^{1/2}}$$

where Δr includes terms relating $\alpha(m_Z)$ to α; top, higgs, etc.

$$m_Z = \frac{m_W}{c_W}$$

$\boxed{\overline{MS}\text{-scheme}}$ on-shell scheme is awkward for discussion of new physics (which can perturb m_W) & RGE

$\overline{MS}$

$$\sin^2\bar\theta_W(\mu) \equiv \frac{\bar g'(\mu)^2}{\bar g^2(\mu) + \bar g'^2(\mu)}, \quad \bar g \text{ \& } \bar g' \text{ in } \overline{MS}$$

$$\bar s_W(m_Z) = (\sin\bar\theta_W)(\mu = m_Z)$$

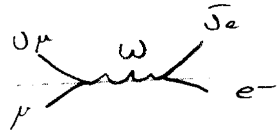
Here

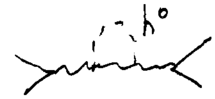
$$m_W = \frac{(\pi\alpha/\sqrt{2}\,G_F)^{1/2}}{\bar s_W(m_Z)\,(1-\Delta\bar r)^{1/2}}$$

$$m_Z = \frac{m_W}{\bar\rho^{1/2}\,\bar c_W(m_Z)}$$

$$\left[\begin{array}{l} \bar g_V^{(f)} = \sqrt{\bar\rho_f}\left(T_3^{(f)} - 2Q^{(f)}\bar s_W^2(m_Z)\right) \\ \bar g_A^{(f)} = \sqrt{\bar\rho_f}\; T_3^{(f)} \end{array}\right.$$

eg. $\mu^- \to e^- \nu_\mu \bar{\nu}_e$

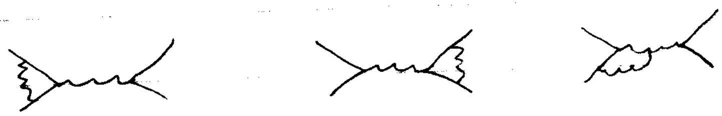




examples

Propagator

Vertex

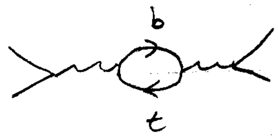


Box Graphs

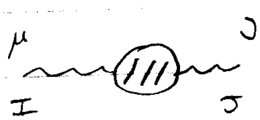
need sum for gauge invariance

Brem Graphs

Heavy Quarks



Focus on last term. It has m_t dependence & is also a universal correction.



$$= -i\,\Pi^{\mu\nu}_{IJ}(q) \qquad I, J \in \{\gamma, W, Z\}$$

$$\Pi^{\mu\nu}_{IJ}(q) = \Pi_{IJ}(q^2)\, g^{\mu\nu} - \Delta(q^2)\, q^\mu q^\nu$$

drops out when dotted into light quark currents

On-shell scheme

$$\begin{array}{rcccc} & \text{bare} & & \text{ren.} & & \text{c.t.} \\ & (M_W^2)_0 & = & M_W^2 & + & \delta M_W^2 \\ & (M_Z^2)_0 & = & M_Z^2 & + & \delta M_Z^2 \\ & e_0 & = & e & - & \delta e \end{array} \qquad y = \frac{c_W^2}{s_W^2}$$

if $g_0 = g - \delta g$

$g_0' = g' - \delta g'$

$v_0^2 = v^2 - \delta v^2$

$$\begin{pmatrix} \frac{\delta g^2}{g^2} \\ \frac{\delta g'^2}{g'^2} \\ \frac{\delta v^2}{v^2} \end{pmatrix} = \begin{pmatrix} y & -y & 1 \\ -1 & 1 & 1 \\ 1-y & y & -1 \end{pmatrix} \begin{pmatrix} \frac{\delta M_W^2}{M_W^2} \\ \frac{\delta M_Z^2}{M_Z^2} \\ \frac{\delta e^2}{e^2} \end{pmatrix}$$

W- propagator (Feyn Gauge) $g^{\mu\nu}$ terms

+inv. prop $= (\;)^{-1} + \overset{t}{\underset{b}{}} + \overset{\delta m^2}{} + \overset{\delta z}{}$

$$-\text{inv. prop} = -ig^{\mu\nu}(q^2 - M_W^2) - i\Pi_{WW}(q^2)g^{\mu\nu} + i\delta M_W^2 g^{\mu\nu} - i\delta Z_W(q^2 - M_W^2)g^{\mu\nu}$$

$$= -ig^{\mu\nu}\left[q^2 - M_W^2 + \Pi_{WW}(q^2) - \delta M_W^2 + \delta Z_W(q^2 - M_W^2) \right]$$

$$\Pi^{ren}_{WW}(q^2) = \Pi_{WW}(q^2) - \delta M_W^2 + \delta Z_W(q^2 - M_W^2)$$

$d = 4 - 2\epsilon$

$$\Pi_{WW}(q^2) = \frac{N_c\, g^2}{24\pi^2}\left[q^2 \left\{ \frac{1}{2\epsilon} + 3\int_0^1 dx\, x(1-x) \ln \frac{\bar{\mu}^2}{M_x^2 - q^2 x(1-x) - i0} \right\} + \frac{3}{2}\left\{ \frac{(m_1^2 + m_2^2)}{2\epsilon} + \int_0^1 dx\, M_x^2 \ln \frac{\bar{\mu}^2}{M_x^2 - q^2 x(1-x) - i0} \right\} \right]$$

where $M_x^2 \equiv m_1^2 x + m_2^2(1-x)$

and we have fermions of mass m_1 & m_2

- on-shell point is $q^2 = M_W^2$. For $M_W^2 > (m_1 + m_2)^2$ we have Imag part in Π_{WW} (finite) is $(\;)(\;)^*$ W-decay to $q_1 \bar{q}_2$

$$(\text{Breit - Wigner})^{-1} = q^2 - M_W^2 + iM_W\Gamma_W$$

so $\text{Im}\,\Pi_{WW}(M_W^2) = M_W \Gamma_W$

- Ren. Conditions - On-shell

$$0 = \text{Re}\,\Pi^{ren}_{WW}(q^2 = M_W^2) \qquad \delta M_W^2 = \text{Re}\,\Pi_{WW}(M_W^2)$$

$$0 = \text{Re}\,\frac{d}{dq^2}\Pi^{ren}_{WW}\Big|_{q^2 = M_W^2} \qquad \delta Z_W = \ldots$$

Z, γ propogator

$$(-\text{inv prop})^{ren} = -i g^{\mu\nu} \begin{pmatrix} q^2 + \Pi^{ren}_{\gamma\gamma}(q^2) & \Pi^{ren}_{\gamma z}(q^2) \\ \Pi^{ren}_{\gamma z}(q^2) & q^2 - M_z^2 + \Pi^{ren}_{zz}(q^2) \end{pmatrix}$$

inverse $\rightarrow$

$$D^{\mu\nu}_{\gamma\gamma} = \frac{-i g^{\mu\nu}}{q^2 + \Pi^{ren}_{\gamma\gamma}(q^2) - \dfrac{\Pi^{ren}_{\gamma z}(q^2)^2}{\left(q^2 - M_z^2 + \Pi^{ren}_{zz}(q^2)\right)}}$$

$$D^{\mu\nu}_{zz} = \frac{-i g^{\mu\nu}}{q^2 - M_z^2 + \Pi^{ren}_{zz} - \dfrac{\Pi^{ren\,2}_{\gamma z}}{(q^2 + \Pi^{ren}_{\gamma\gamma})}}$$

$$D^{\mu\nu}_{z\gamma} = \frac{i g^{\mu\nu} \, \Pi^{ren}_{\gamma z}(q^2)}{(q^2 + \Pi^{ren}_{\gamma\gamma})(q^2 - M_z^2 + \Pi^{ren}_{zz}) - \Pi^{ren\,2}_{\gamma z}}$$

A constant as $q^2 \to 0$

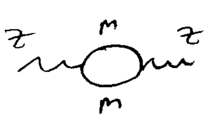
require

$$\left.\begin{aligned} \Pi^{ren}_{\gamma\gamma}(0) &= 0 \\ \Pi^{ren}_{\gamma z}(0) &= 0 \end{aligned}\right\} \quad \text{massless photon}$$

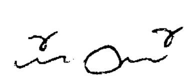
$$\left. \Pi^{ren}_{zz} - \frac{\Pi^{ren\,2}_{\gamma z}}{q^2 + \Pi^{ren}_{\gamma\gamma}} \right|_{q^2 = M_z^2} = 0 \quad \Big\} \quad \text{massive } Z \quad \text{on-shell ren. condition}$$

bare

$$\Pi_{zz}(q^2) = \frac{g^2 N_c}{16\pi^2 c_w^2} \left[\frac{2 q^2}{3} \left(g_v^{(f)2} + g_a^{(f)2}\right) \left\{ \frac{1}{2\epsilon} + 3 \int dx\, x(1-x) \ln \frac{\mu^2}{m^2 - q^2 x(1-x)} \right\} + 4 m^2 g_a^{(f)2} \left\{ \frac{1}{2\epsilon} + \frac{1}{2} \int_0^1 dx \ln \left(\frac{\mu^2}{---} \right) \right\} \right]$$



$$\Pi_{\gamma z}(q^2) = \frac{g_v^{(f)}}{2 c_w s_w Q_f} \Pi_{\gamma\gamma}(q^2)$$



Δ add comment

$\boxed{\Delta\rho}$ low energy ratio of W & Z prop's $(q^2=0)$

$$\rho_0 = \frac{1}{(c_W^2)_0} \frac{D^{\text{bare}}_{ZZ}(q^2=0)}{D^{\text{bare}}_{WW}(q^2=0)}$$

$$= 1+\Delta\rho = \frac{M_Z^2+\delta M_Z^2}{M_W^2+\delta M_W^2} \frac{(-M_W^2-\delta M_W^2+\Pi_{WW}(0))}{(-M_Z^2-\delta M_Z^2+\Pi_{ZZ}(0))}$$

$$\Delta\rho = \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}$$ universal correction

finite (corrects $\rho = 1$)

eg: top $(m_b=0)$

$$\Delta\rho_t = \frac{g^2 N_c}{16\pi^2 M_W^2}\left[4m_t^2 \left(\tfrac{1}{2}\right)^2 \left\{\frac{1}{2\epsilon}+\dots\right\} - \frac{m_t^2}{2\epsilon} + \dots \right]$$ finite terms don't cancel

$$= \frac{g^2 N_c m_t^2}{64\pi^2 M_W^2}$$

$\boxed{\Delta r}$ $\mu^- \to e^- \nu_\mu \bar{\nu}_e$ for G_F (low energy again)

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}(1+\Delta r)$$

heavy fermion corrections from propagator

$$\frac{G_F}{\sqrt{2}} = -\frac{g_0^2}{8\left(q^2-(M_W^2)_0+\Pi_{WW}(0)\right)}\Bigg|_{q^2=0} = \frac{g^2}{8M_W^2}\left(1+\frac{\Pi_{WW}(0)}{M_W^2}-\frac{\delta M_W^2}{M_W^2} - \frac{\delta g^2}{g^2}\right)$$

$$\Delta r = -\frac{\delta e^2}{e^2} - \frac{c_W^2}{s_W^2}\left[\frac{\delta M_Z^2}{M_Z^2}-\frac{\delta M_W^2}{M_W^2}\right] + \frac{\Pi_{WW}(0)-\delta M_W^2}{M_W^2}$$

f....re $\quad$ $\leftarrow$ finite

$$= \dots$$

$$= \left(\frac{\alpha(M_Z)-\alpha}{\alpha}\right) - \frac{c_W^2}{s_W^2}\Delta\rho + \dots$$ + smaller terms

$\boxed{\text{Constraints on new physics}}$ S, T, U , $M_{new} \gg M_Z$

New heavy particles should effect W, Z propagators. Use $\overline{MS}$ scheme

mainly

$$S \equiv \frac{4\bar{s}_Z^2\,\bar{c}_Z^2}{\alpha(m_Z)}\left[\frac{\Pi_{ZZ}^{new}(M_Z^2) - \Pi_{ZZ}^{new}(0)}{M_Z^2}\right] \qquad m_Z$$

$$T \equiv \frac{1}{\alpha(m_Z)}\left[\frac{\Pi_{WW}^{new}(0)}{M_W^2} - \frac{\Pi_{ZZ}^{new}(0)}{M_Z^2}\right] \qquad \Gamma_Z$$

$$U \equiv \frac{4\bar{s}_Z^2}{\alpha(m_Z)}\left[\frac{\Pi_{WW}^{new}(M_W^2) - \Pi_{WW}^{new}(0)}{M_W^2}\right] - S \qquad M_W$$

$S, T, U \sim 1$

as of 2010 e.w. precision data gave

$S = 0.01 \pm .10$	$(-.08)$	
$T = 0.03 \pm .11$	$(+.09)$	
$U = 0.06 \pm .10$	$(+.01)$	
$\uparrow m_H = 117\,\text{GeV}$	$\uparrow m_H = 300\,\text{GeV}$	

eg.

$$\overset{\text{measure}}{M_Z^2} = (M_Z^2)^{SM}\,\frac{(1-\alpha T)}{1 - \frac{G_F (M_Z^2)^{SM}}{2\sqrt{2}\pi} S}$$

$$M_W^2 = (M_W^2)^{SM}\,\frac{1}{1 - \frac{G_F (M_W^2)^{SM}}{2\sqrt{2}\pi}(S+U)}$$

$$\frac{\Gamma_Z}{m_Z^3} = \frac{1}{(1-\alpha T)}\left(\frac{\Gamma_Z}{M_Z^3}\right)^{SM}$$

$$\frac{\Gamma_W}{m_W^3} = \left(\frac{\Gamma_W}{M_W^3}\right)^{SM}$$

$$A^{\text{neutral current}} = \frac{1}{1-\alpha T}\left(A^{\text{neutral current}}\right)^{SM}, \quad \rho = \frac{1}{1-\alpha T}$$

Ch4 Flavor Sector

Quarks Masses & CKM mixing. $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$

Recall

$$\mathcal{L}_{Yukawa} = -g_d^{ij} \bar{d}_R^i H^+ Q_L^j + g_u^{ij} \bar{u}_R^i H^T \epsilon Q_L^j + h.c.$$

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Higgs vev $H = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ gives

$$m_u = \frac{v}{\sqrt{2}} g_u , \quad m_d = \frac{v}{\sqrt{2}} g_d$$ mass matrices

3×3 for 3 families

In weak interaction eigenbasis we have off diagonal terms in mass matrices

Mass Eigenbasis diagonal, positive definite

$$\boxed{m_u = U_u D_u W_u^+} , \quad \boxed{m_d = U_d D_d W_d^+}$$

unitary

Pf: problem is that $m_{u,d}$ need not be Hermitian

Consider $(m_u m_u^+)$. Its Hermitian & Positive Definite

$[\ (z^* m_u) \cdot (m_u^+ z) \geq 0$ for all complex $z\ ]$ so

$$m_u m_u^+ = \underset{\text{unitary}}{U_u} \underset{\text{diagonal}}{D_u^2} U_u^+ = (U_u D_u U_u^+)(U_u D_u U_u^+)$$

$$\begin{pmatrix} d_1^2 & & 0 \\ & d_2^2 & \\ 0 & & \ddots \end{pmatrix} , \quad d_i^2 > 0$$

Let $H_u = U_u D_u U_u^+$, Hermitian & $H_u^2 = m_u m_u^+$

Claim: $\mathcal{U} = H_u^{-1} m_u$ is unitary

$$\mathcal{U}^+ \mathcal{U} = m_u^+ (H_u^{-1})^+ H_u^{-1} m_u = m_u^+ (H_u^2)^{-1} m_u = m_u^+ (m_u^+)^{-1} m_u^{-1} m_u$$

$$= \mathbb{1} \qquad (H_u H_u^{-1} = 1, \quad (H_u^{-1})^+ H_u = 1)$$

so $m_u = H_u \mathcal{U} = U_u D_u \underbrace{(U_u^+ \mathcal{U})}_{\equiv W_u^+}$ unitary

$m_u = U_u D_u W_u^+$ Q.E.D similar for m_d

Change from weak to mass eigenbasis

$u_L \to W_u u_L$

$u_R \to U_u u_R$

$d_L \to W_d d_L$

$d_R \to U_d d_R$

$$\mathcal{L}_{Yukawa} \overset{weak}{=} -\bar{d}_R U_d^\dagger M_d W_d d_L - \bar{u}_R U_u^\dagger M_u W_u u_L + h.c. + (h^0 \text{ interaction})$$

$$\overset{mass}{=} -\bar{d}_R D_d d_L - \bar{u}_R D_u u_L + h.c. + (h^0 \text{ int.})$$

$$= -m_d \bar{d} d - m_s \bar{s} s - \cdots - m_t \bar{t} t + (h^0 \text{ int.})$$

<u>Other $\mathcal{L}$ terms</u>

- γ & Z $\quad \bar{u}_L (\cdots) u_L$, $\bar{d}_L (\cdots) d_L$, $\bar{u}_R (\cdots) u_R$, $\bar{d}_R (\cdots) d_R$

  so unchanged by rotation

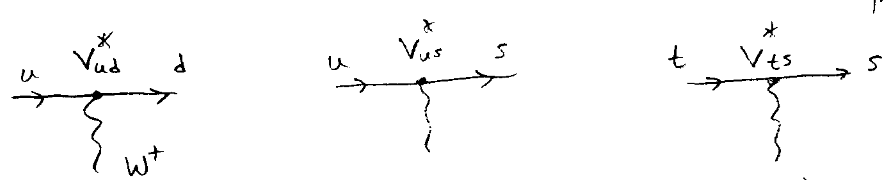
  $\to$ <u>no flavor changing neutral currents</u> at tree level

  unitary gauge : h^0 also couples diagonally
- only $W^\pm$ mixed up & down type quarks

$$\mathcal{L} \overset{weak}{=} \frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L^i \gamma^\mu d_L^i + h.c.$$

$$\overset{mass}{=} \frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L \underbrace{(W_u^\dagger W_d)}_{\text{unitary}} \gamma^\mu d_L + h.c.$$

$\equiv V_{CKM}$ $\quad$ Cabibbo-Kobayashi-Maskawa Matrix (Nobel '08)



$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = V_{ij}$$

i: rows, up-type; j: columns, down-type

[Aside: Leptons $e_L \to W_e e_L$, $e_R \to U_e e_R$, if ν_L massless then <u>pick</u> $\nu_L \to W_e \nu_L$ $\Rightarrow$ diagonal couplings & seperate $e\#$, $\mu\#$, $\tau\#$ conservation]

Indep. Parameters ?

3 families - started with $(g_u, g_d) = 18+18 = 36$ param.

Unitary trnsfms leave

6 masses $\quad M_{u,d,s,c,b,t}$

9 parameters in V_{CKM} (3×3 unitary)

Reduce further : $9 = \underset{\uparrow \, O(3) \text{ rotation}}{3}^{\text{real}} + \overset{\text{phases}}{6}$

Make Phase Redefinitions on quark fields.

Transform L, R together so masses remain real

$\left.\begin{matrix} u^i_{L,R} \\ d^i_{L,R} \end{matrix}\right\}$ 6 phases , but overall global phase change to all fields doesn't help

so $\underline{-5 \text{ phases}}$

V_{CKM} has 3 real parameters & 1 phase which is $\cancel{CP}$

Recall: $\mathcal{L} = \lambda \bar{\psi}_1 W^\mu_+ \gamma_\mu P_L \psi_2 + \lambda^* \bar{\psi}_2 W^-_\mu \gamma^\mu P_L \psi_1$ ← hermitian

$\downarrow CP \qquad\qquad \downarrow CP$

$\mathcal{L}^{CP} = \lambda \bar{\psi}_2 W^\mu_- \gamma_\mu P_L \psi_1 + \lambda^* \bar{\psi}_1 W^+_\mu \gamma^\mu P_L \psi_2$

$\mathcal{L} = \mathcal{L}^{CP}$ iff $\lambda = \text{real}$, complex λ violates CP

Need $\geq$ 3 flavors for $\cancel{CP}$: $\quad$ 2×2 case $\quad 4 = 1 + \underset{\uparrow \text{ phases}}{3}$

minus (4-1) phase rotations

= 1 real parameter

$$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$$

θ_c = Cabbibo angle

no $\cancel{CP}$

$$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1 param   2 param   1 param

$s_{23} = \sin\theta_{23}$ etc $\{\theta_{12}, \theta_{13}, \theta_{23}, \delta\}$

$c_{23} = \cos\theta_{23}$ 4 parameters

Let $s_{12} \equiv \lambda = \dfrac{|V_{us}|}{\sqrt{|V_{us}|^2 + |V_{ud}|^2}}$

$s_{23} \equiv A\lambda^2 = \lambda \left|\dfrac{V_{cb}}{V_{us}}\right|$

$s_{13}e^{i\delta} \equiv A\lambda^3(\rho + i\eta) = V_{ub}^*$ or $\bar{\rho} + i\bar{\eta} = -\dfrac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$

4 parameters : $\{\lambda, A, \rho, \eta\}$ or $\{\lambda, A, \bar{\rho}, \bar{\eta}\}$

Wolfenstein $\quad V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$

$\lambda \simeq 0.226 \pm .002$

$\simeq$ Cabbibo angle

$\Uparrow$

Gets hierarchy right which is crucial for phenomenology

$\underline{\mathcal{O}(1)}$	$\underline{\mathcal{O}(\lambda)}$	$\underline{\mathcal{O}(\lambda^2)}$	$\underline{\mathcal{O}(\lambda^3)}$
V_{ud}	V_{us}	V_{cb}	V_{ub}
V_{cs}	V_{cd}	V_{ts}	V_{td}
V_{tb}			

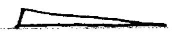
$A, \rho, \eta \sim 1$

Unitary Triangles

$$V^\dagger V = \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6 zeroes

eg. $V_{ud} V_{us}^* + V_{cd} V_{cs}^* + V_{td} V_{ts}^* = 0$ triangle



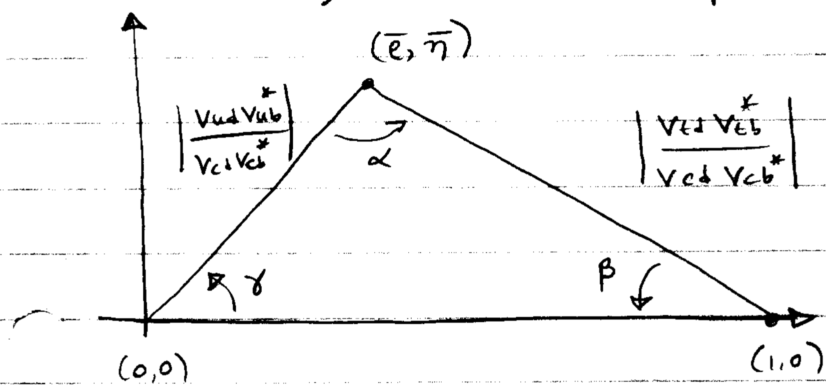
$\quad \sim \lambda \qquad \sim \lambda \qquad \sim \lambda^5$

eg. $V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$ triangle

$\quad \sim \lambda^3 \qquad \sim \lambda^3 \qquad \sim \lambda^3$

(the one you see most in talks)

Rescaled Triangle in complex plane, $V_{cd} V_{cb}^*$ = best known, so



$$\alpha = \arg \left[\frac{-V_{td} V_{tb}^*}{V_{ud} V_{ub}^*} \right]$$

$$\beta = \arg \left[\frac{-V_{cd} V_{cb}^*}{V_{td} V_{tb}^*} \right]$$

$$\gamma = \arg \left[\frac{-V_{ud} V_{ub}^*}{V_{cd} V_{cb}^*} \right]$$

$$= \pi - \alpha - \beta$$

Can Test SM picture of ~~CP~~ by independent measurements of sides and angles

Area of triangle is measure of ~~CP~~. In fact all triangles have area $\frac{|J|}{2}$ where

(more in section)

$$\mathrm{Im} \left[V_{ij} V_{kl} V_{il}^* V_{kj}^* \right] = J \sum_{m,n} \epsilon_{ikm} \epsilon_{jln}$$

Jarlskog invariant (Parameterization indep.)

$$J = c_{12} c_{23} c_{13}^2 s_{12} s_{23} s_{13} \sin \delta$$

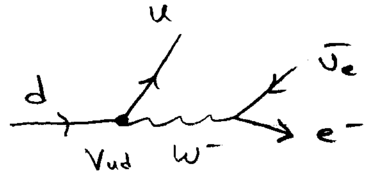
need $\theta_{12}, \theta_{23}, \theta_{13} \neq 0, \pi/2, \pi$ and $\delta \neq 0, \pi$

Also need $(m_t - m_c)(m_c - m_u)(m_t - m_u)(m_b - m_s)(m_s - m_d)(m_b - m_d) \neq 0$

no degeneracy between same charge quarks

$|V_{ij}|$ Measurement

eg. β-decay



$n \to p\, e^- \bar{\nu}_e$

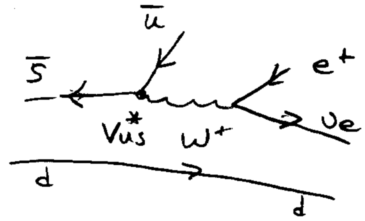
$$\Gamma \propto |V_{ud}|^2$$

$$|V_{ud}| = .97377 \pm .00027$$

(nuclear β-decay)

eg. $K^0 \to \pi^- e^+ \nu_e$

$\bar{s} d \to \bar{u} d \; e^+ \nu_e$



$\Gamma \propto |V_{us}|^2$

decay rate depends on form factors

$$\langle \pi^-(p) | \bar{s} \gamma^\mu u | K^0(k) \rangle = f_+(q^2)\,(k+p)^\mu + f_-(q^2)\,(k-p)^\mu$$

↑ strong interaction m.elt.

constrain with P, T & Lorentz

$f_{+,-}(q^2)$ are real

$$T\,|\pi^-(p)\rangle = -\langle \pi^-(p_T)| \quad , \quad p_T = (p^0, -\vec{p})$$

$$T\,|K^0(k)\rangle = -\langle K^0(k_T)|$$

only k^0, p^0

$$\langle \pi^-(p) | T^\dagger T\, V^0\, T^\dagger T | K^0(k) \rangle = \langle \pi^-(p_T) | V^0 | K^0(k_T) \rangle^*$$

$$f_\pm(q^2) = f_\pm(q^2)^*$$

In SU(3) flavor limit it can be proven that $f_+(0) = -1$, $f_-(0) = 0$. Corrections to this are 2nd order in $\frac{m_s}{\Lambda}$ (Ademollo-Gatto thm. see Donoghue) & 1st order in $(m_u - m_d)$. Chiral Pert. theory gives

$$f_+(0) = 0.961 \pm .008 \quad \&$$

$$|V_{us}| = 0.2257 \pm 0.0021$$

eg. $B^0 \to D^- \, e^+ \nu_e$ $\qquad \Gamma \propto |V_{cb}|^2$

$\bar{b}\, d \to \bar{c}\, d \;\; e^+ \nu_e$

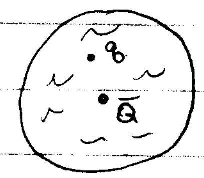
$M_b \sim 4.7\,\text{GeV}$ $\gg \Lambda_{QCD}$ can't use chiral symmetry

$M_c \sim 1.4\,\text{GeV}$ $\qquad M_{u,d,s} \ll \Lambda$

Opposite limit $M_Q \gg \Lambda_{QCD}$ <u>Heavy</u> <u>Quark</u> <u>Symmetry</u>

$\frac{\Lambda}{M_Q}$ expansion

Heavy - Light meson $\bar{Q}\, q$ size $r^{-1} \sim \Lambda_{QCD} \ll M_Q$



typical momentum exchange $p \sim \Lambda_{QCD}$

doesn't change velocity much $\Delta v = \frac{\Delta p}{M_Q}$

Q behave like <u>static</u> external color source

$M_{b,c} \to \infty$

- mass irrelevant to dynamics

  $U(2)$ flavor symmetry

- gluons only interact with chromo electric charge, spin-independent, $SU(2)$

  [think Hydrogen, Coulomb gluons]

Two Methods

① Exclusive $B^0 \to D^- e^+ \nu_e$, $B^0 \to D^{-*} e^+ \nu_e$

$\{ D^-, D^{-*} \}$ related by heavy quark spin flip

so in symmetry multiplet

Symmetry constrains form factors again

② Inclusive $B^0 \to X_c \; e^+ \nu_e$

↑ any mesons with $\bar{c}$

$X_c = D^-, D^- \pi^0, D^0 \pi^- \pi^0$, etc.

summing over all X_c we can use "Operator Product Expansion", and $\Gamma(B^0 \to X_c e^+ \nu_e) = \Gamma(b \to c e^+ \nu_e) + \mathcal{O}(\frac{\Lambda}{M_Q})$

we can compute the leading term exactly.
(More on O.P.E. later.)

