Lectures on Perturbative QCD

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2017

QCD is everywhere

Nuclear Physics
$(N, \pi, \dots)$

~~CP~~ & flavor

BSM search
& Precision SM
(Collider physics)