Heavy Quark Symmetry is encoded in an effective Lagrangian,

$$\mathcal{L}_{HQET} = \bar{h}_v \, i v \cdot D \, h_v \qquad h_v = \begin{pmatrix} c_v^\uparrow \\ c_v^\downarrow \\ b_v^\uparrow \\ b_v^\downarrow \end{pmatrix}$$

$v^\mu = (1,0,0,0)$ in rest frame

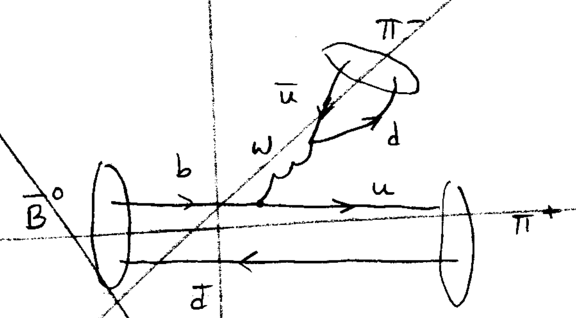
$U(4)$ symmetry

<u>~~CP~~ Measurement</u>

need processes where all 3 generations are active (for 2 gens we can always pick convention where V_{ckm} is real)

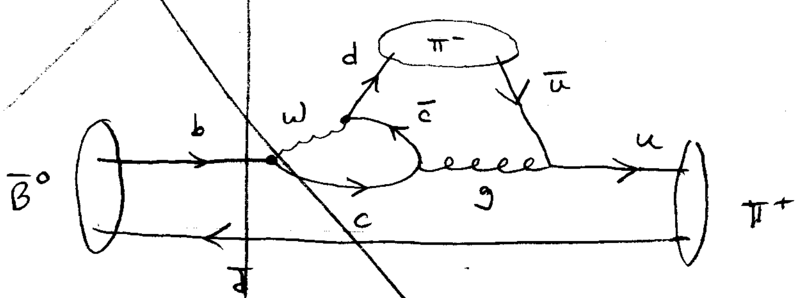
eg. $\bar{B}^0 \to \pi^+ \pi^-$

$b \to u \bar{u} d$



$V_{ub} V_{ud}^*$

$b \to c \bar{c} d$



$V_{cb} V_{cd}^*$

sensitive to ~~CP~~

$\not{CP}$ Measurements

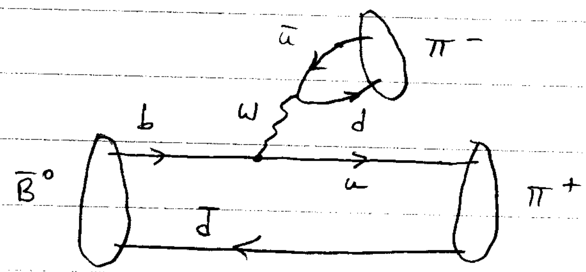
Need processes where all 3 generations are active, since for two generations can always pick convention where V_{CKM} is real

We have three manifestations of $\not{CP}$

① $\not{CP}$ in decay. Amplitude $|A|$ for charged or neutral decay process differs from CP-conjugate $|\bar{A}|$. (This is the only type that occurs for charged mesons.)

eg. $\bar{B}^0 \to \pi^+ \pi^-$

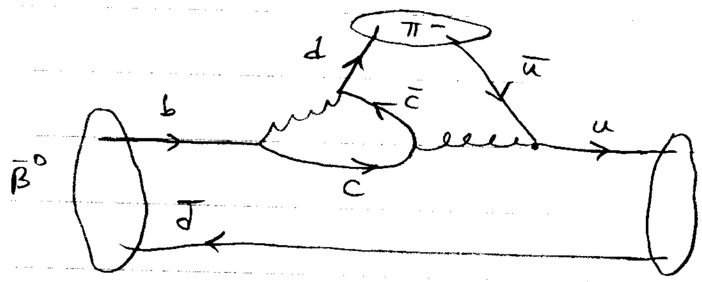
$b \to u \bar{u} d$



$V_{ub} V_{ud}^*$

1^{st} & 3^{rd} gen

$b \to c \bar{c} d$



$V_{cb} V_{cd}^*$

2 & 3   1 & 2

while CP-conjugate $B^0 \to \pi^- \pi^+$ process involves

$\bar{b} d$ $V_{ub}^* V_{ud}$ & $V_{cb}^* V_{cd}$.

Here we can't avoid CP violating phase and

$$|A| = |A(\bar{B}^0 \to \pi^+ \pi^-)| \neq |A(B^0 \to \pi^+ \pi^-)| = |\bar{A}|$$

$$\Gamma(\bar{B}^0 \to \pi^+ \pi^-) \neq \Gamma(B^0 \to \pi^+ \pi^-)$$

② CP violation in <u>mixing</u>. When two neutral mass eigenstate are not CP eigenstates

Neutral	$(K^0, \bar{K}^0)$	$(D^0, \bar{D}^0)$	$(B^0, \bar{B}^0)$	$(B_s^0, \bar{B}_s^0)$
Mesons	$\bar{s}d,\ \bar{d}s$	$\bar{u}c,\ \bar{c}u$	$\bar{b}d,\ \bar{d}b$	$\bar{b}s,\ \bar{s}b$

(no t's)

$$CP\,|B^0\rangle = e^{2i\eta_B}\,|\bar{B}^0\rangle$$
$$CP\,|\bar{B}^0\rangle = e^{-2i\eta_B}\,|B^0\rangle$$

$\uparrow$ convention dependent phase

③ $\cancel{CP}$ in <u>interference</u> between decays with and without mixing. Common final state f_{CP}.

eg
$$\begin{tikzcd} B^0 \arrow[rr] \arrow[dr] & & f_{CP} \\ & \bar{B}^0 \arrow[ur] & \end{tikzcd}$$

<u>Another example</u>

<u>if</u> CP conserved then mass eigenstates K_S, K_L are CP eigenstates:

$$|K_S\rangle = 1/\sqrt{2}\,(|K^0\rangle + |\bar{K}^0\rangle)$$
$$|K_L\rangle = 1/\sqrt{2}\,(|K^0\rangle - |\bar{K}^0\rangle)$$

($\eta_K = -1$ for convenience)

and $K_S \to \pi\pi$ CP-even short lived

$K_L \to \pi\pi\pi$ CP-odd long lived

but see $K_L \to \pi\pi$, so CP is violated

<u>Mixing & Evolution</u> mixture $a|B^0\rangle + b|\bar{B}^0\rangle$ obeys

$$i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = H\begin{pmatrix} a \\ b \end{pmatrix} = \left(M - \frac{i\Gamma}{2}\right)\begin{pmatrix} a \\ b \end{pmatrix} \quad (*)$$

[derive this momentarily]

M, Γ are 2x2, Hermitian, t-independent

<u>onstraints</u>

$$CP\,|B^0, out\rangle = e^{2i\varphi_B}\,|\bar{B}^0, out\rangle$$

$$T\,|B^0, in\rangle = |B^0, out\rangle$$

$$CPT\,|B^0, in\rangle = e^{2i\xi_B}\,|\bar{B}^0, out\rangle$$

$(CPT)^+ H (CPT) = H$ from CPT invariance of QFT

$\Rightarrow$ $H_{11} = H_{22}$ so $\Gamma_{11} = \Gamma_{22}$, $M_{11} = M_{22}$ (which are real)

and particles & antiparticles have same masses and widths

$\Rightarrow$ doesn't constrain $\left.\begin{matrix} M_{12} = M_{21}^* \\ \Gamma_{12} = \Gamma_{21}^* \end{matrix}\right\}$ complex

and imaginary part encodes ~~CP~~. <u>If</u> CP is conserved

$$\langle B^0|H|\bar{B}^0\rangle = \langle B^0|(CP)^{-1}\underbrace{(CP)\,H\,(CP)^{-1}}(CP)|\bar{B}^0\rangle = \langle\bar{B}^0|\,H\,|B^0\rangle,$$

so $H_{12} = H_{21}$ so M_{12} and Γ_{12} are real.

weak Hamiltonian small

<u>Back to (*)</u> consider $S_{\alpha\beta} = \langle\alpha|\,T\,e^{-i\int dt\, H_W(t)}\,|\beta\rangle$

$$\equiv \delta_{\alpha\beta} - 2\pi i\,\delta(E_\alpha - E_\beta)\,H_{\alpha\beta}$$

$$H_{\alpha\beta} = \langle\alpha|H_W|\beta\rangle - \frac{i}{2}\int dt\,\langle\alpha|\,T\,H_W(t)\,H_W(0)\,|\beta\rangle + \ldots$$

$$= \langle\alpha|H_W|\beta\rangle - \frac{i}{2}\int dt\,\sum_n\Big[\theta(t)\,\langle\alpha|H_W(t)|n\rangle\langle n|H_W(0)|\beta\rangle + \theta(-t)\,\langle\alpha|H_W(0)|n\rangle\langle n|H_W(t)|\beta\rangle\Big]$$

now $\displaystyle\int dt\;\theta(t)\,\langle\alpha|H_W(t)|n\rangle = \int dt\left[\left(\frac{-1}{2\pi i}\right)\int\frac{d\omega}{\omega + i0}\,e^{-i\omega t}\right]e^{i(M_\alpha - E_n)t}\,\langle\alpha|H_W(0)|n\rangle$

$$= i\int\frac{d\omega}{\omega + i0}\,\delta(\omega - (M_\alpha - E_n))\,\langle\alpha|H_W(0)|n\rangle = \frac{i}{M_\alpha - E_n + i0}\,\langle\alpha|H_W(0)|n\rangle$$

$$H_{\alpha\beta} = \langle\alpha|H_W(0)|\beta\rangle + \sum_n\frac{\langle\alpha|H_W(0)|n\rangle\langle n|H_W(0)|\beta\rangle}{M_\alpha - E_n + i0}$$

skip

Details on $\not{C}\not{P}$

$$\langle B^0 | (CPT)^{-1} (CPT) H (CPT)^{-1} (CPT) | B^0 \rangle = \langle \bar{B}^0 | H | \bar{B}^0 \rangle$$

$$H_{11} = H_{22}$$

$$\langle B^0 | (CPT)^{-1} (CPT) H (CPT)^{-1} (CPT) | \bar{B}^0 \rangle = \langle B^0 | H | \bar{B}^0 \rangle$$

$$H_{12} = H_{12}$$

Eigenvalue Problem

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{11} \end{pmatrix} \begin{pmatrix} p \\ \pm q \end{pmatrix} = \lambda_{L,H} \begin{pmatrix} p \\ \pm q \end{pmatrix}$$

$$\begin{vmatrix} H_{11}-\lambda & H_{12} \\ H_{21} & H_{11}-\lambda \end{vmatrix} = (H_{11}-\lambda)^2 - H_{12}H_{21} = 0 \qquad \begin{aligned} \lambda_L &= H_{11} - (H_{12}H_{21})^{1/2} \\ \lambda_H &= H_{11} + (H_{12}H_{21})^{1/2} \end{aligned}$$

difference $\quad \lambda_H - \lambda_L = \Delta M_B - \frac{i}{2}\Delta\Gamma_B = 2\sqrt{H_{12}H_{21}}$

$$\Delta M_B = M_H - M_L > 0, \quad \Delta\Gamma_B = \Gamma_H - \Gamma_L$$

$\mathrm{Re}\,(\ )^2$: $\quad (\Delta m)^2 - \frac{1}{4}(\Delta\Gamma)^2 = 4|M_{12}|^2 - |\Gamma_{12}|^2$

$\mathrm{Im}\,(\ )^2$: $\quad -\Delta m\,\Delta\Gamma = 4\,\mathrm{Im}\,(H_{12}H_{21}) = 4\,\mathrm{Im}\left(\left(M_{12} - \frac{i\Gamma_{12}}{2}\right)\left(M_{12}^* - \frac{i\Gamma_{12}^*}{2}\right)\right)$

$$\Delta m\,\Delta\Gamma = 4\,\mathrm{Re}\,(M_{12}\Gamma_{12}^*)$$

λ_L: $\quad \begin{pmatrix} (H_{12}H_{21})^{1/2} & H_{12} \\ H_{21} & (H_{12}H_{21})^{1/2} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 0$

λ_H: $\quad \begin{pmatrix} -(H_{12}H_{21})^{1/2} & H_{12} \\ H_{21} & -(H_{12}H_{21})^{1/2} \end{pmatrix} \begin{pmatrix} p \\ -q \end{pmatrix} = 0$

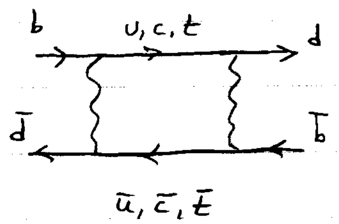
hence $\begin{pmatrix} p \\ \pm q \end{pmatrix}$ is solution

$$\frac{p}{q} = \frac{-H_{12}}{(H_{12}H_{21})^{1/2}} = \frac{-(H_{12}H_{21})^{1/2}}{H_{21}}$$

$$H_{\alpha\beta} = \langle \alpha | H_w | \beta \rangle + \sum_n P \frac{1}{M_\alpha - E_n} \langle \alpha | H_w | n \rangle \langle n | H_w | \beta \rangle \leftarrow M$$

$$- i\pi \sum_n \delta(M_\alpha - E_n) \langle \alpha | H_w | n \rangle \langle n | H_w | \beta \rangle \leftarrow -\frac{i\Gamma}{2}$$

$$H = M - \frac{i\Gamma}{2}$$

eg. 

M_{12} virtual

Γ_{12} on-shell intermediate states

<u>Mass Eigenstates</u>

eigenvalue

light $|B_L\rangle = p\,|B^0\rangle + q\,|\bar{B}^0\rangle$ $\quad \lambda_L = H_{11} - (H_{12} H_{21})^{1/2}$

heavy $|B_H\rangle = p\,|B^0\rangle - q\,|\bar{B}^0\rangle$ $\quad \lambda_H = H_{11} + (H_{12} H_{21})^{1/2}$

$\Delta M_B \equiv M_H - M_L > 0$ $\quad \Delta m - \frac{i}{2}\Delta\Gamma = 2\sqrt{H_{12} H_{21}}$

$\Delta\Gamma_B = \Gamma_H - \Gamma_L$

solution $\quad (\Delta m)^2 - \frac{1}{4}(\Delta\Gamma)^2 = 4|M_{12}|^2 - |\Gamma_{12}|^2$

$$\Delta m\, \Delta\Gamma = 4\, \mathrm{Re}\,(M_{12} \Gamma_{12}^*)$$

and $\quad \dfrac{q}{p} = -\dfrac{(\Delta m - i/2\, \Delta\Gamma)}{2M_{12} - i\Gamma_{12}} = -\dfrac{(2M_{12}^* - i\Gamma_{12}^*)}{\Delta m - \frac{i\Delta\Gamma}{2}} = \dfrac{-H_{21}}{(H_{12}H_{21})^{1/2}}$

$$|q|^2 + |p|^2 = 1$$

Time Dependence: $\quad |B_{H,L}(t)\rangle = e^{-(iM_{H,L} + \Gamma_{H,L}/2)t}\, |B_{H,L}\rangle$

and general solution is quite complicated

$$\frac{d\Gamma}{dt}(M^0(t) \to f) = N_f\, e^{-\Gamma t} \qquad \Gamma \equiv \frac{\Gamma_H + \Gamma_L}{2}$$

$$* \Big\{ \left(|A_f|^2 + \left|\tfrac{q}{p}\bar{A}_f\right|^2\right) \cosh\left(\tfrac{\Delta\Gamma}{2} t\right)$$
$$+ \left(|A_f|^2 - |q/p\, \bar{A}_f|^2\right) \cos(\Delta m\, t)$$
$$+ 2\, \mathrm{Re}\left(q/p\, A_f^* \bar{A}_f\right) \sinh\left(\tfrac{\Delta\Gamma}{2} t\right)$$
$$- 2\, \mathrm{Im}\left(\tfrac{q}{p} A_f^* \bar{A}_f\right) \sin(\Delta m\, t) \Big\}$$

where Decay amplitudes

$$A_f = \langle f | H | B^0 \rangle$$
$$\bar{A}_f = \langle f | H | \bar{B}^0 \rangle$$

also define $\quad \bar{A}_{\bar{f}} = \langle \bar{f} | H | \bar{B}^0 \rangle$

Phase convention allows $\quad |B^0\rangle \to e^{-i\vartheta} |B^0\rangle$

$$|\bar{B}^0\rangle \to e^{i\vartheta} |\bar{B}^0\rangle$$

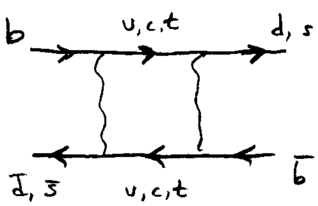
which takes $\quad M_{12} \to e^{2i\vartheta} M_{12}, \quad \Gamma_{12} \to e^{2i\vartheta}\Gamma_{12}, \quad \frac{q}{p} \to e^{-2i\vartheta}\left(\frac{q}{p}\right)$

$$M_{21} \to e^{-2i\vartheta} M_{21}, \quad \Gamma_{21} \to e^{-2i\vartheta}\Gamma_{21}$$

$$A_f \to e^{-i\vartheta} A_f, \quad \bar{A}_f \to e^{i\vartheta}\bar{A}_f, \quad \bar{A}_{\bar{f}} \to e^{i\vartheta}\bar{A}_{\bar{f}}$$

but $\underbrace{\left|\frac{q}{p}\right|, \quad \left|\frac{\bar{A}_{\bar{f}}}{A_f}\right|, \quad \lambda = \frac{q}{p}\frac{\bar{A}_f}{A_f}}_{\text{these show up in observables}}$ independent of convention

Boxes



$$\sim \sum_{i,j = u,c,t} \lambda_i \lambda_j \; F(x_i, x_j)$$

where $\quad x_i = \frac{m_i^2}{M_W^2}, \quad \lambda_i = \begin{cases} V_{is} V_{id}^* & K \\ V_{ib} V_{id}^* & B \\ V_{ib} V_{is}^* & B_s \end{cases}$

unitarity $\quad \lambda_u + \lambda_c + \lambda_t = 0$

so a mass indep. term would drop out

$\boxed{\text{for } B}$

top dominates $\quad (V_{tb} V_{td}^*)^2 \; F\left(\frac{m_t^2}{M_W^2}\right)$ real

$$\Delta M_B \simeq \underset{\substack{\uparrow \\ 1/100}}{0.73\,\Gamma_B} \gg \underset{\substack{\uparrow \\ \text{CKM suppressed}}}{\Delta\Gamma}$$

so $q/p \simeq -\frac{M_{12}^*}{|M_{12}|} = -\frac{V_{tb}^* V_{td}}{V_{tb} V_{td}^*}$, $\left|\frac{q}{p}\right| \simeq 1$

$\Delta m_B \simeq 2|M_{12}|$

$$\begin{pmatrix} |B^0(t)\rangle \\ |\bar{B}^0(t)\rangle \end{pmatrix} = e^{-imt} e^{-\Gamma t/2} \begin{pmatrix} \cos(\Delta m t) & \frac{q}{p} i \sin(\Delta m t) \\ \frac{p}{q} i \sin(\Delta m t) & \cos(\Delta m t) \end{pmatrix} \begin{pmatrix} |B^0\rangle \\ |\bar{B}^0\rangle \end{pmatrix}$$

$\boxed{\text{for } B_s}$ top dominates $V_{ts} \sim \lambda^2$ instead of $V_{td} \sim \lambda^3$

so $\frac{\Delta M_{B_s}}{\Gamma_{B_s}} \sim 17$ fast oscillations

[measured recently]

$\boxed{\text{for } K}$ $\Delta\Gamma_K \sim \Gamma_K \sim 2\Delta M_K$ $\tau_S \sim .9 \times 10^{-10} s$

$\tau_L \sim 5 \times 10^{-8} s$

$\boxed{\text{for } D}$ $\frac{\Delta\Gamma}{\Gamma} \sim 5 \times 10^{-3}$, $\frac{\Delta M}{\Gamma} \lesssim 10^{-3}$

[measured at 3σ recently]

Back to 3 types of $\cancel{CP}$

① $\cancel{CP}$ in decay $B \to f$, $\bar{B} \to \bar{f}$ $\left|\frac{A_f}{\bar{A}_{\bar{f}}}\right| \neq 1$

$$A_f = \langle f | H | B \rangle = \sum_K A_K \underbrace{e^{i\delta_K}}_{\text{strong phase}} \underbrace{e^{i\phi_K}}_{\text{weak phase}}$$

$$\bar{A}_{\bar{f}} = \langle \bar{f} | H | \bar{B} \rangle = \sum_K \underbrace{A_K e^{i\delta_K}}_{CP \text{ even}} \underbrace{e^{-i\phi_K}}_{\text{sign change from } CP}$$

for real $\mathcal{L}$ phases can appear due to rescattering through intermediate states into desired final state, strong interactions dominate $e^{i\delta_K}$.

measure asymmetry

$$A_{CP} = \frac{\Gamma(B \to f) - \Gamma(\bar{B} \to \bar{f})}{\text{"} \quad + \quad \text{"}} = \frac{|A|^2 - |\bar{A}|^2}{|A|^2 + |\bar{A}|^2}$$

here

$$|A|^2 - |\bar{A}|^2 = -2 \sum_{i,j} A_i A_j \sin(\phi_i - \phi_j) \sin(\delta_i - \delta_j)$$

$\Rightarrow$ require two interfering amplitudes with different weak phases & different strong phases

e.g. replace $d \to s$ in earlier $B \to \pi\pi$ graphs

$\bar{B}^0 \to K^- \pi^+$ $\qquad A_{CP} = -.097 \pm .012$

2) $\cancel{CP}$ in mixing $\qquad \left|\frac{q}{p}\right| \neq 1$ $\qquad$ mass & CP eigenstates differ

take semileptonic asymmetry

$$A_{sl} = \frac{\Gamma(\bar{B}^0(t) \to \ell^+ \nu X) - \Gamma(B^0(t) \to \ell^- \nu X)}{\text{"} \quad + \quad \text{"}}$$

use $|A(\ell^- X)| = |\bar{A}(\ell^+ X)|$

wrong sign lepton allowed due to osc!

$$= \frac{1 - |q/p|^4}{1 + |q/p|^4}$$

for B gives $\quad \left|\frac{q}{p}\right| = 1.0015 \pm .0039$

③ $\cancel{CP}$ in Interference

take $f = f_{CP}$ (eg $\pi^+\pi^-$)

$$\lambda = \frac{q}{p}\frac{\bar{A}_f}{A_f} = \eta_{CP}\frac{q}{p}\frac{\bar{A}_{\bar{f}}}{A_f}$$

eigenvalue $\eta_{CP} = \pm 1$

even if $|\lambda| = 1$ we can still have $\cancel{CP}$ from $\mathrm{Im}\,\lambda \neq 0$

$$\begin{tikzcd} B^0 \arrow[rr] \arrow[dr] & & f_{CP} \\ & \bar{B}^0 \arrow[ur] & \end{tikzcd}$$

$$a_{CP}(t) = \frac{\Gamma(\bar{B}^0(t) \to f) - \Gamma(B^0(t) \to f)}{\text{"} \quad + \quad \text{"}}$$

$$= -\frac{(1-|\lambda|^2)}{(1+|\lambda|^2)}\cos(\Delta m t) + \frac{2\,\mathrm{Im}\,\lambda}{1+|\lambda|^2}\sin(\Delta m t)$$

$$\equiv -C_f \cos(\Delta m t) + S_f \sin(\Delta m t)$$

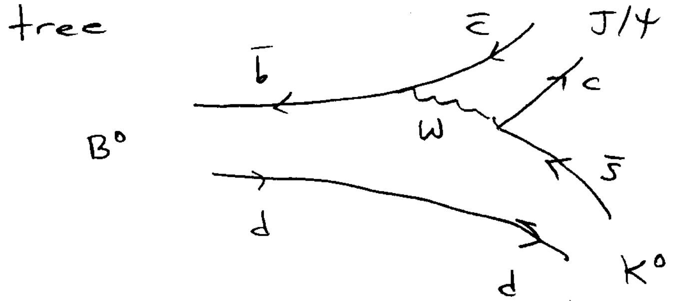
For B-decay if a single weak phase dominates (eg. other's CKM suppressed) then $\left|\frac{\bar{A}}{A}\right| = 1$ & $|\lambda| = 1$

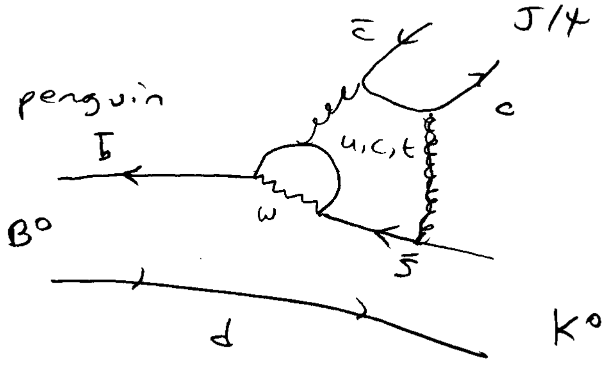
$$a_{CP}(t) = \mathrm{Im}\,\lambda \,\sin(\Delta m t)$$

cleanly extract $\mathrm{Im}\,\lambda_f$

eg $\sin(2\beta)$ from $B \to \underbrace{J/\psi\, K_S}_{\text{CP eigenstate}}$ (clean)

$b \to c\bar{c}\, s$





interference is possible despite $B^0 \to J/\psi\, K^0$

$\bar{B}^0 \to J/\psi\, \bar{K}^0$

due to $K\text{-}\bar{K}$ mixing

$$\left(\frac{p}{q}\right)_K = \frac{V_{cs} V_{cd}^*}{V_{cs}^* V_{cd}}$$

dominated by charm, again $|p/q| \simeq 1$

graphs

$\bar{A}_T = V_{cb} V_{cs}^* \quad T_{c\bar{c}s}$ tree

$\bar{A}_P = V_{tb} V_{ts}^* \; P_t + V_{cb} V_{cs}^* \; P_c + V_{ub} V_{us}^* \; P_u$ penguin

total

$$\bar{A} = V_{cb} V_{cs}^* \underbrace{(T_{c\bar{c}s} + P_c - P_t)}_{T} + V_{ub} V_{us}^* \underbrace{(P_u - P_t)}_{P}$$

$$\left|\frac{\overset{\lambda^3}{V_{ub}}\overset{\lambda}{V_{us}^*}}{\underset{\lambda^2}{V_{cb}}\underset{1}{V_{cs}^*}}\right| \sim \frac{1}{50}$$

so single weak phase dominates $\left|\frac{\bar{A}}{A}\right| - 1 \lesssim 10^{-2}$

effective

$$\lambda = \left(\frac{q}{p}\right)_B \frac{\bar{A}_{\bar{f}}}{A_f} \left(\frac{p}{q}\right)_K = -\left(\frac{V_{tb}^* V_{td}}{V_{tb} V_{td}^*}\right)\left(\frac{V_{cb} \cancel{V_{cs}^*}}{V_{cb}^* \cancel{V_{cs}}}\right)\left(\frac{\cancel{V_{cs}} V_{cd}^*}{\cancel{V_{cs}^*} V_{cd}}\right)$$

$$= -\frac{e^{-i\beta}}{e^{i\beta}} = -e^{-2i\beta}$$

$$\mathrm{Im}\,\lambda = \sin(2\beta) = 0.680 \pm 0.025$$

$\Sigma m_i < .3\,\text{eV}$ cosmology
$m_i < 2\,\text{eV}$ β-decay

Neutrino Masses & Mixing

we know neutrino's have mass small $\rightarrow$ $\Delta M^2_{atm} \sim 2.4\times 10^{-3}$, $\Delta m^2_{\odot} \sim 7.9\times 10^{-5}\,\text{eV}^2$

Neutrino in (old) Standard Model is a <u>massless</u> Weyl fermion

ν_L left-handed, two-component, $P_L\,\nu_L = \nu_L$

annih. L-handed neutrino, create R-handed $\bar\nu$

under CP $\quad \nu_L(\vec{x},t) \rightarrow \gamma^0\, \nu_R^c(-\vec{x},t)$

where $\nu_R^c \equiv C\,\overline{\nu_L}^T = C\,\gamma_0^T\,(\nu_L^+)^T$

$C = i\gamma^2\gamma^0$ in Pauli-Dirac Rep

ν_R^c ann. R-handed $\bar\nu$, creates L-handed ν

also $\nu_L = C\,\overline{\nu_R^c}^T$

To generate a Dirac mass term we need R-handed ν, "N_R", however we know it must be <u>sterile</u> since the number of <u>active</u> neutrinos is 3 (Z-width).

<u>Dirac Mass</u> (1 family to begin)

$$\mathcal{L}_D = -m_D\left(\bar\nu_L N_R + \overline{N_R}\,\nu_L\right) = -m_D\,\bar\nu\,\nu$$

for $\nu \equiv \nu_L + N_R$ Dirac field

generate this from Higgs v.e.v.

$$\mathcal{L}_{\nu_R} = \overline{N_R}\, i\partial\!\!\!/\, N_R + g_\nu\, \overline{N_R}\, H^T \epsilon\, L_L + \text{h.c.}$$

sterile, ie no gauge couplings

$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\quad L_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$

similar to quarks

Majorana Mass

sterile neutrino can have another type of mass

$$\mathcal{L}_{maj} = -\frac{m_S}{2}\left(\overline{N_L^c}\, N_R + \overline{N_R}\, N_L^c\right)$$

N_R N_L^c or N_R N_R

$$= -\frac{m_S}{2}\ \overline{N} N$$

majorana field

$N \equiv N_L + N_R^c$

so $N = N^c$

think of $m_S\, \epsilon^{ab} N_R^a N_R^b$ + h.c.

- if N_R had a $U(1)$ charge then $\mathcal{L}_{maj}$ violates it by 2 units (so no $\mathcal{L}_{maj}$ for charged particles).

- not useful by itself, we need mass for active ν_L.

Can get $\mathcal{L}_{maj}$ for ν_L using Higgs triplet under $SU(2)_L$

$2 \otimes 2 = 1 \oplus 3$

has neutral component , $g_T\, \vec{H} \cdot \left(\overline{L_L^c}\, \vec{\sigma}\, L_L\right)$

constrained by coupling to other fields

Seesaw Mechanism

$$\mathcal{L} = -\frac{1}{2}\left(\overline{\nu_L}\ \overline{N_L^c}\right)\begin{pmatrix} 0 & m_D \\ m_D & m_S \end{pmatrix}\begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix} + \text{h.c.}$$

Let $m_D \sim \mathcal{O}(m_e)$, $m_S \sim \mathcal{O}(10^{14}\ \text{GeV})$

$\sim g_D v$

<u>eigenstate</u> <u>eigenvalue</u>

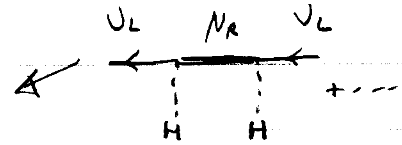
$\nu_1 \simeq \frac{m_D}{m_S}\nu_L + N_L^c$ $\qquad m_1 \simeq m_S$ heavy

$\nu_2 \simeq \left(1 + \frac{m_D^2}{m_S^2}\right)\nu_L - \frac{m_D}{m_S} N_L^c$ $\qquad m_2 \simeq \frac{m_D^2}{m_S}$ Maj. mass term.

- Naturally light ν_L with heavy (unobserved) N_R.

- can forget about N_R and just add higher dim-op to S.M.

$$\mathcal{L}_{dim5} = \frac{g_5}{\Lambda} \left(\overline{L}_L^c \, \epsilon \, H\right)\left(H^T \epsilon \, L_L\right)$$

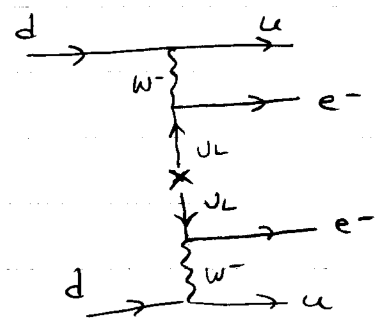


which gives $M_{\nu_L}^{maj} \simeq \dfrac{g_5 v^2}{\Lambda}$ when H gets v.e.v.

This is unique dim-5 operator that is consistent with gauge symmetry of S.M.

Its difficult to distinguish Majorana vs. Dirac, one-way is ν-less dble β-decay

$nn \to pp \, e^- e^-$



<u>More Families</u>

$$\mathcal{L}_{\nu_R} = \overline{N}_R \, i \not\partial N_R + g_\nu^{ij} \, \overline{N}_R^i H^T \epsilon \, L_L^j + h.c.$$
$$- \frac{1}{2} \overline{N}_R^c \underset{\substack{\uparrow \\ \text{matrix}}}{M_S} N_R + h.c.$$

$M_S = M_S^T$

$\underbrace{\text{number of } N_R\text{'s}}_{n_s}$ need not equal $\underbrace{\# \text{ of } L_L\text{'s}}_{3}$

$$\mathcal{L}_{\nu_R}^{mass} = -\frac{1}{2} \left(\bar{\nu}_L \ \overline{N_L^c} \right) \begin{pmatrix} 0 & m_D \\ m_D^T & m_S \end{pmatrix} \begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix} + h.c.$$

$\uparrow \mathcal{M}$ symmetric $\qquad m_D = \frac{v}{\sqrt{2}} g_\nu$

diagonalize

$$\begin{pmatrix} \nu_L \\ N_L^c \end{pmatrix} \to U_L^{\nu\dagger} \begin{pmatrix} \nu_L \\ N_L^c \end{pmatrix}, \quad \begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix} \to U_R^{\nu\dagger} \begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix}$$

$\uparrow$ $(3+n_S) \times (3+n_S)$, unitary

$(U_R^\nu) = (U_L^\nu)^*$

$$\mathcal{M} = U_R^{\nu T} \, D \, U_R^\nu$$

$\uparrow$ diagonal

Let $U_L^\nu = \begin{pmatrix} U_{11} & \delta U_{12} \\ \delta U_{21} & U_{22} \end{pmatrix}$ where δ denotes suppression by "m_D/m_S"

$$D = \begin{pmatrix} \delta^2 D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

W-couplings

$$\mathcal{L} = \frac{g}{\sqrt{2}} W_\mu^+ \, \bar{e}_L \, (W_e^\dagger)(U_{11}^\dagger \nu_L + \underbrace{\delta U_{21}^\dagger N_L^c}_{\text{suppressed}}) + h.c.$$

also $\quad U_{11} U_{11}^\dagger + \delta^2 U_{12} U_{21}^\dagger = 1 \quad$ so $\quad U_{11} \simeq$ unitary

3×3: $$W_e^\dagger U_{11}^\dagger = \underbrace{V_{MNS}}_{\substack{\text{analog of CKM,} \\ \text{3 angles} \\ \text{1 phase}}} \underbrace{\begin{pmatrix} e^{i\beta_1} & & 0 \\ & e^{i\beta_2} & \\ 0 & & 1 \end{pmatrix}}_{\text{Majorana phases } \cancel{CP}}$$

$\leftarrow$ these phases can't be removed because of Majorana mass term. Can't rephase ν's, $6-3 = 3$ phases

η- Phenomenology → see reading material

Ch 5. Anomalies

Classically we used Noether's procedure to relate symmetries to conserved currents $\partial^\mu J_\mu = 0$

eg $\psi \to e^{-i\theta}\psi \qquad J^\mu = \bar\psi \gamma^\mu \psi$

$\psi \to e^{-i\theta\gamma_5}\psi \qquad J^{\mu 5} = \bar\psi \gamma^\mu \gamma_5 \psi$

However Quantum corr. can <u>destroy</u> classical symmetries → not symmetries of eff. action $\Gamma[\bar\phi]$

<u>Noether:</u> $\phi \to \phi' = \phi + \epsilon(x)\,\Delta\phi(x)$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\Delta\phi(x)$$

$$\frac{\delta}{\delta\epsilon(x)} \int d^4x\ \mathcal{L}(\phi + \epsilon\Delta\phi) = -\partial_\mu J^\mu$$

so $\mathcal{L}(\phi) = \mathcal{L}(\phi + \epsilon\Delta\phi)$ gives $\partial_\mu J^\mu = 0$

<u>Quantum:</u>

$$\langle 0|\, T\, \phi(x_0) \cdots \phi(x_n)\, |0\rangle_{conn} = (-i)^k \frac{\delta^k}{\delta j(x_n) \cdots \delta j(x_0)} \ln Z[j] \Big|_{j=0}$$

$$Z[j] = \int [d\phi]\, \exp i \int d^4x\, (\mathcal{L} + j\phi)$$

$$\Gamma[\bar\phi] = \ln Z[j] - \int d^4x\ j(x)\,\bar\phi(x)$$

$$j(x) = -\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi(x)}, \qquad \bar\phi(x) = \frac{\delta \ln Z[j]}{\delta j(x)}$$

if $\int d^4x \, \mathcal{L}' = \int d^4x \, \mathcal{L}$ <u>and</u> $[d\phi] = [d\phi']$

imagine unitary, so $\det \mathcal{U} = 1$

$$Z[j] = \int [d\phi'] \exp i \int d^4x \, (\mathcal{L}' + j\phi')$$
$$= \int [d\phi] \exp i \int d^4x \, \left(\mathcal{L} + j\phi + j(x)\,\epsilon(x)\,\Delta\phi(x)\right)$$

to 1st order

$$\int d^4x \int [d\phi] \; j(x)\,\epsilon(x)\,\Delta\phi(x) \; e^{i\int d^4x\,(\mathcal{L}+j\phi)} = 0$$
$$\int d^4x \; j(x) \; \langle \epsilon \Delta\phi \rangle_j = 0$$
$$\int d^4x \; \frac{\delta \Gamma[\bar\phi]}{\delta \bar\phi(x)} \; \langle \epsilon \Delta\phi \rangle_{j_{\bar\phi}} = 0$$

so $\Gamma[\bar\phi]$ invariant under $\bar\phi \to \bar\phi + \langle \epsilon \Delta \phi \rangle$

and if $\Delta\phi$ is linear in ϕ, $\langle\phi\rangle = \bar\phi$, so

$\bar\phi \to \bar\phi + \epsilon \Delta \bar\phi$, same as classical symmetry

L17

Consider matrix elts with symmetry current by adding source for J^μ

$$Z[v, j] = \int [d\phi] \exp\left(i \int d^4x \, (\mathcal{L} + j\phi + v_\mu J^\mu)\right)$$

functional $\mathcal{J}^\mu(x) \equiv -i \dfrac{\delta}{\delta v_\mu(x)} \ln Z[v, j] \Big|_{v=0}$ describes

$\langle T \, J^\mu(x) \, \phi(x_1) \cdots \phi(x_n) \rangle$ through $\dfrac{\delta^n}{\delta j(x_1) \cdots \delta j(x_n)} \mathcal{J}^\mu(x)[j]$

inverse: $\delta \ln Z[v_\mu] = \ln Z[v_\mu + \delta v_\mu] - \ln Z[v_\mu]$

$$= i \int d^4x \; \mathcal{J}^\mu(x) \, \delta v_\mu(x)$$

let $\delta v_\mu = \partial_\mu \epsilon$ int. by parts.

$$\ln Z[v_\mu + \partial_\mu \epsilon] = \ln Z[v_\mu] - i \int d^4x \; \epsilon(x) \, \partial_\mu \mathcal{J}^\mu(x)$$

if $\quad Z[v_\mu + \partial_\mu \epsilon] = Z[v_\mu] \quad$ then $\quad \partial_\mu \bar{J}^\mu = 0$

conserved

hence $\quad \partial_\mu \langle T\, J^\mu(x)\, \phi(x_1) \cdots \phi(x_n) \rangle = 0$

$$0 = \langle T\, \partial_\mu J^\mu(x)\, \phi \cdots \phi \rangle - \underbrace{i\, \delta(x_i - x_i) \, \langle \cdots \rangle \text{ terms}}_{\text{from } \partial_\mu \theta(x - x_i).}$$

Won't be needed in our computation of on-shell graphs

[see pg. 255 of Pokorski for a more detailed derivation]

<u>However</u> if $[d\phi] \neq [d\phi']$ then classical symmetry is ruined

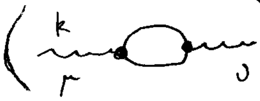
For axial trnsfm's with chiral fermions we'll find this is often the case, and then

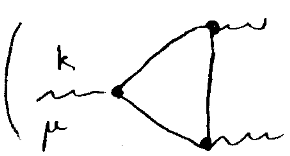
$\partial_\mu \bar{J}_\mu \neq 0 \quad , \quad \partial_\mu J^\mu \neq 0 \qquad J_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi$

$[d\phi] \to [d\psi][d\bar{\psi}]$

Study this with Feyn. Graphs.

The integration $[d\psi][d\bar{\psi}]$ should play a role, so we want fermion loop

$\langle 0 | T\, \partial^\mu J_{\mu 5}(x)\, J_\nu(y) | 0 \rangle \quad$ is $\quad k^\mu$ ()

$\langle 0 | T\, \partial^\mu J_{\mu 5}(x)\, J_\nu(y)\, J_\lambda(z) | 0 \rangle \quad$ is $\quad k^\mu$ ()

Simple Example 2-d massless QED , Schwinger Model

$iD_\mu = i\partial_\mu - eA_\mu$, $[A]=0$

$$\mathcal{L} = \bar\psi i \not{D} \psi - \frac{1}{4}(F_{\mu\nu})^2$$

$\mu,\nu = 0,1$

$[e] = 1$

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(Pauli Matrices)}$$

γ_5 eigenstates

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \qquad D_\pm \equiv D_0 \pm D_1$$

$$\mathcal{L} = \underbrace{\psi_+^\dagger iD_+ \psi_+} + \underbrace{\psi_-^\dagger iD_- \psi_-}$$

free $(\partial_0 + \partial_1)\psi_+ = 0$ left-moving

right-moving $\psi_-(t+x)$

$\psi_+(t-x)$

Currents

$J^\mu = \bar\psi \gamma^\mu \psi$ $\qquad N = \int dx\, J^0 = N_+ + N_-$

$J^{\mu 5} = \bar\psi \gamma^\mu \gamma_5 \psi$ $\qquad N_5 = \int dx\, J^{05} = N_+ - N_-$

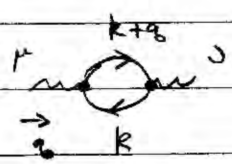
classically conserved $\qquad N_\pm = \int dx\, \psi_\pm^\dagger \psi_\pm$

$= \#$ right/left movers

Currents related

$$\gamma^\mu\gamma_5 = -\epsilon^{\mu\nu}\gamma_\nu , \quad \epsilon^{01} = +1$$

so $\quad J^{\mu 5} = -\epsilon^{\mu\nu} J_\nu$ $\qquad$ look at vector



$$(-ie)^2 \langle 0 | T J_\mu(x) J_\nu(0) | 0 \rangle = i \int \frac{d^2q}{(2\pi)^2} e^{-iq\cdot x}\, \Pi_{\mu\nu}(q)$$

$$i\Pi_{\mu\nu}(q) = -(-ie)^2 \int d^dk\, \mathrm{tr}\left[\gamma^\mu \frac{i\not{k}}{k^2} \gamma^\nu \frac{i(\not{k}+\not{q})}{(k+q)^2} \right]$$

$d = 2$ log divergent

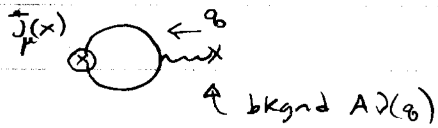
$$i\Pi_{\mu\nu}(q) = \underbrace{\frac{-4ie^2}{(4\pi)^{d/2}}}_{} \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \int_0^1 dx \; x(1-x) \frac{\Gamma(2-d/2)}{\left[-x(1-x)q^2 - i0\right]^{2-d/2}}$$

$\uparrow$ $tr[1] = 2$

$$= i \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \frac{e^2}{\pi}$$

$\uparrow$ photon mass term $M_\gamma^2 = \frac{e^2}{\pi}$ [exact, Schwinger]

$$q^\mu \, \Pi_{\mu\nu}(q) = 0$$



$$= \int d^2x \; e^{iq\cdot x} \langle J^\mu(x) \rangle = \frac{i}{e}\left(i\Pi^{\mu\nu}(q)\right) A_\nu(q)$$

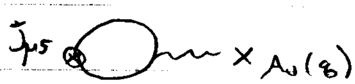
$$\partial^\mu J_\mu(x) = 0 \qquad \text{conserved}$$

<u>Now</u>

$$\Pi_{\mu 5 \nu}(q) = -\epsilon_{\mu\mu'} \Pi^{\mu'}{}_\nu(q)$$

$$q^\mu \Pi_{\mu 5\nu} = -i q^\mu \epsilon_{\mu\mu'} \left(g^{\mu'}{}_\nu - \frac{q^{\mu'} q_\nu}{q^2}\right) \frac{e^2}{\pi}$$

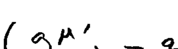
$$= \frac{-ie^2}{\pi} q^\mu \epsilon_{\mu\nu} \neq 0 \qquad \partial^\mu J_{\mu 5} \neq 0$$



$$\int d^2x \; e^{iq\cdot x} \langle \partial^\mu J_{\mu 5}(x) \rangle = \frac{e}{\pi} q^\mu \epsilon_{\mu\nu} A^\nu(q)$$

$$\langle \partial^\mu J_{\mu 5}(x) \rangle = \frac{e}{2\pi} \langle \epsilon^{\mu\nu} F_{\mu\nu} \rangle$$

$$\partial^\mu J_{\mu 5}(x) = \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} \qquad \text{not conserved}$$

Here dim. reg. regulated  so as to preserve gauge invariance, but $\partial^\mu J_{\mu 5} \neq 0$.
If $i\Pi^{\mu\nu} = A g^{\mu\nu} - B \frac{q^\mu q^\nu}{q^2}$ then some regulator which sets $A = 0$ would have $\partial^\mu J_{\mu 5} = 0$ <u>but</u> $\partial^\mu J_\mu \neq 0$
(not what we want.)

Conservation Law?

$$\int_0^T dt \int dx \; \partial_\mu J^{\mu 5} = \Delta(N_+ - N_-) = \frac{e}{\pi} \int dt\, dx \; \partial_\mu \left(\epsilon^{\mu\nu} A_\nu\right)$$

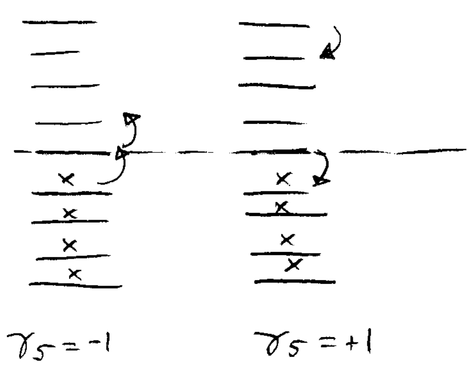
non-zero

Consider A_1 constant in space in box of length L (periodic b.c.)

eigenstates of Hamiltonian $e^{ik_n x}$, $k_n = \frac{2\pi n}{L}$ $n \in \mathbb{Z}$

Ψ_+ : $E_n = (k_n - eA^1)$

Ψ_- : $E_n = -(k_n - eA^1)$



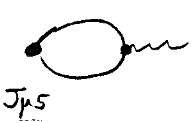
adiabatically change

$$\Delta A^1 = \frac{2\pi}{eL}$$

vacuum looses + fermion, but gains − fermion

$$\Delta(N_+ - N_-) = -2 = \int dt\, dx \; \frac{e}{\pi} \partial_0 A_1$$

<u>QED in $d=4$</u>

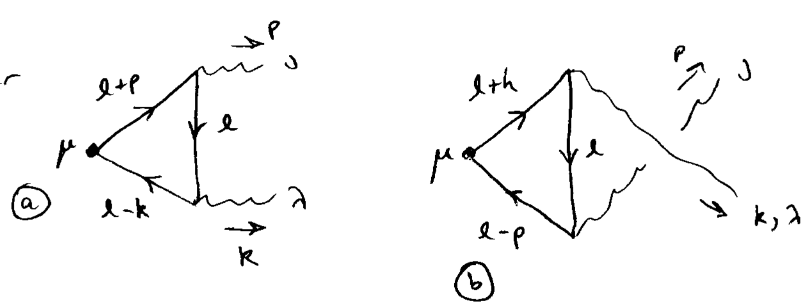
Now 

$= 0$.

$\mathrm{tr}\left[\gamma^\mu \gamma_5 \gamma^\alpha \gamma^\beta \gamma^\nu\right] \propto \epsilon^{\mu\nu\alpha\beta}$

but no way to contract to get $q^\mu \Pi_{\mu 5 \nu} \neq 0$

Consider



$$ⓐ = (-1)(-ie)^2 \int d^4 l \ \mathrm{tr}\left[\gamma^\mu \gamma_5 \frac{i(\not{l}-\not{k})}{(l-k)^2} \gamma^\lambda \frac{i\not{l}}{l^2} \gamma^\nu \frac{i(\not{l}+\not{p})}{(l+p)^2} \right]$$

$$ⓑ = ⓐ\big((p,\nu) \leftrightarrow (k,\lambda) \big)$$

$$ⓐ + ⓑ = T^{\mu\nu\lambda}(k,p) = T_a^{\mu\nu\lambda} + T_b^{\mu\nu\lambda} , \quad q = k+p$$

<u>A Too Formal</u> <u>Calc</u>

$$q_\mu T^{\mu\nu\lambda}(k,p) = 0 \quad ?$$

$$\not{q}\gamma_5 = (\not{l}+\not{p})\gamma_5 + \gamma^5(\not{l}-\not{k}) \qquad \text{using } \{\gamma_5, \gamma^\mu\} = 0$$

$$q_\mu T_a^{\mu\nu\lambda} = -ie^2 \int d^4 l \ \mathrm{tr}\left[\gamma_5 \frac{(\not{l}-\not{k})}{(l-k)^2} \gamma^\lambda \frac{\not{l}}{l^2} \gamma^\nu + \gamma^5 \gamma^\lambda \frac{\not{l}}{l^2} \gamma^\nu \frac{(\not{l}+\not{p})}{(l+p)^2} \right]$$

$\uparrow$ shift $l \to l+k$

$$= -ie^2 \int d^4 l \ \mathrm{tr}\left[\gamma_5 \frac{\not{l}}{l^2} \gamma^\lambda \frac{(\not{l}+\not{k})}{(l+k)^2} \gamma^\nu - \gamma^5 \frac{\not{l}}{l^2} \gamma^\nu \frac{(\not{l}+\not{p})}{(l+p)^2} \gamma^\lambda \right]$$

manifestly antisymm under $(p,\nu) \leftrightarrow (k,\lambda)$

so $q^\mu (T_a + T_b)_{\mu\nu\lambda} = 0$

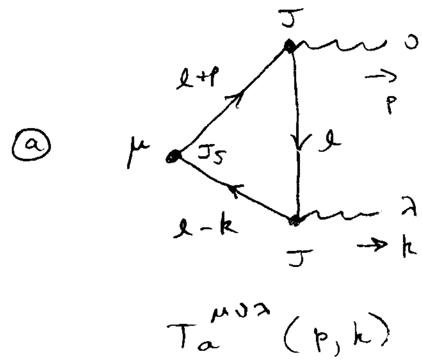
<u>But</u>

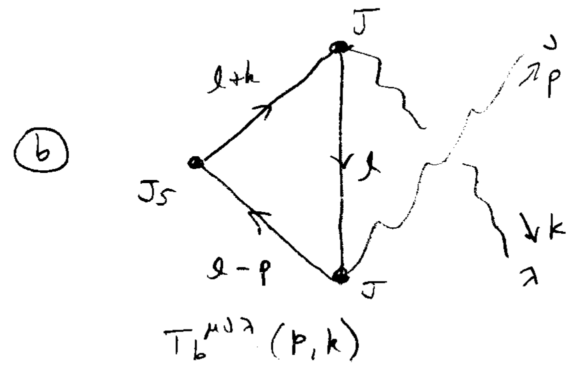
- integral is linearly divergent, shift is apriori not well defined
- in dim. reg. it is, but then $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is inherently 4-dim and we must be careful with use of $\{\gamma_5, \gamma^\mu\} = 0$. If $d = 4-2\epsilon$ then γ_5 commutes with the γ^μ's in the (-2ϵ) dimensions. (HV-scheme for γ_5)

see Peskin

Massless QED, d=4   Adler - Bell - Jackiw anomaly

recall





$T_b^{\mu\nu\lambda}(p,k) = T_a^{\mu\lambda\nu}(k,p)$ bose symmetry for photons

$T^{\mu\nu\lambda}(p,k) \equiv T_a^{\mu\nu\lambda} + T_b^{\mu\nu\lambda}$

$q = p+k$

$q_\mu T^{\mu\nu\lambda} = 0$?   axial ward identity

$p_\nu T^{\mu\nu\lambda} = 0$?   vector ward identity

$k_\lambda T^{\mu\nu\lambda} = 0$

$$T^{\mu\nu\lambda}(p,k) = -ie^2 \int d^4 l \, \frac{tr\left[\gamma^\mu\gamma_5 (\not{l}-\not{k})\gamma^\lambda \not{l} \gamma^\nu (\not{l}+\not{p})\right]}{(l-k)^2 (l^2)(l+p)^2} + \text{symm. term}$$

- linear divergent, careful about shifting l
- $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, careful about d-dimensions.

eg. Consider 1-dim

$$\Delta(a) = \int_{-\infty}^{\infty} dx \left[f(x+a) - f(x)\right] \qquad \text{zero if we can shift}$$

$$= \int dx \left[a f'(x) + \frac{a^2}{2} f''(x) + \cdots\right]$$

$$= a f\Big|_{-\infty}^{\infty} + \underbrace{\frac{a^2}{2} f'\Big|_{-\infty}^{\infty}}_{0} + \cdots$$

linear div f is const at $\pm\infty$

$$= a\left[f(\infty) - f(-\infty)\right]$$

we just need to look at linear divergent behavior to evaluate $\Delta(a)$

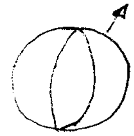
4-dim

$$\Delta(a) = \int d^4\ell \, [\, F(\ell + a) - F(\ell)\,]$$

$$\overset{\text{Eucl.}}{=} \quad i \int d^4\ell_E \, [\, a^\mu \partial_\mu F(\ell) + \cdots \,]$$

$$= i \int_{|\ell_E| \to \infty} d^3S_\mu \, (a^\mu F(\ell)) \qquad \text{Gauss' Law} \qquad \ell = |\ell_E|$$

$$= i\, a^\mu \int_{\ell \to \infty} d^3S \, \frac{\ell_\mu}{\ell} F(\ell) \qquad \text{drop } E$$



normal vector $d^3S_\mu = \frac{\ell^\mu}{\ell} d^3S$

ℓ^μ on surface

A linear divergent $F^\alpha(\ell) = \dfrac{\ell^\alpha}{\ell^4}$, $\ell^\mu \ell^\alpha = \dfrac{\ell^2}{4} g_E^{\mu\alpha}$

$$\Delta(a) = i\, a^\alpha \int_{\ell \to \infty} d^3S \, \frac{1}{4\ell^3} = i\, a^\alpha \lim_{\ell \to \infty} (2\pi^2 \ell^3) \frac{1}{4\ell^3}$$

$$\boxed{\Delta(a) = \frac{i\, a^\alpha \pi^2}{2}} \qquad (*) \qquad [\text{a regulating procedure}] \text{ due to limit/sphere}$$

Lets use this to evaluate <u>vector</u> ward identity

$$p_\nu T^{\mu\nu\lambda} = -ie^2 p_\nu \int d^4\ell \; \mathrm{tr}\left[\frac{\gamma^\mu \gamma_5 (\not\ell - \not k)\gamma^\lambda \not\ell \gamma^\nu (\not\ell + \not p)}{(\ell-k)^2 \ell^2 (\ell+p)^2} + \frac{\gamma^\mu \gamma_5 (\not\ell - \not p)\gamma^\nu \not\ell \gamma^\lambda (\not\ell + \not k)}{(\ell-p)^2 \ell^2 (\ell+k)^2} \right]$$

use $\not p = (\not\ell + \not p) - (\not\ell)$ $\qquad$ $\not p = \not\ell - (\not\ell - \not p)$

$$= -ie^2 \int d^4\ell \; \mathrm{tr}\left[\frac{\gamma^\mu \gamma_5 (\not\ell - \not k)\gamma^\lambda}{(\ell-k)^2} \left(\frac{\not\ell}{\ell^2} - \frac{(\not\ell + \not p)}{(\ell+p)^2} \right) + \gamma^\mu \gamma_5 \left(\frac{(\not\ell - \not p)}{(\ell-p)^2} - \frac{\not\ell}{\ell^2} \right) \frac{\gamma^\lambda (\not\ell + \not k)}{(\ell+k)^2} \right]$$

use $\mathrm{tr}[\gamma_5 \gamma^\sigma \gamma^\lambda \gamma^\tau \gamma^\mu] = -4i\,\epsilon^{\sigma\lambda\tau\mu}$

$$= -\frac{4e^2}{(2\pi)^4} \epsilon^{\mu\lambda\sigma\tau} \int d^4\ell \left[\underbrace{\frac{(\ell-k)^\sigma \ell^\tau}{(\ell-k)^2 \ell^2}}_{H(\ell-k)} - \underbrace{\frac{(\ell-k)^\sigma (\ell+p)^\tau}{(\ell-k)^2 (\ell+p)^2}}_{G(\ell)} + \underbrace{\frac{(\ell-p)^\sigma (\ell+k)^\tau}{(\ell-p)^2 (\ell+k)^2}}_{G(\ell-p+k)} - \underbrace{\frac{\ell^\sigma (\ell+k)^\tau}{\ell^2 (\ell+k)^2}}_{H(\ell)} \right]$$

H: $\quad a^\alpha = (-k)^\alpha$ $\qquad$ use $(*)$

G: $\quad a^\alpha = (k-p)^\alpha$

large l

$$H(l) = -\frac{l^\sigma \cancel{l^\tau}}{l^4} - \frac{l^\sigma k^\tau}{l^4} + \text{more } \cancel{l^\sigma l^\tau \text{ terms}} + \text{convergent}$$

0 due to $\epsilon^{\mu\lambda\sigma\tau}$ 0

$$G(l) = 0 - \frac{l^\sigma p^\tau + k^\sigma l^\tau}{l^4} = (-g^{\sigma\alpha} p^\tau + g^{\tau\alpha} k^\sigma)\frac{l^\alpha}{l^4}$$

under $\int d^4 l$ take $\frac{l^\alpha}{l^4} \rightarrow \frac{i\pi^2}{2} a^\alpha$

$$p_\nu T^{\mu\nu\lambda} = -\frac{4e^2}{(2\pi)^4}\epsilon^{\mu\lambda\sigma\tau}\frac{i\pi^2}{2}\Big[\underbrace{\cancel{(-k^\tau)(-k^\sigma)}}_{0} + (-g^{\sigma\alpha}p^\tau + g^{\tau\alpha}k^\sigma)(k_\alpha - p_\alpha)\Big]$$

$$= \frac{-ie^2}{8\pi^2}\epsilon^{\mu\lambda\sigma\tau}[-k^\sigma p^\tau - k^\sigma p^\tau]$$

$$= \frac{ie^2}{4\pi^2}\epsilon^{\mu\lambda\sigma\tau} k_\sigma p_\tau$$

Axial

$(p+k)_\mu T^{\mu\nu\lambda}$, $(\not p + \not k)\gamma_5 = (\not p + \not l)\gamma_5 + \gamma_5(\not l - \not k)$

$(\not p + \not k)\gamma_5 = (\not l + \not k)\gamma_5 + \gamma_5(\not l - \not p)$

$$(p+k)_\mu T^{\mu\nu\lambda} = \frac{-4e^2}{(2\pi)^4}\epsilon^{\nu\lambda\sigma\tau}\int d^4 l\Big[\underbrace{\frac{(l-k)^\sigma l^\tau}{(l-k)^2 l^2}}_{H(l-k)} - \underbrace{\frac{(l+p)^\sigma l^\tau}{(l+p)^2 l^2}}_{I(l)} + \underbrace{\frac{l^\sigma (l-p)^\tau}{l^2(l-p)^2}}_{I(l-p)} - \underbrace{\frac{l^\sigma (l+k)^\tau}{l^2(l+k)^2}}_{H(l)}\Big]$$

H: $a^\alpha = (-k)^\alpha$ $H(l) = -k^\tau l^\sigma / l^4$

I: $a^\alpha = (-p)^\alpha$ $I(l) = -p^\sigma l^\tau / l^4$

$$(p+k)_\mu T^{\mu\nu\lambda} = \frac{-4e^2}{(2\pi)^4}\epsilon^{\nu\lambda\sigma\tau}\frac{i\pi^2}{2}\Big[\cancel{(-k^\tau)(-k^\sigma)} + \cancel{(-p^\sigma)(-p^\tau)}\Big]$$

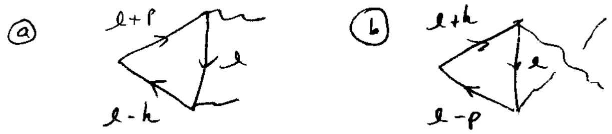
$$= 0$$

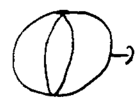
$p_\nu T^{\mu\nu\lambda} \neq 0$ vector broken

$(p+k)_\mu T^{\mu\nu\lambda} = 0$ axial conserved

What's going on?

Integral was linearly divergent, and we regulated in a way so that axial W.I. is conserved. However, we started with a particular choice for momentum routing





sphere was not shift symmetric

and any other choice is equally good (and gives a different regulator). An infinite family of choices is

$l \to l + b$ in T_a $\qquad b^\mu = b_1 k^\mu + b_2 p^\mu$

$l \to l + \tilde{b}$ in T_b $\qquad \tilde{b}^\mu = b_1 p^\mu + b_2 k^\mu$

where we fixed $\tilde{b}$ so $T_b^{\mu\nu\lambda}(p,k) = T_a^{\mu\lambda\nu}(k,p)$ still.

Now (b_1, b_2) regulator gives

$$p_\nu T^{\mu\nu\lambda} = \frac{-4e^2}{(2\pi)^4}\,\epsilon^{\mu\lambda\sigma\tau}\int d^4l\,\Big[\underbrace{H(l-k+b) - G(l+b)}_{a^\alpha = -k+b-\tilde{b}} + \underbrace{G(l-p+k+\tilde{b}) - H(l+\tilde{b})}_{a^\alpha = k-p+\tilde{b}-b}\Big]$$

$$= \frac{-ie^2}{8\pi^2}\,\epsilon^{\mu\lambda\sigma\tau}\Big[\underbrace{(-k^\tau)}(-k^\sigma+b^\sigma-\tilde{b}^\sigma) + \underbrace{(-g^{\sigma\alpha}p^\tau + g^{\tau\alpha}k^\sigma)}(k-p+\tilde{b}-b)_\alpha\Big]$$

same as before because shift gives something symmetric in $\sigma\tau$

$$= \frac{-ie^2}{8\pi^2}\,\epsilon^{\mu\lambda\sigma\tau}\Big[(b_1-b_2)k_\tau p_\sigma - 2k_\sigma p_\tau + \cancel{(b_1-b_2)}k_\sigma p_\tau + (b_1-b_2)p_\tau k_\sigma\Big]$$

$$= \frac{-ie^2}{8\pi^2}\,\epsilon^{\mu\lambda\sigma\tau}\,k_\sigma p_\tau\,(-2+b_1-b_2)$$

$$(p+k)_\mu T^{\mu\nu\lambda} = \frac{-4e^2}{(2\pi)^4} \epsilon^{\mu\lambda\sigma\tau} \int d^4\ell \left[H(\ell-k+b) - I(\ell+b) + I(\ell-p+\tilde{b}) - H(\ell+\tilde{b}) \right]$$

$a = -k+b-\tilde{b}$ $\qquad$ $a = -p+\tilde{b}-b$

$$= \frac{-ie^2}{8\pi^2} \epsilon^{\mu\lambda\sigma\tau} \left[(-k^\tau)(-\cancel{k^\sigma}+b^\sigma-\tilde{b}^\sigma) + (-p^\sigma)(-\cancel{p^\tau}+\tilde{b}^\tau-b^\tau) \right]$$

$$= \frac{-ie^2}{8\pi^2} \epsilon^{\mu\lambda\sigma\tau} p_\sigma k_\tau \; 2\,(b_1-b_2)$$

So pick $b_1 - b_2 = 2$ then vector current conserved (photon couples with e.m. charge)

and $(p+k)_\mu T^{\mu\nu\lambda} = \dfrac{-ie^2}{2\pi^2} \epsilon^{\mu\lambda\sigma\tau} p_\sigma k_\tau$ ← $\mu\lambda$ are $\epsilon^\mu \epsilon^\lambda$ so $\sim F_{\mu\nu}\tilde{F}^{\mu\nu}$

For operators $\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$ putting back mass m

$\partial_\mu J^\mu = 0$ $\qquad$ $J^\mu = \bar{\psi}\gamma^\mu\psi$

$i\vec{D}^\mu = i\vec{\partial}^\mu - eA^\mu$

$i\overleftarrow{D}^\mu = i\overleftarrow{\partial}^\mu + eA^\mu$

naive $\partial_\mu J^{\mu 5} = \bar{\psi}\overleftarrow{\partial}\gamma_5\psi + \bar{\psi}\vec{\partial}\gamma_5\psi$

$= \bar{\psi}\overleftarrow{\cancel{\partial}}\gamma_5\psi - \bar{\psi}\gamma_5\cancel{\partial}\psi$ $\qquad$ $\cancel{\partial}\psi = -im\psi$

$= 2im\,\bar{\psi}\gamma_5\psi$ $\qquad$ explicit breaking by m

now
$$\boxed{\partial_\mu J^{\mu 5} = 2im\,\bar{\psi}\gamma_5\psi - \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}}$$

$\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau}$

[Symmetry factor $\frac{1}{2}$ due to identical γ's]

$\epsilon^{0123} = +1$

as in Peskin

Functional Integral Method (massless again)

Recall $\psi \to \psi' = \psi + \epsilon(x)\,\Delta\psi(x) = (1 - i\epsilon\gamma_5)\psi$

$$\ln Z[v_\mu - \partial_\mu \epsilon] - \ln Z[v_\mu] = +i \int d^4x\, \epsilon(x)\, \partial_\mu \bar{J}_5^{\,\mu}(x)$$

v_μ source for J_5^μ

$$\bar\psi i \partial\!\!\!/ \psi + i\partial_\mu\epsilon\, \bar\psi \gamma^\mu \gamma_5 \psi = \bar\psi(1 - i\epsilon\gamma_5)\, i\partial\!\!\!/\, (1 - i\epsilon\gamma_5)\psi$$

$$= \bar\psi' i\partial\!\!\!/ \psi'$$

we can absorb $\partial_\mu \epsilon$ into change $\psi \to \psi'$ in path integral

But $\ln Z[v - \partial\epsilon] \neq \ln Z[v]$ if measure transforms

$$\int [d\psi][d\bar\psi] = \int [d\psi'][d\bar\psi']\, J^{-2}$$

Jacobian $J = \det(\cdots)$

$J = \bar{J}$

We will see that J^{-2} is independent of $\psi, \bar\psi$, so

$$\ln Z[v - \partial\epsilon] - \ln Z[v] = \ln[J^{-2}] = +i \int d^4x\, \epsilon(x)\, \partial^\mu \bar{J}_{5\mu}(x)$$

Compute J

$$\psi'(x) = \int d^4y\, V(x,y)\, \psi(y)$$

$$V(x,y) = \delta^4(x-y)(1 - i\epsilon(x)\gamma_5) = 1 + C$$

$$J = \det(1 + C) = e^{\mathrm{tr}\ln(1+C)} = e^{\mathrm{tr}C + \cdots}$$

trace in spin & position space

Naively:

$$\ln J = \mathrm{tr}\,C = -i\, \mathrm{tr}[\gamma_5] \int d^4x\, \epsilon(x)\, \delta^4(x-x)$$

is $0 * \infty$, need regulator

Use eigenstates of $\not{D}$:

$$i\not{D}\,\phi_m = \overset{\text{real}}{\lambda_m}\,\phi_m$$

$$\hat{\phi}_m\, i\overleftarrow{\not{D}} = -iD_\mu\,\hat{\phi}_m\gamma^\mu = \lambda_m\,\hat{\phi}_m$$

$$\psi(x) = \sum_m a_m\,\phi_m(x)$$

$$\bar{\psi}(x) = \sum_m \hat{a}_m\,\hat{\phi}_m(x) \qquad \int[d\psi][d\bar{\psi}] = \int\prod_m da_m\,d\hat{a}_m$$

$$\ln J = -i\int d^4x\,\epsilon(x)\sum_n \phi_n^+(x)\,\gamma_5\,\phi_n(x)$$

add gauge inv. regulator (any inv. reg. will do)

$$= -i\int d^4x\,\epsilon(x)\lim_{M\to\infty}\sum_n \phi_n^+(x)\,\gamma_5\,\phi_n(x)\,e^{\lambda_n^2/M^2}$$

Note: $\lambda_n^2 < 0$
eg. Free Field, $\lambda_n^2 = k^{0\,2} - \vec{k}^2$
and is negative when continued to Euclidean

$$= \lim_{M\to\infty} -i\int d^4x\,\epsilon(x)\sum_n \phi_n^+\,\gamma_5\,e^{(i\not{D})^2/M^2}\phi_n$$

$$= -i\int d^4x\,\epsilon\lim_{M\to\infty}\langle x|\,\mathrm{tr}[\gamma_5\,e^{(i\not{D})^2/M^2}]\,|x\rangle$$

now $(i\not{D})^2 = -D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$. Since $M\to\infty$ assume $k\sim$ large and expand in A_μ field. Need 4 γ's to get non-zero trace.

$$= -i\int d^4x\,\epsilon\lim_{M\to\infty}\mathrm{tr}\left[\gamma_5\,\frac{1}{2}\left(\frac{e}{2M^2}\sigma\cdot F\right)^2\right]\underbrace{\langle x|\,e^{-\partial^2/M^2}|x\rangle}_{iM^4/16\pi^2}$$

use Fourier space

$$= +i\int d^4x\,\epsilon(x)\,\frac{e^2}{32\pi^2}\,\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}(x)\,F_{\mu\nu}(x)$$

(indep $\psi,\bar{\psi}$)

So

$$\log J^{-2} = \frac{-ie^2}{16\pi^2}\int d^4x\,\epsilon(x)\,F_{\alpha\beta}(x)\,\tilde{F}^{\alpha\beta}(x) = i\int d^4x\,\epsilon\,\partial^\mu J_{5\mu}$$

$$\partial^\mu J_{5\mu} = \frac{-e^2}{16\pi^2}F_{\alpha\beta}\tilde{F}^{\alpha\beta}$$

✓

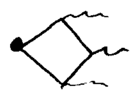
One-Loop Exact, no more operators.

Comments

- Role of regulator: In both Δ-graph & Functional Int. computations there was no regulator which was simultaneously gauge inv (vector current conserved) and chiral invariant

- No new operators are induced in anomaly eqtn beyond one-loop. It arose from surface term in lin. divergent fermion loop, but at higher orders the fermion loop is more convergent

- When we extend vector currents to non-abelian case $F^{\mu\nu}\tilde{F}_{\mu\nu} \to G^a_{\mu\nu}\tilde{G}^{\mu\nu a}$, which has 3-gluon terms

but results are connected by gauge invariance

- Total derivative

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu \,, \qquad K^\mu = 4\epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta$$

$$G^a_{\mu\nu}\tilde{G}^{\mu\nu a} = \partial_\mu K^\mu \,, \qquad K^\mu = 2\epsilon^{\mu\nu\alpha\beta}\left[A^a_\nu G^a_{\alpha\beta} - \frac{g}{6} f^{abc} A^a_\nu A^b_\alpha A^c_\beta\right]$$

could define new current

$$\tilde{J}_{5\mu} = J_{5\mu} + \frac{e^2}{16\pi^2} K_\mu \,, \qquad \partial^\mu \tilde{J}_{5\mu} = 0$$

K^μ and $\tilde{J}^{5\mu}$ are not gauge invariant

Chiral Currents in QCD

$$\mathcal{L} = \bar{u}(i\not{D} - m_u)u + \bar{d}(i\not{D} - m_d)d - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a}$$

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

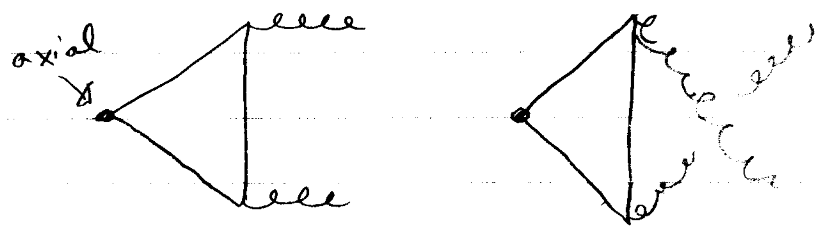
τ^a = Pauli Matrices in flavor

Global Symmetry Currents

$J^\mu = \bar{\psi}\gamma^\mu\psi$	$U(1)$	baryon #
$J^{\mu a} = \bar{\psi}\gamma^\mu \frac{\tau^a}{2}\psi$	$SU_V(2)$	isospin
$J^{\mu 5} = \bar{\psi}\gamma^\mu\gamma_5\psi$	$U(1)_A$	isosinglet
$J^{\mu 5 a} = \bar{\psi}\gamma^\mu\gamma_5\frac{\tau^a}{2}\psi$	"$SU_A(2)$"	isotripet

<u>QCD</u> - vector gluons

$G\tilde{G}$



Just change matrices at the vertices. For $J^{\mu 5}$ or $J^{\mu 5 a}$ both graphs give same color & flavor traces

- $$\partial_\mu J^{\mu 5 a} = \frac{-g^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} G^c_{\alpha\beta} G^d_{\mu\nu} \underbrace{\mathrm{tr}\left[\frac{\tau^a}{2}\right]}_{0} \underbrace{\mathrm{tr}\left[T^c T^d\right]}_{\delta^{cd}/2} + \text{mass terms}$$

  conserved for $m_u = m_d = 0$

  which we used when considering $SU(2)_L \times SU(2)_R \xrightarrow{\text{spont.}} SU(2)_V$

- $$\partial_\mu J^{\mu 5} = \frac{-g^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} G^c_{\alpha\beta} G^d_{\mu\nu} \underbrace{\mathrm{tr}[\mathbb{1}]}_{n_f = \#\text{ of flavors}} \frac{\delta^{cd}}{2} + \text{mass terms}$$

  $$= \frac{-g^2 n_f}{32\pi^2} G^A_{\alpha\beta} \tilde{G}^{\alpha\beta A}$$

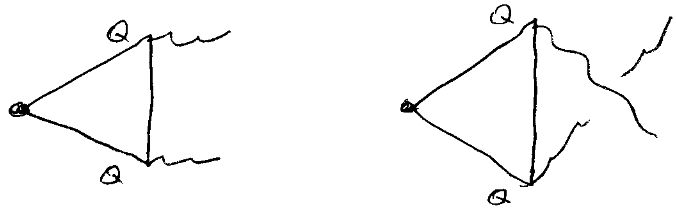
no $U(1)_A$ symmetry

no Goldstone boson (η')

[more later]

vector photons to quark currents $\{u, d\}$

$F\tilde{F}$



charge $Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} = \frac{\tau^3}{2} + \frac{\mathbb{1}}{6}$, $Q^2 = \frac{\tau^3}{6} + \frac{5}{18}\mathbb{1}$

- $\partial_\mu J^{\mu 5 a} = \frac{-e^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} \; tr\left[\frac{\tau^a}{2} Q^2\right] \underbrace{tr\,[\mathbb{1}]}_{N_c \,=\, \#\text{ of colors}}$

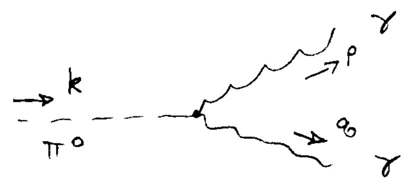
need $a = 3$

$\left(\frac{1}{6}\right)\frac{2}{2} \Rightarrow \frac{N_c}{6}$

$$\partial_\mu J^{\mu 5 3} = \frac{-e^2}{16\pi^2} \frac{N_c}{6} F_{\alpha\beta} \tilde{F}^{\alpha\beta}$$

- $\partial_\mu J^{\mu 5} = -\frac{e^2}{16\pi^2} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \left(2 \cdot \frac{5}{18}\right) N_c$

eg $\pi^0 \to \gamma\gamma$



without anomaly $\mathcal{L}_{\pi\gamma\gamma} = \lambda \underbrace{\partial^2}_{} \pi^0 \; F_{\alpha\beta} \tilde{F}^{\alpha\beta}$

deriv coupled, both derivs on π^0 by int. by parts

$\lambda \sim \underset{\text{loop}}{\frac{e^2}{16\pi^2}} \; \underset{\pi^0/F_\pi \text{ \& dim.}}{\frac{1}{F_\pi (4\pi F_\pi)^2}}$, $\Gamma \sim \frac{4 m_\pi^7 \alpha^2}{\pi (4\pi F_\pi)^6}$, 1000 times too small

[Steinberg: $\lambda \sim \frac{e^2 g_A}{16\pi^2 F_\pi} \times \frac{1}{(4\pi F_\pi)^2}$]

$\boxed{\text{anomaly contribution}}$

general matrix elt

$$\mathcal{M} = \langle p,\epsilon;\, q,\epsilon \mid \pi^0(k)\rangle$$

use LSZ massless π^0

$$= \lim_{k^2\to 0} -ik^2 \int d^4x\, e^{-ik\cdot x} \langle p,\epsilon;\, q,\epsilon \mid \Phi(x) \mid 0\rangle$$

$i\sqrt{Z_\pi}/k^2$

where $\langle 0 \mid \Phi(x) \mid \pi^0(k)\rangle = e^{ik\cdot x}$, Φ-norm is such that $Z_\pi = 1$

Now $\langle 0 \mid J^{\alpha 53}(x) \mid \pi^0(k)\rangle = -i F_\pi k^\alpha e^{ik\cdot x}$

$\langle 0 \mid \partial_\alpha J^{\alpha 53} \mid \pi^0\rangle = F_\pi k^2 e^{ik\cdot x}$

so $\Phi(x) = \dfrac{\partial_\alpha J^{\alpha 53}}{F_\pi k^2}$ has right norm

$$\mathcal{M} = \frac{-i}{F_\pi}\int d^4x\, e^{-ik\cdot x} \langle p,\epsilon;\, q,\epsilon \mid \partial_\mu J^{\mu 53} \mid 0\rangle$$

$$= \frac{-i}{F_\pi}\left(\frac{-e^2 N_c}{96\pi^2}\right)\int d^4x\, e^{-ik\cdot x} \underbrace{\langle p,\epsilon;\, q,\epsilon \mid F_{\mu\nu}\tilde{F}^{\mu\nu}(x) \mid 0\rangle}_{\text{we can compute this matrix elt, \& sum over pol. to get rate}}$$

this gives $\Gamma(\pi^0\to\gamma\gamma)$ in good agreement with experiment [do this, Homework]

Like $\mathcal{L}_{\pi\pi\gamma} \simeq \lambda' \pi^0 F_{\alpha\beta}\tilde{F}^{\alpha\beta}$ $\qquad \lambda' \sim \dfrac{e^2}{16\pi^2 F_\pi}$

↑ no deriv.

So $\Gamma \sim \dfrac{\alpha^2 M_\pi^3}{F_\pi^2}$ [must keep M_π^2 in phase space & kinematics]

Anomalies in Chiral Gauge Theory

So far our axial currents were for global symmetries that are simply broken by anomalies, and we only considered vector gauge symmetries.

For chiral gauge theories, anomaly could make the gauge symmetry anomalous, and hence the theory inconsistent. We require anomalies to be absent.

Consider $\mathcal{L} = \bar{\Psi}\gamma^\mu (i\partial_\mu + g A_\mu^a t_L^a P_L + g' B_\mu^a t_R^a P_R)\Psi$

gens. for some reps of symmetry, $t_L \neq t_R$

if $t_L^a = t_R^a$ symmetry is vector.

chiral rep.

$\mathcal{L} = \psi_L^\dagger \bar{\sigma}\cdot iD_L \psi_L + \psi_R^\dagger \sigma\cdot iD_R \psi_R$, $\quad \psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}$

$\sigma^\mu = (1,\vec{\sigma})$, $\bar{\sigma}^\mu = (1,-\vec{\sigma})$, $\quad \sigma_2 \sigma^{\mu T}\sigma_2 = \bar{\sigma}^\mu$

For convenience rewrite R-handed as L-handed: $\quad \gamma_5 = \begin{pmatrix}-1 & 0 \\ 0 & +1\end{pmatrix}$

$\psi_L' = \sigma^2 \psi_R^*$, $\quad \psi_L'^\dagger = \psi_R^T \sigma^2$

call L-handed antifermion a fermion'

$\psi_R^\dagger i\sigma\cdot(\partial - igA^a t_R^a)\psi_R = \psi_L'^\dagger i\bar{\sigma}\cdot(\partial + igA^a (t_R^a)^T)\psi_L'$

$= \psi_L'^\dagger i\bar{\sigma}\cdot(\partial - igA^a \bar{t}_R^a)\psi_L'$

(minus from fermion interchange, int. by parts) conjugate rep

$\Psi \equiv \begin{pmatrix}\psi_L \\ \psi_L'\end{pmatrix}$, $\quad T^a = \begin{pmatrix} t_L^a & 0 \\ 0 & \bar{t}_R^a\end{pmatrix}$ $\qquad \bar{t}_R^a = -(t_R^a)^T$

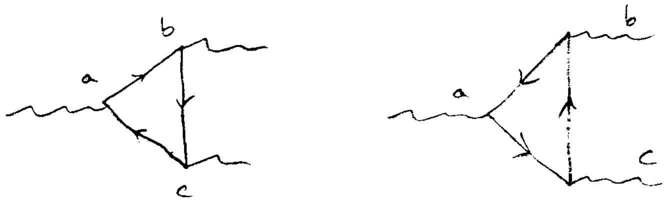
4-comp

$\mathcal{L} = \Psi^\dagger \bar{\sigma}\cdot iD\,\Psi = \bar{\Psi}\, i\cancel{D} P_L \Psi$, $\quad \Psi \equiv \begin{pmatrix}\psi \\ \psi'\end{pmatrix}$

$$J_\mu^a = \bar{\Psi}\, \gamma^\mu P_L T^a\, \Psi$$

consider $\langle 0 | T\, J_\mu^a(0)\, J_\nu^b(x)\, J_\lambda^c(y) | 0 \rangle$

generators T^a, T^b, T^c can be in different groups



• anomaly can be moved to any one current by choice of regulator

For a, b, c in same group (eg $SU(3)$) then $t^a t^b t^c \neq t^a t^c t^b$

$$\mathrm{tr}\left[T^a T^b T^c\right] = D^{abc} + i F^{abc}$$
$$\mathrm{tr}\left[T^a T^c T^b\right] = D^{abc} - i F^{abc}$$

symmetric   antisymmetric

$D^{abc} = \frac{1}{2}\,\mathrm{tr}\left[T^a \{T^b, T^c\}\right]$ carries anomaly

(antisymmetric F^{abc} terms are part of Ward identity)

$$\boxed{\partial_\mu J^\mu_a = \frac{g^2}{32\pi^2}\, D^{abc}\, F^b_{\alpha\beta}\, \tilde{F}^{c\,\alpha\beta}}$$

(subst. g^2 for appropriate couplings)

<u>Check</u> • Pure Vector case $t_L^a = t_R^a$, $T = \begin{pmatrix} t_L^a & \\ & -t_R^{a\,T} \end{pmatrix}$

$$D^{abc} = \frac{1}{2}\,\mathrm{tr}\left[t_L^a \{t_L^b, t_L^c\}\right] - \frac{1}{2}\,\mathrm{tr}\left[t_R^a \{t_R^b, t_R^c\}\right]$$
$$= 0$$

• $U(1)_A\, U(1)^2$ case $g^2 = e^2$

$\gamma_5 \cdot 1 \cdot 1 = (P_R - P_L)(P_L + P_R)(P_L + P_R)$ $T = \begin{pmatrix} 1_L & 0 \\ 0 & -1_R \end{pmatrix}$

$$D^{abc} \Rightarrow -\frac{1}{2}\,\underbrace{\mathrm{tr}}_{\text{no trace here}}\left[(1)(1+1)\right] - \frac{1}{2}\,\mathrm{tr}\left[(1)(1+1)\right] = -2$$

✓

$$\partial J = \frac{-e^2}{16\pi^2}\, F\tilde{F}$$

Standard Model - Check Anomalies

$T^a \; , \; t^a \; , \; Y$

consider 1 generation

$\underbrace{SU(3)}_{\text{vector}} \times \underbrace{SU(2)_L}_{} \times \underbrace{U(1)_Y}_{}$ chiral, different coupling to L & R

$SU(2)^3$: $\frac{1}{2} \, tr[\{t^a, t^b\}, t^c] = \delta^{ab} \, tr[t^c] = 0$

$SU(3)^3$: pure vector , $D^{abc} = 0$

cases with one $SU(3)$ or one $SU(2)$ $\to 0$ too

① $SU(3)^2 U_Y(1)$: $tr[T^a T^b] \, tr[Y] = \frac{\delta^{ab}}{2} \underbrace{\sum_{} \; Y_i}_{i \in \text{quarks}}$

[Left] - [Right] : $N_c \Big[2 \big(\tfrac{1}{6}\big) - \big[\tfrac{2}{3} - \tfrac{1}{3} \big] \Big] = 0$ ✓

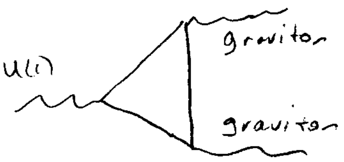
two members of doublet → Q_L $\quad u_R \quad d_R$

② $SU(2)^2 U_Y(1)$: $tr[Y] = \sum_{\substack{i \in L\text{-handed} \\ \text{Fermions}}} Y_i = \underbrace{2\left(-\tfrac{1}{2}\right)}_{L_L} + \underbrace{2 N_c \tfrac{1}{6}}_{Q_L} = 0$

③ $U_Y(1)^3$: $tr[Y^3] = \underbrace{2\left(-\tfrac{1}{2}\right)^3}_{L_L} - \underbrace{(-1)^3}_{e_R} + N_c \Big(\underbrace{2\left(\tfrac{1}{6}\right)^3}_{Q_L} - \underbrace{\left(\tfrac{2}{3}\right)^3}_{u_R} - \underbrace{\left(-\tfrac{1}{3}\right)^3}_{d_R} \Big) = 0$

④ $\text{grav}^2 \times U_Y(1)$:

$tr[Y] = 2\left(-\tfrac{1}{2}\right) - (-1) + N_c \Big(2\left(\tfrac{1}{6}\right) - \left(\tfrac{2}{3}\right) - \left(-\tfrac{1}{3}\right) \Big) = 0$



here $tr[Y] \, \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\kappa\lambda} R_{\rho\sigma}{}^{\kappa\lambda}$

Pretend we didn't know hypercharges: $\{Y_L, Y_e, Y_Q, Y_u, Y_d\}$

① $2Y_Q = Y_u + Y_d$

② $Y_L = -N_c Y_Q$

③ $N_c\,(2Y_Q^3 - Y_u^3 - Y_d^3) = Y_e^3 - 2Y_L^3$

④ $N_c\,(2Y_Q - Y_u - Y_d) = Y_e - 2Y_L$

Fix e_R^-: $Q = T_3 + Y_e = -1$

picking also a convention for $U(1)$ charges

then we have 4 eqn's & 4 unknowns

soln: i) $Y_L = -\frac{1}{2}$, $Y_u = \frac{2}{3}$, $Y_d = -\frac{1}{3}$, $Y_Q = \frac{1}{6}$, $(Y_e = -1)$

Nature did not pick the only other solution

ii) $Y_e = Y_Q = Y_L = 0$, $Y_u = -Y_d$

Values of hypercharge are not arbitrary

Ch 6. Topological Sectors, Instantons, $U(1)_A$

(Briefly)

We have some loose ends to tie up, which will turn out to be related

- When considering quark rotations and V_{CKM}, we used the anomalous $U(1)_A$ rotation to ensure our mass matrix was real.
- We left out terms $\theta\, G^{\mu\nu}_a \tilde{G}^a_{\mu\nu}$ in QCD & $\theta' F^{\mu\nu}\tilde{F}_{\mu\nu}$ in QED when writing down the most general Lagrangians
- The axial-anomaly in QCD for $U(1)_A$

$$\partial_\mu J^{5\mu} \propto G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a = \partial_\mu K^\mu$$

points to an explanation for why there is no Goldstone boson from spont. breaking of the $U(1)_A$. To be sure that the corresponding charge is not conserved we'd like to see a field configuration contribute to $\int d^4x\, \partial_\mu K^\mu$ even though its a total derivative. (Nontrivial configs exist in the non-abelian case so we focus on that. For $U(1)$ $\theta' F\tilde{F}$ is really a neglectible surface term.)

We'll approach this in a round about way by studying $\boxed{\text{QCD vacuum}}$

Classical energy density $\sim G^{\mu\nu}G_{\mu\nu} \sim \vec{E}^2 + \vec{B}^2$. Field should be zero $A^\mu = 0$, or in a pure gauge config. where $G^{\mu\nu} = 0$
$A^\mu = -i\,\partial^\mu U\, U^{-1}$

[Notation: absorb g into A, $A = gA'$, $\mathcal{L} = -\frac{1}{4g^2}GG = -\frac{1}{4}G'G'$]

To classify gauge trasfm's pick $A_0 = 0$
(we can always get to this gauge by Wilson line trnsfm)

since $A_0 \to \mathcal{U} A_0 \mathcal{U}^+ - i\partial_0 \mathcal{U}\mathcal{U}^+$ we want $\partial_0 \mathcal{U} = 0$

time independent transformations

We're still free to make transformations $\mathcal{U}(\vec{x})$

so we study $A_i(\vec{x}) = -i\partial_i \mathcal{U}(\vec{x})\, \mathcal{U}^{-1}(\vec{x})$

$$\mathcal{U}(\vec{x}) = e^{i\alpha^A(\vec{x})T^A}$$

$\alpha^A(\vec{x})$ is a map : $\mathbb{R}^3 \to SU(3)$ or $SU(2)$

Impose $\lim_{|\vec{x}|\to\infty} \mathcal{U}(\vec{x}) = 1$

This condition will ensure the $A_i(\vec{x})$ are connected to $A = 0$ by finite energy configurations

We can identify pts at ∞

so $\alpha^A(\vec{x}) : S^3 \to SU(3)$ or $SU(2)$

We'll study $SU(2)$ for simplicity ($SU(2) \subset SU(3)$)

$SU(2) \sim S^3$ ($\mathcal{U} = a_0 + i\vec{\sigma}\cdot\vec{a}$, $a_0^2 + \vec{a}^2 = 1$)

so we map sphere to sphere

- small gauge trnsfms $\lim_{|\vec{x}|\to\infty} \alpha^A(\vec{x}) = 0$ & can be continuously deformed to zero

- large gauge trnsfms involve wrapping one sphere around the other with one or more windings

Winding number in 3-dim

$$n = \frac{1}{24\pi^2} \int d^3x \;\; \epsilon_{ijk} \;\; \mathrm{tr}\left(A_i(\vec{x})\, A_j(\vec{x})\, A_k(\vec{x}) \right)$$

eg $\quad U: \; S^1 \to S^1 \sim U(1) \qquad$ # times you wrap around in mapping $\bigcirc \to \bigcirc$

$0 \le \theta \le 2\pi \qquad U(\theta) = e^{if(\theta)}$

$$f(2\pi) = f(0) + 2\pi n \;, \quad n = \text{winding \#}$$

Here $\quad A_i = -i\, U^{-1} \partial_i U \;\to\; -i\, U^{-1}\, d/d\theta \; U$

$$n = \frac{-i}{2\pi} \int d\theta \;\; U^{-1}\, d/d\theta\, U = \frac{1}{2\pi} \int_0^{2\pi} \frac{df}{d\theta}\, d\theta = \frac{1}{2\pi}\left[f(2\pi) - f(0) \right] \checkmark$$

$n = 0$ are small gauge transfms

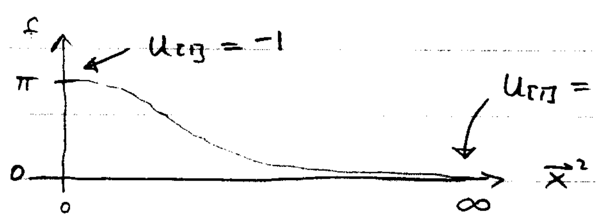
$$n(U_1 U_2) = n(U_1) + n(U_2)$$

$S^3 \to SU(2)$

$$U_{[1]} = \frac{\vec{x}^2 - a^2}{\vec{x}^2 + a^2} + \frac{2ia\,\vec{\sigma}\cdot\vec{x}}{\vec{x}^2 + a^2} = \exp\left(i\, f(\vec{x}^2)\, \vec{\sigma}\cdot\hat{x} \right)$$

$n[U_{[1]}] = 1$

can't be deformed to identity



$$U_{[k]} = [U_{[1]}]^k \;, \quad n(U_{[k]}) = k$$

Within each n-sector different $A_i(\vec{x})$ are related by small gauge transformations

Consider physical states $|\Psi\rangle$

Gauss' Law $\quad 0 = D_\mu F^{\mu 0} = -D_i E^i \quad$ but in quantum

$[A^i, E^i] \neq 0 \quad$ so instead require $\quad D_i E^i \, |\Psi\rangle = 0$

This makes theory gauge invariant

For $n=0$ the operator on Hilbert space is

$$U(\alpha) = e^{iQ} = \exp i \int d^3x \, [\, \underbrace{D_i \alpha^a(\vec{x})}_{\delta A_a^i} \; \underbrace{E_a^i(\vec{x})}_{\Pi_a^i} \,] \qquad \text{int. by parts}$$

$$U(\alpha)\,|\Psi\rangle = \exp\Big[-i \underbrace{\int d^3x \, \mathrm{tr}[\alpha \, D_i E_i]}_{\text{zero}} + i \int_{\partial\mathbb{R}^3} d^3 S_i \, \mathrm{tr}\,(\alpha(\vec{x})\, E^i) \Big] |\Psi\rangle$$

$$\lim_{|\vec{x}| \to \infty} \alpha(\vec{x}) = 0$$

$$= |\Psi\rangle$$

Lets label physical states in different sectors by n, $|n\rangle$, then $U(\alpha)\,|n\rangle = |n\rangle$. But these are not physical vacua since

① We can tunnel between them (instantons)

② They are not invariant under U_1,
$$U_1 \, |n\rangle = |n+1\rangle$$

Study ② require superposition to get gauge invariant vacuum state

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} \, |n\rangle \quad , \qquad U_1 \, |\theta\rangle = e^{i\theta} \, |\theta\rangle$$

For any gauge inv. operator B, $[U_1, B] = 0$

$0 = \langle \theta | [U_1, B] | \theta' \rangle = (e^{i\theta} - e^{i\theta'}) \langle \theta | B | \theta' \rangle$

so $\langle \theta | B | \theta' \rangle = 0$ for $\theta \neq \theta'$

Distinct θ-vacua. (transfn. by overall phase cancels in m.elt. squared)

Study ① We'll study definition of instanton, but not the tunneling rate

A classical finite action config where $F^{\mu\nu} \to 0$ as euclidean $r = \sqrt{x_4^2 + \vec{x}^2} \to \infty$, so $A_\mu \to -i g^\dagger \partial_\mu g$ (pure gauge) as $r \to \infty$, and it is a map $S^3 \to SU(2)$ with winding number

$$\nu = \frac{i}{32\pi^2} \int d^4x_E \; \mathrm{tr}\,(G_{\mu\nu} \tilde{G}^{\mu\nu}) = \frac{i}{32\pi^2} \int d^4x_E \; \partial^\mu K_\mu$$

$$= \int_{\substack{\text{sphere}\\ \infty}} \frac{d^3x}{24\pi^2} \; \hat{r}_\mu \, \epsilon^{\mu\alpha\beta\gamma} \, \mathrm{tr}\,[A_\alpha A_\beta A_\alpha] = n(g)$$

instanton has right topological charge to cause transition

Instanton minimizes action & hence is classical sol'n of eqtn. of motion, $D_\mu G^{\mu\nu} = 0$. Let $\tilde{G} \equiv \frac{1}{2} G^{\mu\nu\alpha\beta} G_{\alpha\beta}$ here.

trick: $$0 \leq \int d^4x \, (G^a_{\mu\nu} \pm \tilde{G}^a_{\mu\nu})^2$$

$$\int d^4x \; \mathrm{tr}\,(G\tilde{G}) \geq \left| \int d^4x \; \mathrm{tr}\,(G\tilde{G}) \right|$$

$$2 g^2 S_E \geq 16\pi^2 |\nu|$$

saturated if $G = \tilde{G}$ (easier than solving $D^\mu G_{\mu\nu} = 0$)

or $A_\mu = \frac{r^2}{r^2+\rho^2}(-i g^\dagger \partial_\mu g)$, $g = \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{r}$ is a solution (more on Hmwk #7)

makes transition $\Delta\nu = 1$ "instantaneously" (time $\sim \rho$)

In matrix element

$${}_{out}\langle \theta | X | \theta \rangle_{IN} = \sum_{m,n} e^{i(m-n)\theta} \; {}_{out}\langle m | X | n \rangle_{IN}$$

configs exist which cause $n \to m$ transition

Recall $\quad tr[G_{\mu\nu} \tilde{G}^{\mu\nu}] = \partial_\mu K^\mu$

$$K^\mu = 2 \epsilon^{\mu\alpha\beta\gamma} \, tr\left[G_{\alpha\beta} A_\gamma + \frac{2i}{3} A_\alpha A_\beta A_\gamma \right]$$

$$\nu = \frac{1}{32\pi^2} \int d^4x \; tr(G\tilde{G}) = \frac{i}{32\pi^2} \int d^4x_E \left(\partial^0 K_0 + \partial_i K^i \right)$$

$\to 0$ in $A^0 = 0$ gauge as $|\vec{x}| \to \infty$

$$= \frac{i}{32\pi^2} \int d^3x \; K^0 \Big|_{x_4 = -T}^{T}$$

transition over $x_4 = -T$ to T $\quad$ as $T \to \infty$

now $r = \sqrt{x_4^2 + \vec{x}^2} \to \infty$, $\quad A_\mu \to -i g^\dagger \partial_\mu g$, $\quad G_{\mu\nu} = 0$

$$= \frac{-1}{24\pi^2} \int d^3x \; \epsilon^{ijk} \, tr[A^i A^j A^k] \Big|_{-T}^{T}$$

$$= -(m-n) \qquad \text{change in winding number}$$

The $e^{i(m-n)\theta}$ phase can be accounted for by

$$\langle 0|X|0\rangle = \int [dA][d\psi][d\bar\psi] \times e^{iS_{QCD} - i\frac{\theta}{32\pi^2}\int tr(G\tilde G)}$$

$tr\, G\tilde G = \frac{1}{2} G^{\mu\nu a}\tilde G^{\mu\nu a}$ a new term in QCD action

$$\mathcal{L} = \mathcal{L}_{QCD}^{\theta=0} - \frac{\theta}{64\pi^2} g^2 \int d^4x\, G^a_{\mu\nu}\tilde G^{a\mu\nu}$$

$\vec E\cdot\vec B$ P-odd & T-odd

Can think of θ as a parameter labeling the theory

Besides coupling (Λ_{QCD}) & masses we must specify θ (vacuum label) to fix QCD

<u>Connection to Chiral Rotations</u>

$$\partial_\mu J^{5\mu} = -\frac{n_f g^2}{32\pi^2} G^a_{\mu\nu}\tilde G^{\mu\nu a}$$

recall when $\psi \to e^{i\alpha\gamma_5}\psi$, $\bar\psi \to \bar\psi e^{i\alpha\gamma_5}$

$$[d\psi][d\bar\psi] \to [d\psi][d\bar\psi]\, J^{-2} = [d\psi][d\bar\psi] \exp\left(\frac{-i\alpha}{32\pi^2}\int d^4x\, G^a_{\mu\nu}\tilde G^{\mu\nu a}\right)$$

so $\theta \to \theta + 2\alpha n_f$

for n_f flavors

Note. $\psi \to e^{i\alpha\gamma_5}\psi$ is just change of var in path integral.

If we have at least one <u>massless</u> quark then its rotation only changes θ. Since physics is indep. of var. choice its indep. of θ.

In SM we have 6 massive quarks, and $U(1)_A$ rotation changes mass terms. In fact we used the $U(1)_A$ to ensure our diagonal mass matrix was real

$$\mathcal{L} = -\sum_f M_f \bar{\Psi}_f P_R \Psi_f - \sum_f m_f^* \bar{\Psi}_f P_L \Psi_f$$ [diagonal m-matrix]

then $\Psi_f \to e^{i\alpha_f \gamma_5} \Psi_f$ changes

$\bar{\Psi}_f \to \bar{\Psi}_f e^{i\alpha_f \gamma_5}$ $\qquad m_f \to e^{2i\alpha_f} m_f$

<u>and</u> $\theta \to \theta + \sum_f 2\alpha_f$

However $\bar{\theta} \equiv \theta - \arg \det m$ is <u>invariant</u>.

- If we pick α_f to make mass matrix real then $\sum_f 2\alpha_f = -\arg \det m_{\text{original}}$, and $\bar{\theta}$ is induced θ-term in the action

$$\bar{\theta} = \theta_{\text{orig}} - \arg \det m_{\text{orig}}$$

↑ we can include off diagonal terms since $\det U = 1$ for unitary tranformations used to diagonalize it.

Strong CP Problem

$$|\bar{\theta}| < 10^{-10}$$

from neutron electric dipole moment

Ch 7. Collider Physics

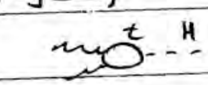
At the LHC we use high energy collisions (center of mass energy $\sqrt{s} = 7 \to 14$ TeV) to probe nature at short distances.

Features

- we collide protons not quarks, so confinement plays a role in the initial state
- its high energy, $\alpha_s \ll 1$, QCD perturbation theory will apply to hard scattering interactions
- we observe hadrons and jets, confinement and pert. theory appear again in the final state

Theoretically the key idea is to factorize those 3 things, by distinguishing long and short distance processes.

eg $pp \to H \to (b\text{-jet})(\bar{b}\text{-jet})$



short $\quad gg \to H \;,\quad H \to b\bar{b}$

long $\quad pp \to ggX \;,\quad b\bar{b} \to \begin{pmatrix}\text{hadrons}\\ \text{in}\\ b\text{-jet}\end{pmatrix}\begin{pmatrix}\text{hadrons}\\ \text{in } \bar{b}\text{-jet}\end{pmatrix}$ ← no "H" here!

universal

Tools

- IR divergences. Signal sensitivity to long distances. We can pick observables to minimize the sensitivity (IR safe cross-section) or control it as much as possible (eg. ensure only initial state IR divergences)
- use inclusive observables. DIS $e^- p \to e^- X$ ← anything hadronic

  $e^+e^- \to X$ ← any hadrons

The idea of summing over hadronic states X is that having computed an amplitude with an energetic quark/gluon we then include all possibilities for their subsequent evolution (Σ Prob $= 1$). into hadrons at long dist.

- Operator Product Expansion (OPE)
- Factorization Theorems (SCET)

} Based on Wilsonian idea of scale seperation

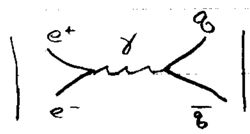
$\boxed{\text{eg. } e^+e^- \to \text{hadrons}}$

massless

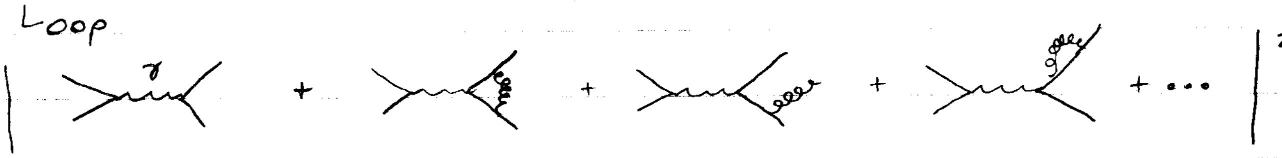
$e^+e^- \to q\bar{q}\,,\ q\bar{q}g\,,\ \ldots$ through virtual γ (for simplicity ignore Z)

Tree level

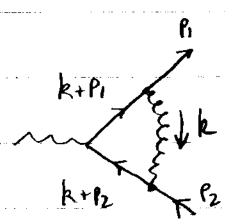
Born Cross-Section $\sigma_B = \dfrac{4\pi\alpha_s}{3s} N_c \sum_i Q_i^2$ from



One Loop



Lets focus on IR divergences (suppress color, spin, UV renormalyation)



most IR singular

$$\int \frac{d^d k}{k^2 (k+p_1)^2 (k+p_2)^2} \underset{\text{soft}}{\overset{k\to 0}{\sim}} \frac{d^4 k}{k^2\, p_1\cdot k\ p_2\cdot k} \sim \frac{d^4 k}{k^4}$$

log IR divergent

also $p_1^2 = 0$ so $k \xrightarrow{\text{collinear}} p_1$ $\quad k^2 \to 0\,,\ (k+p_1)^2 \to 0$

$p_2^2 = 0$ $\quad k \xrightarrow{\text{collin}} p_2$ $\quad k^2 \to 0\ \ (k+p_2)^2 \to 0$

} 4 powers, IR div. again

Add a mass $P_1^2 = m^2$

$$[(k+P_1)^2 - m^2] \underset{\sim}{\xrightarrow{k\to 0}} P_1\cdot k \qquad \text{still divergent in soft limit}$$

$$\underset{\sim}{\xrightarrow{k\to P_1}} (2P_1)^2 - m^2 \sim 3m^2 \qquad \text{regulated by } m^2$$

Keeping momenta offshell $P_1^2 \neq 0$, $P_2^2 \neq 0$ will regulate both types of singularity

Dim. Reg $d = 4-2\epsilon$, $\epsilon < 0$ also regulates them:
to see this let $n = 3-2\epsilon$ do k^0 by contours (always possible since TOPT exists)

Soft integrand

$$\int \frac{dk^0\, d^n k}{(k^0 - |k| + i\epsilon)(k^0 + |k| - i\epsilon)(E_1 k^0 - E_1 |k| \cos\theta + i\epsilon)(k^2 + 2k\cdot P_2)}$$

$$\sim \frac{d^n k}{|k|\, E_1 |k| (1-\cos\theta) |k| \cdots} + \cdots$$

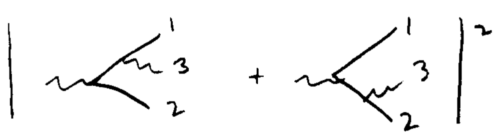
$$d^n k = dk\, k^{n-1}\, d\Omega_n = dk\, k^{n-1}\, d\cos\theta\, (\sin\theta)^{n-3}\, d\Omega_{n-1}$$

$$\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

$$\sim \int_0 \underbrace{\frac{dk\, k^{n-1}}{k^3}}_{\propto \frac{1}{n-3} = \frac{1}{-2\epsilon_{IR}}} \underbrace{\frac{dx\, (1-x^2)^{\frac{n-3}{2}}}{(1-x) \cdots}}_{1/\epsilon_{IR}} + \cdots$$

soft   collinear

We also have IR divergences in Bremstrahlung graphs



which can be regulated by dim. reg. by doing phase space integrals in d-dimensions note overall $(\text{mass})^{2n-4}$

$$\prod_{i=1}^{3} \frac{d^n p_i}{2p_i^0} (2\pi)^d \quad \delta^d(p_1+p_2+p_3-q) \quad |\mathcal{M}_{brem}^{tree}|^2$$

$n = d-1 = 3-2\epsilon$

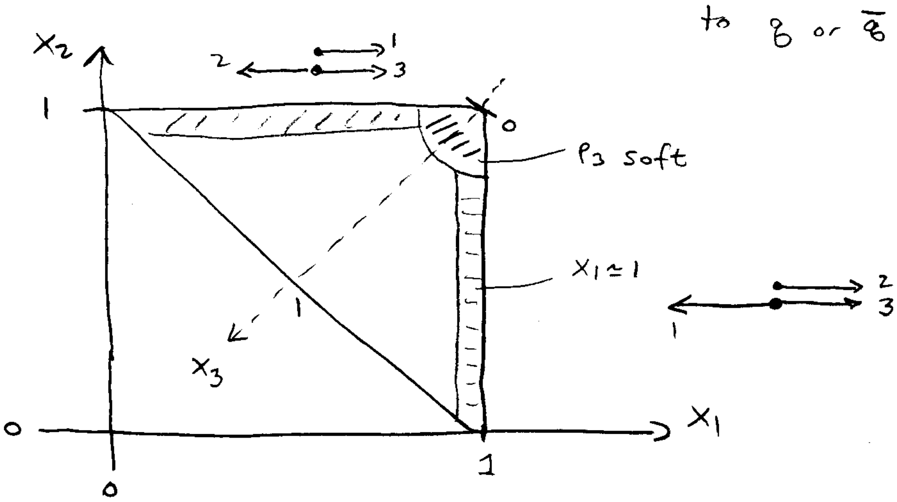
if $\quad x_i \equiv \dfrac{2 p_i \cdot q}{q^2} \quad , \quad x_1+x_2+x_3 = 2$ $\quad$ [derive in section]

get $\displaystyle\int_0^1 dx_1\, dx_2\, dx_3 \; \frac{\delta(2-x_1-x_2-x_3)}{(1-x_1)^\epsilon (1-x_2)^\epsilon (1-x_3)^\epsilon} \left[\frac{x_1^2 + x_2^2 - \epsilon x_3^2}{(1-x_1)(1-x_2)} \right]$

from $|\mathcal{M}|^2$

IR divergences from $\quad p_3 \to 0 \quad$ soft $\quad\quad x_3 \to 0,\ x_1 \,\&\, x_2 \to 1$

$p_3 \to p_1,\ p_3 \to p_2 \quad$ gluon collinear to q or $\bar{q}$ $\quad\quad x_2 \to 1$ or $x_1 \to 1$



IR singularities occur at edges of phase space

One-Loop Result $\quad e^+e^- \to q\bar{q},\ q\bar{q}g$

real $\quad |\ \ +\ \ |^2$

$$\sigma = \sigma_B (1 + A_V + A_R)$$

virtual, vertex & w.fn $\quad (\ \ +\ \ )(\ \ )^* + c.c.$

$$\sigma_B = \frac{4\pi\alpha^2}{3s} \left(\sum_i Q_i^2\right) N_c \left(\frac{4\pi}{s}\right)^\epsilon \left[\frac{3(1-\epsilon)\Gamma(2-\epsilon)}{(3-2\epsilon)\Gamma(2-2\epsilon)}\right]$$

Born with p.space in d-dimensions

$\overline{MS}$ -scheme for μ $\qquad C_F = \frac{4}{3} = \frac{N_c^2-1}{2N_c}$

$$A_V = \frac{\alpha_s C_F}{\pi}\left(\frac{\mu^2}{s}\right)^\epsilon \frac{\cos(\pi\epsilon)\, e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - 4 + \cdots\right)$$

$$A_R = \frac{\alpha_s C_F}{\pi}\left(\frac{\mu^2}{s}\right)^\epsilon \frac{\cos(\pi\epsilon)\, e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4} + \cdots\right)$$

$A_R + A_V$ IR safe

$$\sigma = \sigma_B\left(1 + \frac{3}{4}\,\frac{C_F\alpha_s}{\pi}\right)$$

$\boxed{\text{OR}}$ include small mass $p_1^2 = p_2^2 = m^2 \ll s$

collin sing.

$$A_V \simeq \frac{\alpha_s C_F}{\pi}\left(\frac{1}{\epsilon}\ln\left(\frac{s}{m^2}\right) - \ln\left(\frac{s}{\mu^2}\right)\ln\left(\frac{s}{m^2}\right) + \cdots\right)$$

$$A_R \simeq \frac{\alpha_s C_F}{\pi}\left(-\frac{1}{\epsilon}\ln\left(\frac{s}{m^2}\right) + \ln\left(\frac{s}{\mu^2}\right)\ln\left(\frac{s}{m^2}\right) + \cdots\right)$$

$\boxed{\text{OR}}$ make it less inclusive by restricting to soft brem

$E_g < \Delta$

$$A_R^{soft} \simeq \frac{\alpha_s C_F}{\pi}\left(-\frac{1}{\epsilon}\ln\left(\frac{s}{m^2}\right) + \ln\left(\frac{\Delta^2}{\mu^2}\right)\ln\left(\frac{s}{m^2}\right) + \cdots\right)$$

$$A_V + A_R^{soft} \simeq \frac{\alpha_s C_F}{\pi}\ln\left(\frac{s}{\Delta^2}\right)\ln\left(\frac{s}{m^2}\right)$$

no soft sing. $\qquad$ collinear sing. for $m \to 0$

Typically in

QED $\quad$ soft div. cancel in sum over final states

collin div. cutoff by m_e^2

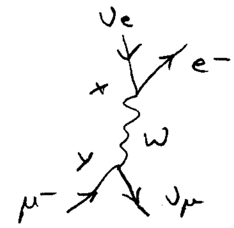
QCD $\quad$ soft div. cancel (suitable obs.)

collin div. cancel or are absorbed into initial state parton dist'n or removed by jet angle cut in final state

[more later]

Operator Product Expansion

We met an example earlier



$$\frac{-g^{\mu\nu}}{k^2 - M_W^2} \simeq \frac{g^{\mu\nu}}{M_W^2} + \cdots \quad \text{at low momentum}$$

Position space $\quad \Delta^{\mu\nu}(x,y) = \dfrac{g^{\mu\nu}}{M_W^2}\,\delta^4(x-y) + \cdots$

$$\int d^4x\, d^4y\; J^-_\mu(x)\,\Delta^{\mu\nu}(x-y)\,J^+_\nu(y) \simeq \frac{1}{M_W^2}\int d^4x\; \underbrace{J^-_\mu(x)\,J^{+\mu}(x)}_{\text{new 4-fermion operator } O(x)}$$

$H_W = C\;O$ ← long-dist

↑ short-dist W

Generically

$$\langle\; J_1(x)\, J_2(0)\, \phi(y_1)\, \phi(y_2) \cdots \phi(y_m)\;\rangle$$

• y_i

and as $x \to 0$

x •

o •

$$J_1(x)\, J_2(0) \to \sum_n C^n(x)\; O_n(0)$$

↑ any sing. as $x \to 0$ (UV effects)

OPE in $e^+e^- \to$ hadrons

$$\sigma = \frac{1}{2s}\;\mathrm{Im}\;\mathcal{M}\,(e^+e^- \to e^+e^-)$$

↑ optical Thm, sum hadrons is $\mathrm{Im}\,\Pi_h$

$$= -\frac{4\pi\alpha}{s}\;\mathrm{Im}\;\Pi_h(s)$$

where $\quad \mathcal{M} = e^2\,(\bar{u}\gamma_\mu v)\,\dfrac{1}{s}\,\Pi_h^{\mu\nu}(q)\,\dfrac{1}{s}\,(\bar{v}\gamma_\nu u)$

$$\Pi_h^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\;\Pi_h(q^2)$$

$$i\,\Pi_h^{\mu\nu}(q) = -e^2\int d^4x\; e^{iq\cdot x}\;\langle 0|\; T\{\, J^\mu(x)\, J^\nu(0)\,\}\,|0\rangle$$

E.M. current $\quad J^\mu = Q_f\;\bar{\Psi}_f\,\gamma^\mu\,\Psi_f$

for $x \to 0$

OPE $\quad J_\mu(x) J_\nu(0) = C^{1}_{\mu\nu}(x)\, \mathbb{1} + C^{q\bar{q}}_{\mu\nu}(x)\; m\bar{q}q(0) + C^{G^2}_{\mu\nu}\,(G^a_{\alpha\beta})^2(0) + \cdots$

op. dim $\qquad 0 \qquad\qquad 4 \qquad\qquad 4$

$\boxed{\text{IF}}$ Fourier transfm dominated by short distances

$$\overset{-4}{\sim e^2 \int d^4x}\ \overset{+3}{J_\mu(x)}\ \overset{+3}{J_\nu(0)}\ e^{iq\cdot x}$$

$$= -ie^2 \underset{+2}{(q^2 g_{\mu\nu} - q_\mu q_\nu)} \Big[C^1(q^2)\,\mathbb{1} + C^{q\bar{q}}(q^2)\, m\bar{q}q + C^{G^2}(q^2)(G^2) + \cdots \Big]$$

dim analysis $\qquad C^1 \sim (q^2)^0 \qquad C^{q\bar{q}} \sim (q^{-4}) \quad , \quad G^{G^2} \sim q^{-4}$

matrix elements $\qquad \langle 0|\mathbb{1}|0\rangle = 1 \quad , \quad \underbrace{\langle m\bar{q}q\rangle \sim m\Lambda^3_{QCD} \quad , \quad \langle G^2\rangle \sim \Lambda^4}_{}$

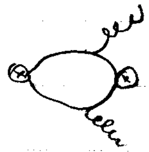
for large q^2 ~~these~~ these are suppressed

Compute C's in OPE

C^1  $+$  $+ \cdots$

$\qquad 1 \qquad\qquad \alpha_s$

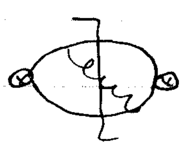
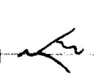
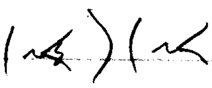
$C^{q\bar{q}}$  $+ \cdots$

C^{GG}  $+ \cdots$

 $\to \quad C^1(q^2) = -(\mathrm{tr}\,\mathbb{1}) \sum_f Q_f^2 \dfrac{1}{3\pi} \ln(-q^2 - i0)$

$$\sigma(e^+e^- \to \text{had}) = \frac{4\pi\alpha_s}{s}\ \mathrm{Im}\, C^1\ \langle 0|\mathbb{1}|0\rangle + \cdots$$

$$= \frac{4\pi\alpha_s}{s} \sum_f Q_f^2 \qquad \text{as before}$$

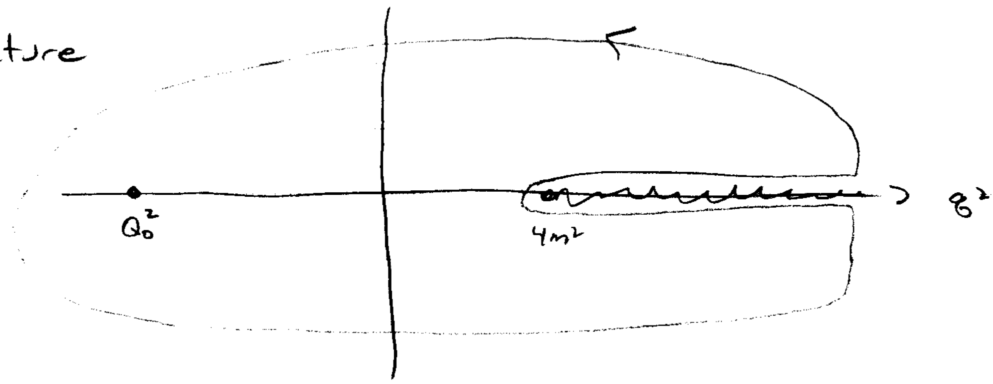
cuts:  is $\Big|$  $\Big|^2$,  is $\Big($  $\Big)^*$

and sum is IR finite , compute α_s corrections

Back to $\boxed{1F}$

- want $\Pi_h(q^2)$ for large timelike q^2 where product of currents is dominated by high E int. states with large # of hadrons
- OPE is valid for $q^2 = -Q_0^2$, int. states are spacelike and for off shell ($x_E \to 0$) like massive W at 0-momentum

Analytic structure of Π_h



Consider

$$I_1 = -4\pi\alpha \oint \frac{dq^2}{2\pi i} \frac{\Pi_h(q^2)}{(q^2+Q_0^2)^2}$$

around pole use OPE $\quad I_1 = -4\pi\alpha \frac{d}{dq^2} \Pi_h \Big|_{q^2=-Q_0^2} \sim \frac{1}{Q_0^2} \sum_f Q_f^2$

Pinching cut as $R \to \infty$

$$I_1 = -4\pi\alpha_s \int \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^2} \underbrace{\text{Disc}\, \Pi_h(q^2)}_{2i\,\text{Im}\,\Pi_h}$$

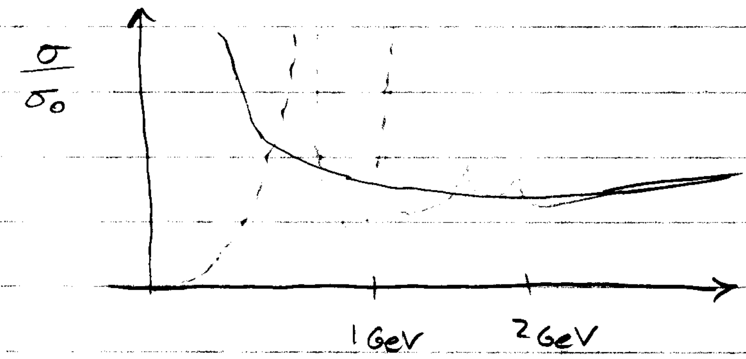
$$= \frac{1}{\pi} \int_{4m^2 \to 0}^{\infty} ds \frac{s}{(s+Q_0^2)^2} \sigma(s)$$

- integral $\underline{\text{avgs}}$ over quark-gluon thresholds in pert theory & hadronic resonances in data
  - our quark $\sigma(s)$ gives correct result

could smear in other ways

eg $\quad \bar{\sigma}(s,\Delta) = \frac{\Delta}{\pi} \int_0^\infty \frac{ds' \, \sigma(s')}{(s'-s)^2+\Delta^2}$

picks region around desired value "s"



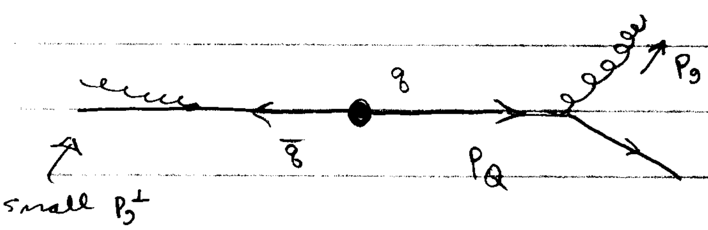
expect we need $\Delta \sim$ few GeV to sum over enough states

averages with width Δ $\rightarrow$ "global quark-hadron duality"

at large s more & more states in given Δ, unaveraged comparison "local duality"

Jet cross sections

so far we have no kinematic info about final state

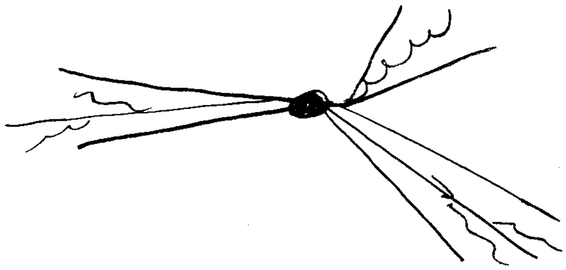


large transv. momentum then α_s is small

α_s - big & enhanced by collinear IR singularities

(which are cutoff by hadronization, Λ_{QCD} in nature but still give large logs)

P_Q^2 will be far off shell, so looks like



3 jets

$e^+e^- \to q\bar{q}$ is 2-jets (which dominate in data)

$e^+e^- \to q\bar{q}g$
$$\frac{1}{\sigma_B}\frac{d^2\sigma}{dx_1 dx_2} = \frac{\alpha_s C_F}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

expect this to describe 3^{rd} jet as long as we stay away from collinear regions $x_1 \to 1$, $x_2 \to 1$

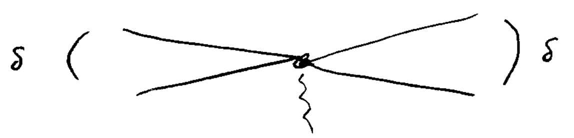
To do this we define a <u>jet algorithm</u>, a means to classify a final state of hadrons (experimentally) or quarks & gluons (theoretically) as having some number of jets.

<u>Sterman-Weinberg jets</u> ('77)

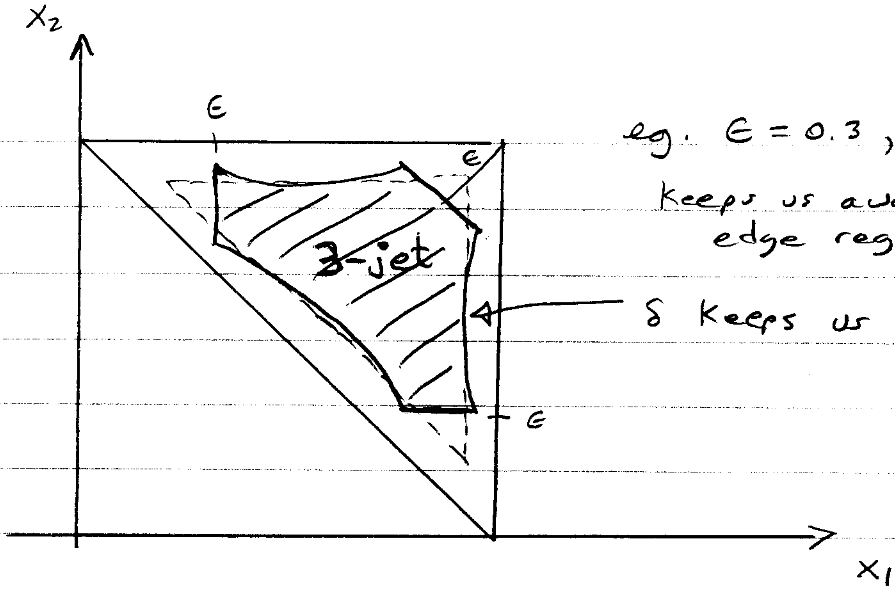
initial energy Q

two-jet $\equiv$ final state with all but energy ϵQ inside two cones of angular size δ

$|\text{Two Jets}(\epsilon,\delta)\rangle$



any particles out here must have $\sum E_i < \epsilon Q$

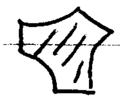


e.g. $\epsilon = 0.3$, $\delta = 30°$

keeps us away from IR singular edge regions

δ keeps us away from sides

At this order $\quad \sigma_{TOT} = \sigma_{2\text{-jet}} + \sigma_{3\text{-jet}}$

We can compute $\sigma_{3\text{-jet}}$ by integrating over  region which is an IR finite integral.

Then $\quad \sigma_{2\text{-jet}} = \sigma_{TOT} - \sigma_{3\text{-jet}}$

negative as expected

$$\sigma_{2\text{-jet}} = \frac{4\pi\alpha^2}{3s} N_c \sum_i Q_i^2 \left[1 - \frac{C_F \alpha_s}{\pi} \left\{ 4 \ln \delta^{-1} \ln\left(\frac{\epsilon^{-1}}{2}\right) - 3 \ln \delta^{-1} + \frac{\pi^2}{3} - \frac{5}{2} + \mathcal{O}(\delta, \epsilon) \right\} + \cdots \right]$$

for $\delta, \epsilon \ll 1$

Jet Algorithms

a set of rules for grouping particles into jets (partons for theorists, hadrons for experimentalists)

- involve a measure d_{ij} for how close i & j particles are and parameters for when they are combined into a jet
- recombination scheme to give a momentum for $i+j$
- Should be IR SAFE insensitive to
  - adding an infinitesimally soft parton
  - replacing a massless parton by two exactly collinear partons

Two types: ① Sequential Recombination ("K_T algorithms")
② Cone Algorithms

Type① e^+e^- to start

JADE: $d_{ij} = 2E_iE_j(1-\cos\theta_{ij}) = \text{inv. mass}(i+j)^2$

Merge i,j if $d_{ij} < y_{cut}\,Q^2$

↑ replaces ϵ, δ (gives dashed curve in figure on prev. page)

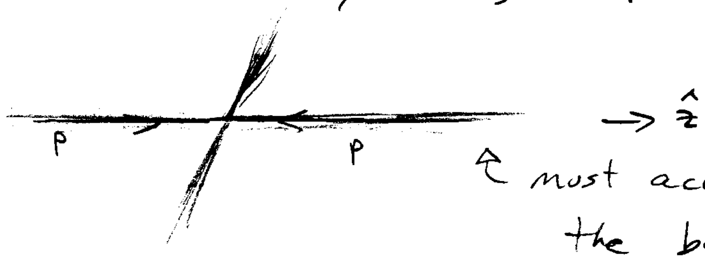
Start with smallest y_{ij}, combine p_i+p_j into new particle in list, repeat until y_{cut} bound is reached

Problem – phantom jets from wide angle soft partons 

K_T/Durham: $d_{ij} = 2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})$

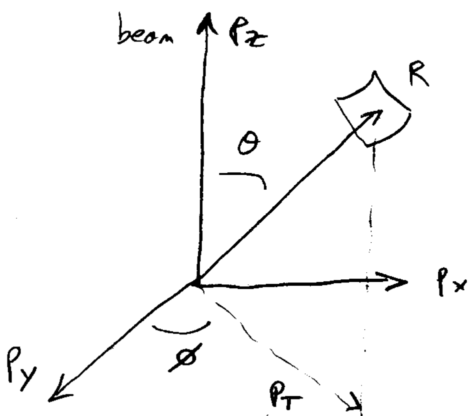
combines soft parton with nearby energetic parton

Hadron Collider Vars



↑ must account for the beam

$$p^\mu = (E, p_x, p_y, p_z) = (M_T\cosh y,\ p_T\sin\phi,\ p_T\cos\phi,\ M_T\sinh y)$$



transverse momentum p_T

" mass $M_T = \sqrt{p_T^2+m^2}$

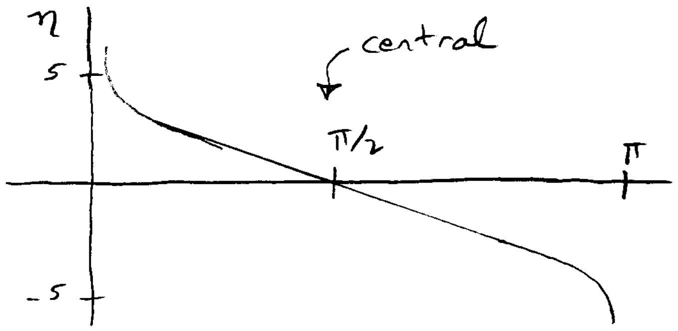
" energy $E_T = E\sin\theta$

rapidity $y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right)$

Δy invariant under longitudinal boosts

more expt. convenient: pseudorapidity $\eta = -\ln\tan\theta/2 \overset{m=0}{=} y$

$R = \sqrt{(\Delta\eta)^2 + (\Delta\phi)^2}$



hadron-hadron jet algorithms

- introduce particle-beam distance d_{iB}
- use vars invariant under longitudinal boosts
  p_T^i , $\Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$

eg. $d_{ij} = \min(p_{Ti}^{2r}, p_{Tj}^{2r}) \dfrac{\Delta R_{ij}^2}{R^2}$, $d_{iB} = p_{Ti}^{2r}$ ← parameter "R"

consider min d_{ij}, d_{iB} , if d_{iB} smallest combine with "beam"

...

$r = 1$ k_T algorithm ← cluster soft particles first
$r = 0$ Cambridge / Aachen alg. ← cluster by geometry
$r = -1$ anti-k_T alg. ← clusters hard particles first

② <u>Cone Algorithms</u>

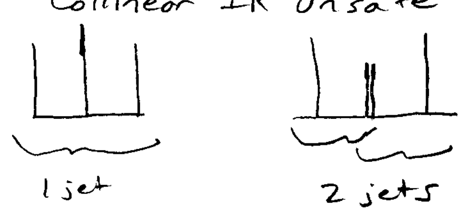
sum up momenta of particles j in cone about "seed" i
$\Delta R_{ij}^2 < R^2$

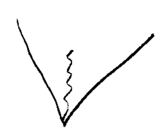
use resulting momentum as new seed, iterate until stable

issues
- how to start, what seeds
- how to deal with overlapping cones, particles in more than one jet

discard low energy jets

<u>problems</u> Collinear IR unsafe Soft IR unsafe





soft particle provides new seed that changes outcome

[Dark Towers : hard energy flow that are not clustered into any jet (never end up in stable cone)]

Cone Algorithms have been favored by expt's (often with some IR issues, but modern "SISCone" resolves these)
CMS & ATLAS use anti-k_T.

Event-Shape Variables

rather than using jets we use a variable "e" that characterizes "shape" of an event

eg. e^+e^- Thrust $\displaystyle T = \max_{\hat{t}} \frac{\sum_i |\vec{p}_i \cdot \hat{t}|}{\sum_i |\vec{p}_i|}$ (Eddie)

- IR Safe
- two-jet like $T \to 1$ $\quad p_1 \longleftarrow \bullet \longrightarrow p_2 \quad \vec{\hat{t}}$

$$T = \frac{|p_1| + |p_2|}{|p_1| + |p_2|} = 1$$

spherical  $T \to 1/2$

$$\frac{1}{\sigma_B} \frac{d\sigma}{dT} = \delta(1-T) + \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)} \right]$$

↑ technically $T \to 1$ singularities are +-function distributions (more on these later)

Event Shapes are studied with Factorization Thms

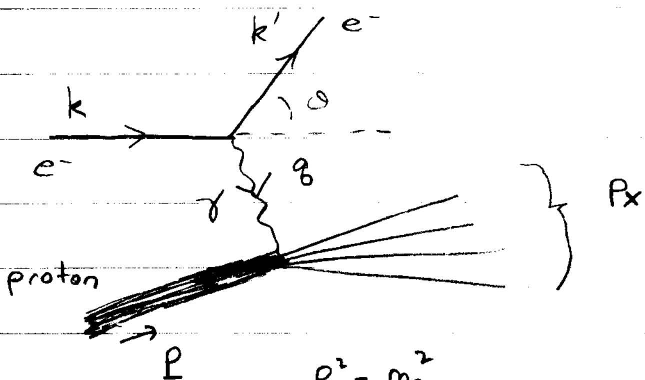
eg Hadron-Hadron

beam-thrust $\displaystyle \tau_B = \sum_k |\vec{p}_{kT}| \min\{ e^{-\eta_k}, e^{+\eta_k} \}$

(thrust along beam axis)

Our next goal is to understand the complications arising from initial state hadrons. To do so we'll study a classic QCD process

<u>DIS</u> Deep-Inelastic Scattering $e^- p \to e^- X$



$P^2 = m_p^2$

$k^2 = k'^2 = 0$

Kinematics $q = k - k' = P_X - P$

$q^2 = -Q^2$ space like, $Q^2 > 0$

$Q^2 = 2 k^0 k'^0 (1 - \cos\theta)$

$$y = \frac{q \cdot P}{k \cdot P} \underset{\substack{\uparrow \\ \text{proton} \\ \text{rest frame}}}{=} 1 - \frac{k^{0\prime}}{k^0} \quad \text{fractional energy loss of } e^-$$

$0 \le y \le 1$

$$x = \frac{Q^2}{2 P \cdot q} = \frac{Q^2}{2 m_p (k^0 - k^{0\prime})} > 0$$

all these variables can be measured with lepton info.

$$P_X^2 = (q + P)^2 = -Q^2 + \frac{Q^2}{x} + m_p^2 = \frac{Q^2(1-x)}{x} + m_p^2$$

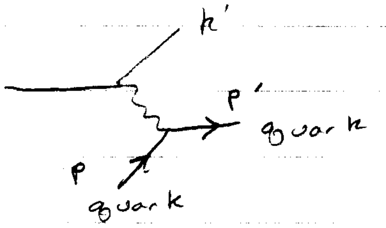
$P_X^2 \ge m_p^2$ so $0 \le x \le 1$

If $Q^2 \gg m_p^2 \sim \Lambda_{QCD}^2$ and $x \ne 1$, P_X^2 is large and proton is blown apart (deep inelastic)

Observed σ agreed with expectation for scattering from an elementary particle, which we now know are asymptotically free quarks.

atrix Element

$$iM(ep\to eX) = +ie^2\,\bar{u}(k')\gamma_\mu u(k)\,\frac{1}{Q^2}\underbrace{\int d^4x\, e^{iq\cdot x}\langle X|J^\mu_{em}(x)|P\rangle}_{Q_f\,\bar{u}(p')\gamma^\mu u(p)\quad \text{tree level}}$$



Let's see how far we can get without using quarks.

$$d\sigma = \frac{1}{2s}\int \frac{d^3k'}{2k_0'}\sum_X (2\pi)^4\,\delta^4(k+P-k'-P_X)\ \sum_{\text{pol.}}\ |M|^2$$

avg. over initial pol.

$$= \frac{2\pi e^4}{sQ^4}\int\frac{d^3k'}{2k^{o\prime}}\ \underbrace{L^{\mu\nu}(k,k')}_{\text{leptonic tensor}}\ \underbrace{W_{\mu\nu}(P,q)}_{\text{hadronic tensor}}$$

where $L^{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\not{k}'\gamma_\mu\not{k}\gamma_\nu] = (k_\mu' k_\nu + k_\mu k_\nu' - g_{\mu\nu}\,k\cdot k')$

and

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X (2\pi)^4\,\delta^4(q+P-P_X)\,\langle P|J^\mu(0)|X\rangle\langle X|J^\nu(0)|P\rangle$$

- current conservation implies $q^\mu W_{\mu\nu} = 0 = q^\nu W_{\mu\nu}$
- Parity $W^{\mu\nu}(P,q) = W^{\mu_P\nu_P}(P_P, q_P)$
- Time Reversal $W^{\mu\nu}(P,q) = W^{\mu_T\nu_T}(P_T,q_T)^*$
- Hermitian $J^+ = J$

$$W^{\mu\nu}(P,q)^* = W^{\nu\mu}(P,q)$$

Solution

$$W^{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)F_1 + \left(P_\mu - \frac{P\cdot q\, q_\mu}{q^2}\right)\left(P_\nu - \frac{P\cdot q\, q_\nu}{q^2}\right)\frac{F_2}{P\cdot q}$$

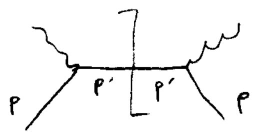
$F_{1,2} = F_{1,2}(x, Q^2)$ are dimensionless structure functions

they contain all hadronic effects.

$$\int \frac{d^3k'}{2k'} = \int \frac{dx\,dy}{(4\pi)^2}\, y\, s$$

$$\frac{d\sigma}{dx\,dy} = \frac{8\pi\alpha^2}{Q^4}\left[\left(\frac{1+(1-y)^2}{2}\right) 2xF_1 + (1-y)\left(F_2 - 2xF_1\right)\right]$$

<u>Tree-Level</u> with quarks



← Im. part of tree graph much easier than

$$W^{(0)}_{\mu\nu} = \frac{1}{8\pi}\int \frac{d^3k'}{2k^{0\prime}}\, Q_f^2\, \mathrm{tr}(\gamma_\mu \not{p}' \gamma_\nu \not{p})\, (2\pi)^4\, \delta^4(p'-p-q)$$

$$\ldots \Rightarrow \qquad F_1 = \frac{Q_f^2}{2}\,\delta(1-x) \quad , \quad F_2 = Q_f^2\,\delta(1-x)$$

<u>only</u> functions of x, not Q^2 (scaling)

Here we've assumed $p^\mu = P^\mu$, then $p'^2 = 0 = (P+q)^2 = 2P\cdot q - Q^2$
so $x = 1$. Too naive.

<u>Parton Model</u> in cm frame proton is energetic
$P^\mu = (P, 0, 0, P)$, and so are its constituents.
Let struck quark carry a fraction ξ of its momentum
$p^\mu \simeq \xi P^\mu$

now $\quad 2p\cdot q = Q^2 = 2\xi P\cdot q \quad$ so $\quad \xi = x$

$$\delta(1-x) \Rightarrow \delta(\xi - x)$$

(large $Q^2 \Leftrightarrow$ free quarks)

$0 \leq \xi \leq 1$

Probability of finding constituent "q" with mom. fraction ξ in proton p is $f_{q/p}(\xi)\,d\xi$

$$F_1 = \int d\xi\, \underbrace{f_{q/p}(\xi)}_{\substack{\text{hadronic info,}\\ \text{above calc } f^{\text{tree}}(\xi) = \delta(1-\xi)}}\; \underbrace{\frac{Q_f^2}{2}\,\delta(\xi - x)}_{\text{calc. in pert. theory}} = \frac{Q_f^2}{2}\, f_{q/p}(x)$$

↑ a function, but only x still!

$F_2 = \quad Q_f^2 \quad x \quad f_{q/p}(x)$

$\uparrow$ extra x because prefactor of F_2 in $W^{\mu\nu}$ went linearly with P & must convert $p = \xi P$

$F_2 = 2x F_1 \Rightarrow$ spin-½ quarks   Callan-Gross Relation

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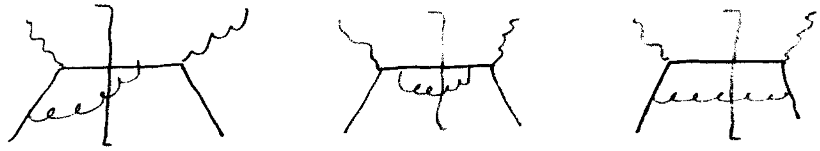
Lets Look at IR divergences at one-loop

V ≡ virtual

, ...

R ≡ real emission



$\nwarrow$ only this graph & its IR finite, corrects C.G.

Look at ① $p^\mu p^\nu W_{\mu\nu} = \frac{Q^2}{4x^2}\left(\frac{F_2}{2x} - F_1\right)$

$$= \frac{Q^2}{4x^2}\left(2x \; Q_f^2 \; \frac{\alpha_s(Q^2)\, C_F}{4\pi}\right) \leftarrow$$

② $-g^{\mu\nu} W_{\mu\nu}$ (all graphs)

$$\left(-g^{\mu\nu} W_{\mu\nu}^{(V)}\right) = -\frac{\alpha_s C_F \, Q_f^2}{\pi} \left(\frac{e^{\gamma_E} \bar{\mu}^2}{Q^2}\right)^{\epsilon} \left[\frac{(1-\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4\right) \delta(1-x)$$

dim. reg for IR again $\nearrow$

$$\left(-g^{\mu\nu} W_{\mu\nu}^{(R)}\right) = \frac{\alpha_s C_F}{2\pi} Q_f^2 \left(\frac{\bar{\mu}^2 e^{\gamma_E}}{Q^2}\right)^{\epsilon} \left(\frac{x}{(1-x)}\right)^{\epsilon} \left[\frac{(1-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]$$

$$* \left\{ -\frac{(1-\epsilon)}{\epsilon}\left[(1-x) + \frac{2x}{(1-x)(1-2\epsilon)}\right] + \frac{(1-\epsilon)}{2(1-2\epsilon)(1-x)} + \frac{2\epsilon}{1-2\epsilon} \right\}$$

For any $x \neq 1$ its obvious that IR divergences <u>do</u> <u>not</u> cancel in virtual + real!

In $W_{\mu\nu}^{(R)}$ the $1/\epsilon$ divergences come from k^μ_{gluon} collin to p^μ_{IN} & $\frac{1}{(1-x)}$ from k^μ collin p'^μ_{OUT} or $k^\mu \to 0$

To see what happens as $x \to 1$ we must treat result as a distribution. Recall $f^{tree} = \delta(1-x)$ was a dist'n.
Consider $\frac{1}{(1-x)^{1+\epsilon}}$ and integrate with test function $g(x)$:

$$\int_0^1 dx \frac{1}{(1-x)^{1+\epsilon}} \left[g(x) - g(1) + g(1) \right] = \frac{g(1)}{-\epsilon} + \int_0^1 dx \frac{[g(x) - g(1)]}{(1-x)} + \ldots$$

So
$$\boxed{\frac{1}{(1-x)^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(1-x) + \frac{1}{[1-x]_+} - \epsilon \left[\frac{\ln(1-x)}{1-x}\right]_+ + \cdots} \quad (*)$$

<u>Plus-function</u> is a distribution, much like δ-fn and can defined as
$$\frac{1}{(1-x)_+} = \lim_{\eta \to 0} \left[\frac{\theta(1-x-\eta)}{(1-x)} - \delta(1-x) \int_0^{1-\eta} \frac{dx'}{(1-x')} \right]$$

Note that $\int_0^1 dx \frac{1}{(1-x)_+} = 0$

Using $(*)$ $1/\epsilon^2\, \delta(1-x)$ term cancels in $-g^{\mu\nu}\left(W^{(R)}_{\mu\nu} + W^{(V)}_{\mu\nu}\right)$ and expanding in ϵ we're left with

$$(-g^{\mu\nu})\left(W^{(V)}_{\mu\nu} + W^{(R)}_{\mu\nu}\right) = \frac{\alpha_s}{2\pi} Q_f^2 \Bigg[-\frac{1}{\epsilon} P_{qq}(x) - \ln\left(\frac{\mu^2}{Q^2}\right) P_{qq}(x)$$
$$+ C_F \left\{ (1+x^2)\left(\frac{\ln(1-x)}{(1-x)}\right)_+ - \frac{3}{2}\frac{1}{(1-x)_+} + \text{more finite terms} \right\} \Bigg]$$

ie $P_{qq}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$ is called the

<u>quark</u> <u>splitting</u> <u>function</u>

When will $1/\epsilon_{IR}$'s cancel?

<u>Block-Nordsiek Mechanism</u> In QED virtual + real cancellation works to all orders in α when we sum over indistinguishable ("degenerate") final states. [masses regulate collinear sing.]

<u>Kinoshita-Lee-Navenberg Thm</u> More generally (theory with massless charged fields) transition rates are IR safe if we sum over degenerate initial & final states.

In DIS @ 1-loop, the remaining $1/\epsilon$ collin div. is due to an insufficient avg. over initial states, so we absorb it into

$$f_{i/p}(\xi)^{\text{partonic}} = \delta(1-\xi) - \frac{\alpha_s(\mu)}{2\pi\epsilon} P_{qq}(\xi) + \frac{\alpha_s}{\pi}(\cdots)$$

scheme dependent. In "$\overline{MS}$" $f_{i/p}$ we set $(\cdots)=0$.

& put rest in a perturbative coefficient

$$C_1\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right) = \frac{Q_f^2}{2}\left[\delta\left(1-\frac{x}{\xi}\right) - \frac{\alpha_s}{2\pi}\ln\frac{\mu^2}{Q^2} P_{qq}\left(\frac{x}{\xi}\right) + \frac{\alpha_s}{\pi}\tilde{C}_1\left(\frac{x}{\xi}\right)\right]$$

Log violation of scaling

where
$$F_1 = \int_x^1 \frac{d\xi}{\xi}\, f_{q/p}(\xi,\mu)\, C_1\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right)$$

<u>Factorization Theorem</u>

universal hadronic function, extract from data

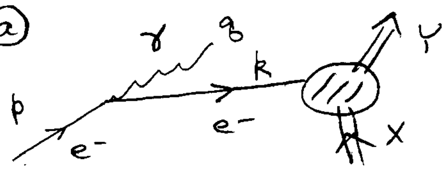
perturbatively calculable coefficient

Factorization states that <u>all</u> IR divergences (to any order in α_s) can be absorbed into $f_{q/p}(\xi)$ in this way.

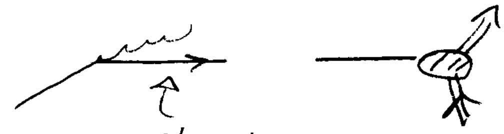
Parton Splitting & Evolution

Consider QED:

(a)



looks like

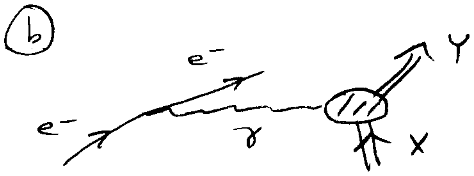


almost on-shell

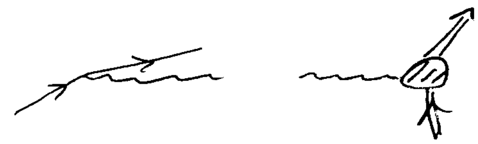
with $q \to p$, $k^2 \to 0$

$$p^\mu = (p, 0, 0, p), \quad q^\mu = (zp, 0, 0, zp) + \mathcal{O}\left(\frac{p_\perp^2}{p}\right)$$

(b)



looks like



and in fact

$$\sigma_{(b)} = \sigma(e^- X \to e^- Y) = \underbrace{\int_0^1 dz \, \frac{\alpha}{2\pi} \ln\frac{\mu^2}{m_e^2} \left[\frac{1+(1-z)^2}{z}\right]}_{} \sigma(\gamma X \to Y)$$

$f_{\gamma/e^-}(z)$ = prob. of finding γ of mom. fraction z inside e^-

$$\sigma_{(a)} = \sigma(e^- X \to \gamma Y) = \int_0^1 dx \, \frac{\alpha}{2\pi} \ln\frac{\mu^2}{m_e^2} \left[\frac{1+x^2}{1-x}\right] \sigma(e^- X \to Y)$$

here $x = 1 - z$, momentum fraction for e^-

$$f_{e/e}(x) = \delta(1-x) + \frac{\alpha}{2\pi} \ln\frac{\mu^2}{m_e^2} \left[\frac{1+x^2}{1-x} - A\,\delta(1-x)\right]$$

tree level — above, singular — e^- prob. removed from $\delta(1-x)$, virtual graph

$$\int_0^1 dx \, f_{e/e}(x) = 1$$

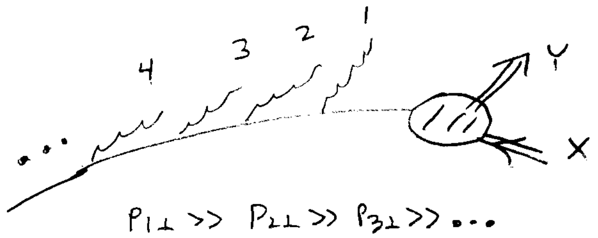
since γ radiation changes x, but doesn't remove the e^-

so $f_{e/e}(x) = \delta(1-x) + \frac{\alpha}{2\pi} \ln \mu^2/m_e^2 \; P_{ee}(x)$ splitting fn.

$$P_{ee}(x) = \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)$$

Multiple Splittings are described by evolution equation

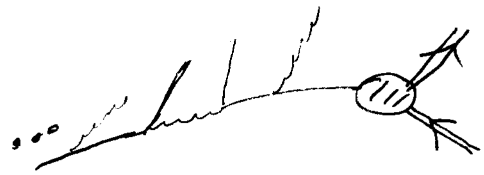
$$\mu \frac{d}{d\mu} f_{e/e}(x,\mu^2) = \frac{\alpha}{\pi} \int_x^1 \frac{dz}{z} \left[P_{ee}(z)\, f_e\left(\frac{x}{z},\mu^2\right) + P_{e\gamma}(z)\, f_\gamma\left(\frac{x}{z},\mu^2\right) \right]$$



$p_{1\perp} \gg p_{2\perp} \gg p_{3\perp} \gg \cdots$

from iterating P_{ee} term

&



mixed P_{ee}, $P_{e\gamma}$ term

sums large logs $\left(\frac{\alpha}{\pi}\right)^n \ln^n\left(\frac{\mu^2}{m^2}\right)$

QCD analog

quark in proton

$$\mu \frac{d}{d\mu} f_{q/P}(x,\mu) = \frac{\alpha_s(\mu)}{\pi} \int_x^1 \frac{dz}{z} \left\{ P_{qq}(z)\, f_q\left(\frac{x}{z},\mu\right) + P_{qg}(z)\, f_g\left(\frac{x}{z},\mu\right) \right\}$$

here we can't compute initial condition (value at small μ is nonperturbative input) but we can compute how $f_{q/P}$ changes as we include partons with larger $\perp$-momenta

Operator Product Expansion for DIS

We need to compute

$$W^{\mu\nu} = \frac{1}{4\pi} \sum_X (2\pi)^4 \, \delta^4(q + p - p_X) \langle p | J^\mu(0) | X \rangle \langle X | J^\nu(0) | p \rangle$$

We know how to do an OPE for

$$t^{\mu\nu} = i \int d^4x \, e^{iq\cdot x} \, T[\, J^\mu(x) J^\nu(0) \,]$$

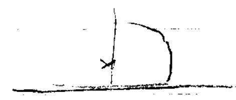
$$T^{\mu\nu} = \langle p | t^{\mu\nu} | p \rangle$$

Connect them:

$$T^{\mu\nu} = i \int d^4x \, e^{iq\cdot x} \left[\theta(x_0) \langle p | J^\mu(x) J^\nu(0) | p \rangle + \theta(-x_0) \langle p | J^\nu(0) J^\mu(x) | p \rangle \right]$$

$$= i \sum_X \int d^4x \, e^{iq\cdot x} \left[\theta(x_0) \underbrace{\langle p | J^\mu(x) | X \rangle}_{= e^{i(p - p_X)\cdot x} \langle p | J^\mu(0) | X \rangle} \langle X | J^\nu(0) | p \rangle + \theta(-x_0) (\cdots) \right]$$

$$\theta(x^0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, \frac{e^{i\omega x^0}}{\omega - i\epsilon}$$



$$\int d^3x \Rightarrow (2\pi)^3 \delta^3(\vec{q} + \vec{p} - \vec{p}_X)$$

$$\int dx^0 \Rightarrow (2\pi) \, \delta(\omega + q^0 + p^0 - p_X^0) \, ,$$

do $\int d\omega$

$$T^{\mu\nu} = \frac{-1}{2\pi} \sum_X (2\pi)^4 \frac{\delta^3(\vec{q} + \vec{p} - \vec{p}_X) \langle p | J^\mu(0) | X \rangle \langle X | J^\nu(0) | p \rangle}{q^0 + p^0 - p_X^0 + i\epsilon} + \begin{pmatrix} \theta(-x_0) \text{ induced} \\ \text{term} \end{pmatrix}$$

$$\mathrm{Im}\, T^{\mu\nu} = \frac{1}{2} \sum_X (2\pi)^4 \, \delta^4(q + p - p_X) \langle p | J^\mu | X \rangle \langle X | J^\nu | p \rangle$$

$$- \frac{1}{2} \sum_X (2\pi)^4 \, \delta^4(q - p + p_X) \langle p | J^\nu | X \rangle \langle X | J^\mu | p \rangle$$

$p_X^0 \geq p^0$, $q^0 > 0$ $\quad$ 2^{nd} δ never satisfied

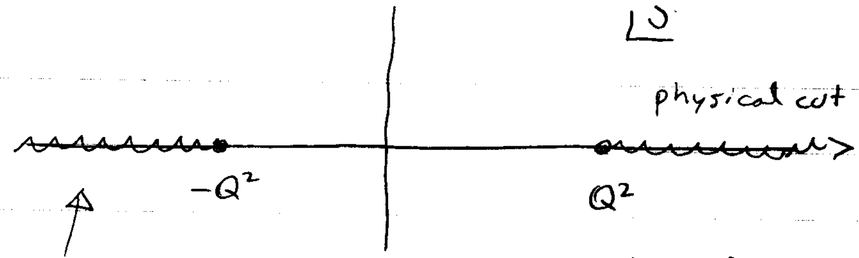
$$\boxed{\begin{aligned} \mathrm{Im}\, T^{\mu\nu} &= 2\pi \, W^{\mu\nu} \\ \mathrm{Im}\, T_{1,2} &= 2\pi \, F_{1,2} \end{aligned}}$$

(just the optical thm)

(same tensor decomposition for $T^{\mu\nu}$)

Analytic Structure of $T^{\mu\nu}$

let $\nu \equiv 2P\cdot q$



$(q,\mu) \leftrightarrow (-q,\nu)$

$T_2(\nu, Q^2) = T_2(-\nu, Q^2)$

this cut can be deduced via symmetry

$\nu = \dfrac{Q^2}{x}$

$0<x<1$

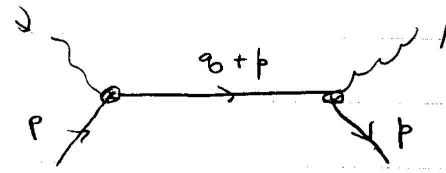
OPE for $T^{\mu\nu}$

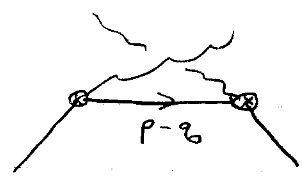
expand about $x_E \to 0$. symmetrize in $\mu \leftrightarrow \nu$

$W^{\{\mu\nu\}} = \frac{1}{2}(W^{\mu\nu} + W^{\nu\mu}) = W^{\mu\nu}$

$[W^{\mu\nu}] = 2$

$$\int d^4x\, e^{iq\cdot x}\left[\bar\psi\, \gamma^{\{\mu}\, \overbrace{\psi(x)\ \bar\psi(0)}\, \gamma^{\nu\}}\psi(0) \;+\; \overbrace{\bar\psi\,\gamma^{\{\mu}\psi(x)\ \bar\psi(0)\,\gamma^{\nu\}}\psi(0)}\right.$$





$$= \bar\psi(0)\,\frac{\gamma^{\{\mu}\, i\,(\not q + i\not\partial)\,\gamma^{\nu\}}}{(q+i\partial)^2}\,\psi(0) \qquad + \qquad (q \to -q)$$

$p \leftrightarrow i\partial$

$$= i\bar\psi\,\frac{(-1)}{Q^2}\underbrace{\sum_{n=0}^{\infty}\left(\frac{2iq\cdot\partial - \partial^2}{Q^2}\right)^n}_{\text{OPE expansion } \partial^2 \ll Q^2}\left(2\gamma^{\{\mu}i\partial^{\nu\}} - \underbrace{ig^{\mu\nu}\not\partial}_{\substack{0 \text{ by equations of motion}\\ \not\partial\psi=0}} + \underbrace{2\gamma^{\{\mu}q^{\nu\}}}_{\substack{\text{just builds}\\ \text{up tensor prefactor}}} - g^{\mu\nu}\not q\right)\psi$$

$+ (q \to -q)$

Which terms are needed?

consider

$$\bar\psi\ i\partial^{\mu_1}\cdots i\partial^{\mu_k}\,(\partial^2)^\ell\,\gamma^\alpha\,\psi$$

indices $\{\mu_1,\ldots,\mu_k,\alpha\} = \mu,\nu$ or are dotted with q's

k is odd ($q \to -q$ term kills k-even terms)

rearrange, and make operator gauge invariant

$$O^{d,s} \equiv \bar\psi\ S\left(\gamma^{\mu_1} iD^{\mu_2}\cdots iD^{\mu_s}\right)(D^2)^{\frac{1}{2}(d-s-2)}\,\psi$$

S ← decompose under Lorentz group, term we need has S = symmetric, traceless gives highest spin-s

$d = \text{dimension} = \frac{3}{2} + (s-1) + (d-s-2) + \frac{3}{2} = d$ ✓

To see which terms we need consider proton matrix elt.

$$\langle P | O^{d,s}_{\mu_1 \ldots \mu_s} | P \rangle = \underbrace{A^{d,s}}_{} \, 2 \, P^{\mu_1} \ldots P^{\mu_s}$$

dim -1 $\quad d \quad$ -1 $\quad \sim m_p^{d-s-2} \sim \Lambda_{QCD}^{d-s-2}$

indices $\mu\nu$ in $L^{\mu\nu}$ count as $q^\mu \sim Q$, so all vectors dotted into $O^{d,s}_{\mu_1 \ldots \mu_s}$ count as Q

$$T_1 \sim C^{\mu_1 \ldots \mu_s} \langle O^{d,s}_{\mu_1 \ldots \mu_s} \rangle$$

$$\sim \underbrace{\left(\frac{2 q \cdot P}{Q^2} \right)^s}_{\mathcal{O}(1) \text{ for } x \sim 1} \left(\frac{m_p}{Q} \right)^{d-s-2}$$

want "twist" $t \equiv d - s = 2$ for Leading term

$A^{2+s,s} \equiv A^s$

Leading Result

$$i \int d^4x \, e^{iq \cdot x} \, T \, J^\mu(x) J^\nu(0)$$

$P_\mu P_\nu$ term

$$= 4 \sum_{n=2}^{\infty} \frac{(2q^{\mu_1}) \cdots (2q^{\mu_{n-2}})}{(Q^2)^{n-1}} O^{t=2}_{\mu\nu\mu_1 \ldots \mu_{n-2}}$$

$$- g_{\mu\nu} \sum_{n=2}^{\infty} \frac{(2q^{\mu_1}) \cdots (2q^{\mu_n})}{(Q^2)^n} O^{t=2}_{\mu_1 \ldots \mu_n} + \mathcal{O}\left(\frac{M_p^2}{Q^2} \right)$$

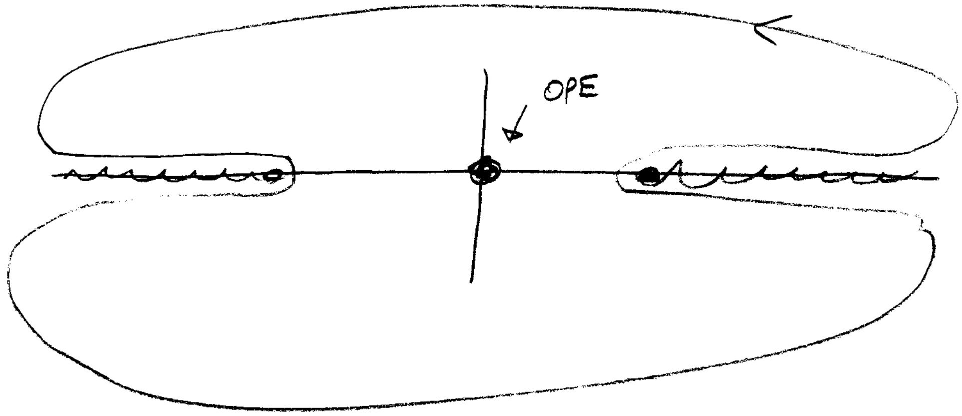
$$T_1 = Q_f^2 \sum_{n=2}^{\infty} 2 \left(\frac{2 q \cdot P}{Q^2} \right)^n A_f^n = Q_f^2 \underbrace{\sum_n \frac{2}{x^n} A_f^n}$$

flavor f $\qquad$ flavor f

function of x, just like $f_{q/p}(x)$ in parton model

Connect to Hadrons

Consider $\quad I_n \equiv \oint \frac{d\nu}{2\pi i} \frac{1}{\nu^{n+1}} T_1(\nu, Q^2)$



ν

$$I_n = Q_f^2 \frac{2}{Q^{2n}} A_f^n \qquad \text{residue thm}$$

$$= \underset{\substack{\uparrow \\ \text{same for} \\ \text{both cuts}}}{2} \int_{Q^2}^{\infty} \frac{d\nu}{2\pi i} \frac{1}{\nu^{n+1}} \underbrace{2i \operatorname{Im} T_1}_{\text{Disc. across cut}}$$

$$T_1(-\nu, Q^2) = T_1(\nu, Q^2)$$

$$(\operatorname{Im} T_1 = 2\pi F_1)$$

$$I_n = 4 \int_{Q^2}^{\infty} \frac{d\nu}{\nu^{n+1}} F_1(\nu, Q^2) = \frac{4}{Q^{2n}} \int_0^1 dx\, x^{n-1} F_1(x, Q^2)$$

so $\displaystyle\int_0^1 dx\, x^{n-1} F_1(x, Q^2) = \frac{Q_f^2}{2} A_f^n$

Trade moments $\{A_f^n\}$ for parton dist'n function $f_{q/p}(x)$

$$\circledast \quad \int_0^1 dx\, x^{n-1} f_{q/p}(x) \equiv A_f^n \qquad \text{OPE def'n}$$

then

$$F_1(x, Q^2) = \frac{Q_f^2}{2} f_{q/p}(x) \qquad \text{just as in parton model}$$

So we've derived parton model from OPE

The solution to $\circledast$ gives operator def'n for PDF

$$f_{q/p}(\xi) = \int \frac{dy}{2\pi} e^{-2i(\xi \bar{n}\cdot p)y} \langle p | \bar{\psi}(\bar{n}^\mu y) \underbrace{W(\bar{n}^\mu y, -\bar{n}^\mu y)}_{\substack{\text{Wilson line} \\ \text{along light-cone}}} \psi(-\bar{n}^\mu y) | p \rangle$$

$\bar{n}^2 = 0 \leftrightarrow$ traceless

Proof: with momentum space fields this says

$$f_{i/p}(\xi) = \langle p | \bar{\tilde{\Psi}} \tilde{W} \not{\bar{n}}\, \delta\!\left(\xi - \frac{\bar{n}\cdot\hat{\mathcal{P}}^{\dagger}}{\bar{n}\cdot p}\right) \tilde{W}^{\dagger} \tilde{\Psi} | p \rangle$$

momentum operator

Fourier transformed Wilson line

$$\int d\xi\, \xi^{n-1} f_{i/p}(\xi) \propto \underbrace{\bar{n}_{\mu_1} \cdots \bar{n}_{\mu_n}}_{\text{symm. traceless part}} \langle p | \bar{\psi} \underbrace{\gamma_{\mu_1} iD_{\mu_2} \cdots iD_{\mu_n} \psi}_{\mathcal{O}^{\text{twist-2}}} | p \rangle$$

$m_p^2 = 0, \quad \bar{n}^2 = 0$

Note: moments Af^n have anom. dimensions consistent with

$$\mu \frac{d}{d\mu} f_{i/p}(\xi,\mu) = \frac{\alpha_s(\mu)}{\pi} \int_{\xi}^{1} \frac{dz}{z} \Big[\underbrace{P_{qq}(z)}_{\text{splitting function}} f_{q/p}\!\left(\frac{\xi}{z},\mu\right) + P_{qg}(z) f_{g/p}\!\left(\frac{\xi}{z},\mu\right) \Big]$$

Summary & Final Comments

- The process $e^- p \to e^- X$ was not IR safe with our standard pert. theory with quarks & gluons, but these initial state IR divergences can be absorbed into $f_{i/p}(\xi)$, a function that describes prob. of finding parton "i" in proton "p" with momentum fraction ξ.

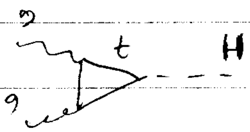
- For DIS we could justify the appearance of $f_{i/p}(\xi)$ with an OPE. We are not always so lucky. However even in more complicated situations we can still derive Factorization Theorems for IR effects analogous to

$$F_1(x,Q^2) = \int_x^1 \frac{d\xi}{\xi} f_{i/p}(\xi, \mu^2/\Lambda^2)\, C_1\!\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right)$$

[the simplest way to do this is to use the soft-collinear effective theory, see hep-ph/0202088]

2) Higgs Production $pp \to HX$ through two gluons

$$\sigma(pp \to HX) = \int dx_1\, dx_2\; f_{g/p}(x_1)\, f_{g/p}(x_2)\; \hat{\sigma}(g_1 g_2 \to HX)$$



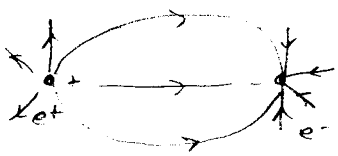
- The key point about $f_{i/p}(\xi)$ is that it is <u>universal</u>, it just describes properties of the proton. The same function occurs in many processes.

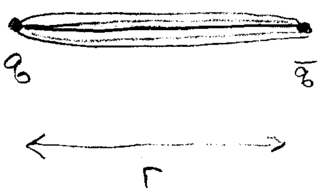
- Proofs of factorization are not always available and are sometimes just assumed.

Ch 8 Confinement

Confinement: isolated colored quarks are not observed,

ie long-dist color electric Coulomb flux from free quarks does not occur (& hence no e.m. flux for $Q = +\frac{2}{3}, -\frac{1}{3}$ either)

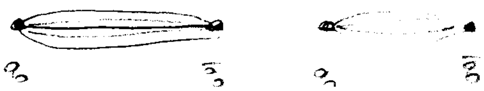
Rather than  $V_{ee}(r) \sim -\frac{e}{r}$, we picture



a color electric flux tube (string).
of width $a \sim \Lambda_{QCD}^{-1}$
energy density $\epsilon \sim tr(E^2)$

energy tube $\sim \epsilon a^2 r$ ie $V(r) \propto r$

As we increase r eventually tube breaks via configuration



being energetically favorable.

This suggest studying the dynamics of flux tubes, ie what prevents flux tube from spreading out in QCD.

Study <u>Wilson Loop</u> 

$$W(C) \equiv P \exp\left[i g \oint_C dx^\mu A_\mu \right]$$

P: path ordering

eg. path $x^\mu(s)$, $0 \le s \le 1$

$$W(C) = P \exp\left[i g \int_0^1 ds \frac{dx^\mu}{ds} A_\mu(x(s)) \right]$$

where $\quad P \; A_\mu(x(s))\, A_\nu(x(s')) = A_\mu^a(x(s))\, A_\nu^b(x(s')) \begin{cases} T^a T^b & s > s' \\ T^b T^a & s' > s \end{cases}$

$s = 0 \quad x = x(0)$ $\qquad$ if $\; x = y \;$ closed Wilson loop

$s = 1 \quad y = x(1)$ $\qquad$ $x \neq y \;$ Wilson line (as in DIS)

write $\; W(y, x)$

<u>Gauge Trnsfm</u> $\qquad U(x) = e^{i\alpha^a(x) T^a}$

$$W(y,x) \;\rightarrow\; U(y)\, W(y,x)\, U^+(x)$$

so

$\hat{W} = \mathrm{tr}\, W(c)$ is gauge inv. for closed loop

$\bar{\Psi}(y)\, W(y,x)\, \Psi(x)$ is " " too.

<u>Composition</u> $\qquad W(y,x)\, W(x, x_0) = W(y, x_0)$

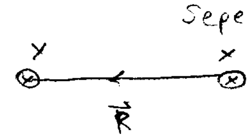
$\qquad W(y,x)\, W(x,y) = W(y,y) = \mathbb{1}$

<u>Unitary</u> $\qquad W(y,x)\, W^+(y,x) = \mathbb{1}$

<u>Eqtn. Motion</u> $\qquad d/ds \; W(x(s), y) = ig \dfrac{dx^\mu}{ds} A_\mu(x(s))\; W(x(s), y)$

ie $\qquad \dfrac{dx^\mu}{ds} D_\mu\, W(x,y) = 0 \qquad$ deriv. along path

$\bar{\Psi}(y)\, W(y,x)\, \Psi(x) = M(y,x)$ $\qquad$ Seperated quarks, gauge inv.



Consider

$$A(T,R) = \langle 0 |\, M^+(T,\vec{0}; T, \vec{R})\; M(0,\vec{0}; 0, \vec{R})\, | 0 \rangle$$
$$= \sum_n \left| \langle n |\, M(0,\vec{0}; 0,\vec{R})\, | 0 \rangle \right|^2 \; e^{-iE_n T}$$

in Euclidean time $\; e^{-E_n T} \;$ and least energetic state dominates for $T \to \infty$

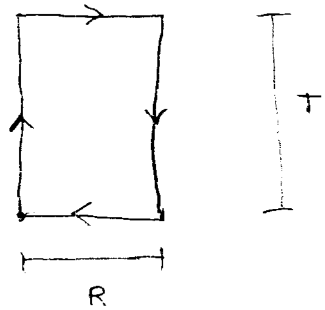
Let quarks be heavy, Mass $\to \infty$; then they act as classical color sources and

$$A(T,R) = \langle 0 | \, \mathrm{tr}\, W^{\dagger}(T,0;T,\vec{R})\, W(0,\vec{0};0,\vec{R}) | 0 \rangle$$

$$= \langle 0 | \hat{W}(c) | 0 \rangle$$

time-like loop in $A^0 = 0$ gauge

(actually any gauge)



So $\quad \langle 0 | \hat{W}(c) | 0 \rangle \xrightarrow[T\to\infty]{} e^{-V(R)\,T}$

$V(R)$ = static ptnl. energy

Confinement $\quad V(R) = K R \quad$ for large R

$$\langle 0 | \hat{W} | 0 \rangle \xrightarrow[\substack{T\to\infty \\ R\to\infty}]{} e^{-KRT} = e^{-K\,(\text{Area})}$$

Area Law

(Can generalize contour to large loop)

No Confinement, expect $V(R) \sim \mu$ a constant

$$\langle 0 | \hat{W} | 0 \rangle \to e^{-\mu T} \sim e^{-(\text{Perimeter})} \qquad (T \gg R)$$

non-local order parameter

In pure gauge QCD Area Law is confirmed by lattice QCD simulations.

With light quarks $A(T,R)$ is not $e^{-(\text{Area})}$ due to breaking of flux tube. Expect Area Law is sufficient but not necessary.

In fact in the Schwinger Model (1+1 electrodynamics)

$$W(c) = \exp\left(-\frac{e^2}{2} A\right)$$ area law when we leave out fermions. But when we add them

$$W(c) = \exp\left(-\frac{\pi\mu}{4} \underset{\uparrow\ \text{perimeter}}{P}\right)$$ where μ is γ-mass

However, even so, theory is <u>confined</u>. Theory is exactly solveable and only involves neutral bosons, which can be thought of as $\bar{\psi}\psi$ bound states.

<u>Wilson Loops & Fluxes</u>

$A^0 = 0$ gauge, canonical commutation relations

$$[A_i^a(\vec{x}), A_j^b(\vec{y})] = [E_i^a, E_j^b] = 0$$

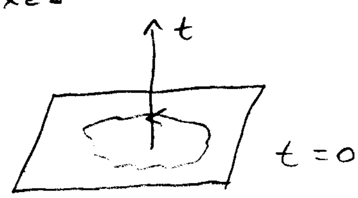
$$[A_i^a(\vec{x}), E_j^b(\vec{y})] = i\,\delta_{ij}\,\delta^{ab}\,\delta^3(\vec{x}-\vec{y})$$

$$U|\psi\rangle = |\psi\rangle \quad \text{as long as} \quad [D_i, E_i]|\psi\rangle = 0$$

Gauss' Law

Consider spacelike loop at fixed time

$$E_i^a(\vec{x}) = -i\frac{\delta}{\delta A_i^a(\vec{x})}$$ so



$$E_i^a(\vec{x})\,\hat{W}(c)\,|\psi\rangle = W(c)\,E_i^a|\psi\rangle - i\underbrace{\frac{\delta W(c)}{\delta A_i^a(\vec{x})}}\,|\psi\rangle$$

$$\mathrm{tr}\, P\, g \underbrace{\left[\frac{\delta}{\delta A_i^a(\vec{x})}\oint_c \vec{A}\cdot d\vec{l}\right]}_{\delta\text{-fn along loop}} e^{ig\oint \vec{A}\cdot d\vec{l}}\,|\psi\rangle$$

Thus action of Wilson loop is to create string of electric flux with infi width and unit strength still expect $\langle 0 | W(c) | 0 \rangle \sim e^{-(\text{Area})} \Rightarrow$ confinement

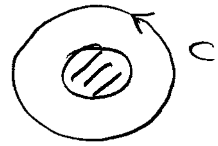
(by Euclidean Inv.)

Here $\langle 0 | \hat{W}(c) | 0 \rangle$ is overlap of vacuum with flux tube. With confinement the Electric Flux tube can only decay by collapse and is locally stable. It differs significantly from $\langle 0 |$ over whole area of loop, hence $e^{-(\text{Area})}$.

Comments

- Confinement can also be seen from studying Magnetic flux, again with Wilson lines

$$\Phi = \int \vec{B} \cdot d\vec{a} = \oint_C \vec{A} \cdot d\vec{l}$$



$$W(c) = e^{ig\Phi}$$

Here confinement is viewed as a consequence of a magnetically disordered vacuum, where random magnetic fields fluctuate freely (uncorrelated vortices $\Rightarrow \langle \hat{W} \rangle \sim e^{-\text{Area}}$). Magnetic Disorder $\Rightarrow$ Electric Confinement

- In pure gauge theory it was useful to reduce the problem of quark confinement to a question about stability of electric flux tubes. With dynamic quarks flux tubes are unstable, and things become more complicated.

Ch 9 SM Problems and Beyond the SM

Gauge Sector

$SU(3)_c \times SU(2)_L \times U(1)_Y$

- direct product with 3 couplings → unification to single gauge group & single coupling
- why is weak part chiral ?
- charge quantization (monopoles ?)
- strong CP problem → Peccei-Quinn symmetry & axion

Fermions

- 3 families
- Fermion mass hierarchy ⟹ Minimal Flavor violation
- CKM Hierarchy
- Neutrino masses, Majorana ? → See-Saw

Higgs

- Is simplest picture of e.w. symm. breaking with 1 Higgs correct ? → 2 Higgs doublets (5 Phys. Higgs) in minimal SUSY
- Naturalness, $M_H \sim v$?
- Complexity of Yukawa sector

Higgs Mass

Unitarity bound in SM:

consider long. gauge boson scattering, $A \sim M_H^2$

eg. $W_L^+ W_L^- \to W_L^+ W_L^-$ at high energy

$$A = -\frac{M_H^2}{v^2}\left(\frac{s}{s-M_H^2} + \frac{t}{t-M_H^2}\right)$$

$$= 16\pi \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\theta)\, a_\ell \quad \text{, part. wave decomp.}$$

$|a_\ell|^2 = \text{Im}\, a_\ell$

$\int_{-1}^{1} d\cos\theta \Rightarrow \int_{-s}^{0} dt$

Unitarity $\Rightarrow$ $|\text{Re}(a_\ell)| \le \frac{1}{2}$

$$a_0 = \frac{1}{16\pi s}\int_{-s}^{0} A\, dt = \frac{-M_H^2}{16\pi v^2}\left[2 + \frac{M_H^2}{s-M_H^2} - \frac{M_H^2}{s}\ln\left(1+\frac{s}{M_H^2}\right)\right]$$

for $M_H^2 \ll s$ $\wedge$ $|\text{Re}\, a_0| \le \frac{1}{2}$ gives $M_H \le 870$ GeV

if we try to decouple Higgs via $M_H \to \infty$, $M_H^2 \gg s$

then $a_0 \to \dfrac{-s}{32\pi v^2}$ and some type of

new physics is needed for $\sqrt{s} < 1.8$ TeV

Vacuum Stability: $\lambda \phi^4$, $M_H^2 = 2\lambda v^2$

P. Set #5: $\mu^2 \dfrac{d}{d\mu^2}\lambda = \dfrac{1}{16\pi^2}\left[12\lambda^2 - 12 y_t^2 + \cdots\right]$

top-yukawa dominates for small λ

need $\lambda(\mu) > 0$ for vacuum to remain stable

$\lambda(\mu) = \lambda(v) - \dfrac{3}{4\pi^2} y_t^2 \ln\left(\dfrac{\mu^2}{v^2}\right) + \ldots$, input M_H, get $\lambda(\mu) < 0$ at some scale (use 2-loop RGE)

Find $\underbrace{\mu \simeq 10^8 - 10^{12}}_{\text{uncertainty from } m_t,\ \alpha_s}$ GeV for $M_H = 124$ GeV

- LEP constraints on SM Higgs $e^+e^- \to H^0 Z$

  $M_H > 114.4$ GeV at 95% confidence

- Global electroweak fit:

  $M_H = 76^{+33}_{-24}$ GeV for $M_t = 170.9 \pm 1.8$ ($\Delta M_t = 2$GeV ⇒ $\Delta M_H/M_H = 15\%$)

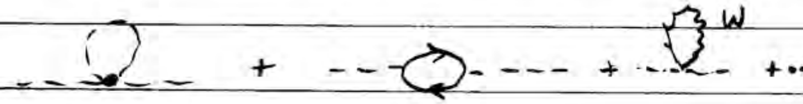
- Tevatron ($10fb^{-1}$) exclude $100 < M_H < 119$ GeV (95% CL)

  $141 < M_H < 184$ GeV

- CMS ($4.6 fb^{-1}$) exclude $127 < M_H < 600$ (95%), excess @ 124 GeV [3.1σ local, 1.5σ global, 2.1σ in 110-145]
- ATLAS ($4.9 fb^{-1}$) exclude $110 < M_H < 117.5$, $118.5 < M_H < 122.5$, $129 < M_H < 539$; excess @ 126 GeV 2.5σ local

Wilsonian RG:

Higgs mass is special, being the only dim-2 operator in SM (other's all dim-4).


are quadratically divergent

eg. $\Sigma(p) = 3\lambda \int^{\Lambda} \frac{d^4k}{k^2 - M_H^2} \Rightarrow M_H^2 = M_H^{0\,2} + \frac{3\lambda}{16\pi^2}\Lambda^2$

- no symmetry we know of in SM forbids quadratic term
- for large Λ we must fine tune M_H^0 to get small observed M_H (at each order in pert theory)

eg. if we introduce new GUT particles at $\Lambda_G \sim 10^{16}$ GeV then generically $M_H^2 \sim \Lambda_G^2$ without fine tuning

Solution:

new physics at TeV scale, where Higgs is Goldstone boson (light) as in little Higgs models or Supersymmetry where loops come in pairs and quadratic div. cancel for $\Lambda > \Lambda_{SUSY\ breaking}$.

Minimal Flavor Violation (MFV)

need $\Lambda^{new} \sim$ few TeV to stabilize M_H but generic flavor violating interactions at $\Lambda \simeq$ TeV are already ruled out by precision K & B data.

$\epsilon_K \Rightarrow \Lambda \gtrsim 10^4$ TeV, $\Delta M_{B_d} \Rightarrow \Lambda \gtrsim 10^3$ TeV, $\Delta M_{B_s} \Rightarrow \Lambda \gtrsim 10^2$ TeV

MFV $\leftrightarrow$ link flavor violating interactions to known SM Yukawa couplings so they can be pushed to higher scales.

This can be done <u>generically</u>:

SM has 2 $SU(2)_L$ doublets (Q_L, L_L), 3 singlets (u_R, d_R, e_R)

3 families.

Largest flavor group that commutes with gauge group is

$$G_F = U(3)^5 = \underbrace{(SU(3)_q)^3}_{SU(3)_{Q_L} \times SU(3)_{u_R} \times SU(3)_{d_R}} \otimes \underbrace{(SU(3)_\ell)^2}_{SU(3)_{L_L} \times SU(3)_{e_R}} \otimes \underbrace{U(1)_B}_{\text{baryon}} \otimes \underbrace{U(1)_L}_{\text{lepton}} \otimes \underbrace{U(1)_Y}_{\text{hypercharge}} \otimes U(1)_{PQ} \otimes U(1)_{E_R}$$

$U(1)_{B,L,Y}$ are respected by S.M. Yukawa's. <u>Rest broken</u>

Let $U(1)_{PQ}$ be simult. rotation to both d_R, e_R

$U(1)_{e_R}$ to e_R only.

Introduce Yukawa Spurion's Y_u, Y_D, Y_E transforming under $SU(3)_q^3 \otimes SU(3)_\ell^2$ as

$Y_u \sim (3, \bar{3}, 1)$ $\qquad$ $Y_E \sim (3, \bar{3})$

$Y_D \sim (3, 1, \bar{3})$

then $\mathcal{L} = \bar{Q}_L Y_D d_R H + \bar{Q}_L Y_u U_R H^c + \bar{L}_L Y_E e_R H + h.c.$
is invariant

Can rotate bkgnd values to

$$Y_D = \lambda_d \quad \Big\} \text{ diagonal} \qquad Y_u = V^\dagger_{CKM} \lambda_u$$
$$Y_L = \lambda_e$$

An EFT satisfies "MFV" if higher dim operators built from SM & Y fields is invariant under CP and G. Setting Y's to bknd values then ensures flavor dynamics is governed by SM Yukawa's

interesting operators have dim = 6

eg. $\mathcal{O}_0 = (\bar{Q}_L \lambda_{FC} \gamma_\mu Q_L)^2$ $\qquad \Delta m_{Bd}$

$\qquad \uparrow (Y_u Y_u^\dagger - \mathbb{1}) \qquad$ FCNC

Can also impose MFV on models like Supersymmetry

GUT's

should have simple group $G \supset SU(3) \times SU(2) \times U(1)$,
must allow complex reps, be compact (Lie Group) chg. quant.

SU(5) (minimal group that works)

$$\text{24 matrices } \lambda^a \; (5\times 5) \qquad \overset{\lambda^{4,\dots,11}}{\begin{pmatrix} SU(3) & \\ & 0 \end{pmatrix}}, \quad \overset{\lambda^{1,2,3}}{\begin{pmatrix} 0 & \\ & SU(2) \end{pmatrix}}, \quad \overset{\lambda^{0}}{\frac{1}{\sqrt{15}}\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}}$$

SU(3) × SU(2) decomp

Irr Reps:

Fund	5	$= (3,1) + (1,2)$	$(\bar{2}=2)$	
$\overline{\text{Fund}}$	$\bar{5}$	$= (\bar{3},1) + (1,2)$	$=$	$\bar{d}_R + L_L$
Anti Symm 5×5 ψ_{ij}	10	$= (\bar{3},1) + (3,2) + (1,1)$	$=$	$\bar{u}_R + Q_L + \bar{e}_R$

Let $$Y = -\sqrt{\frac{5}{3}}\,\lambda^0 = \begin{pmatrix} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1/2 & \\ & & & & 1/2 \end{pmatrix}$$

then
$$Y(\bar{d}_R) = -1/3 \qquad Y(\bar{u}_R) = 2 \times (-1/3)$$
$$Y(L_L) = 1/2 \qquad Y(Q_L) = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$
$$Y(\bar{e}_R) = 2 \times \left(\frac{1}{2}\right)$$

$\boxed{\text{all match}}$

$$Y(\psi_i) = Y_{ii}$$

$$Y(\psi_{ij}) = Y_{ii} + Y_{jj}$$

- eigenvalues of generators of simple non-abelian group are discrete
  $Q = \underbrace{\lambda_3 + Y}_{\text{generators}}$ so charge automatically quantized (unlike U(1))
- $\bar{5} + 10$ has no anomaly

Gauge bosons

adjoint $\quad 24 = (8,1) + (1,3) + (1,1) + (3,2) + (\bar{3},2)$

$$A^\mu_A \quad A^\mu_a \quad B^\mu \quad (X_\mu, Y_\mu), \quad \begin{pmatrix} X^\mu \\ Y^\mu \end{pmatrix}$$

$$K^\mu = A^\mu{}_A \frac{\lambda^A}{2}_{SU(3)} + W^\mu{}_a \frac{\sigma^a}{2} + B^\mu \underbrace{\lambda^0}_{\sqrt{3/5}\,Y} + \dots$$

$$g'/g = \sqrt{3/5} = \tan \theta_W$$

$$\sin^2\theta_W = \frac{3}{8} \qquad \text{prediction (at GUT scale)}$$

running down $\quad \sin^2\theta_W(\mu \sim v) \simeq 0.20$

not bad!

- Spont breaking

$$SU(5) \xrightarrow{\langle \phi \rangle} SU(3) \times SU(2) \times U(1) \xrightarrow{\langle H \rangle} SU(3) \times U(1)$$

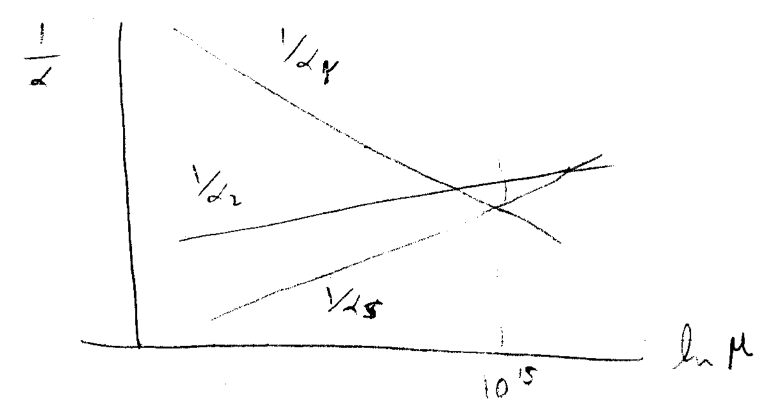
eg, adjoint higgs

$$\phi = \psi_i{}^j$$

$$\langle \phi \rangle = \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \quad \leftarrow \text{Pset!}$$

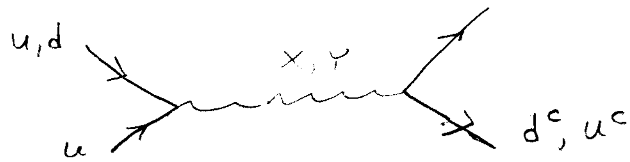
gives mass to X & Y bosons $\sim 10^{15}$ GeV

- Coupling Unification



Meet better in SUSY SU(5).

• Proton decay



$\Delta B = 1$

$p \to e^+ \pi^0$

minimal SU(5) ruled out by proton lifetime

– the end –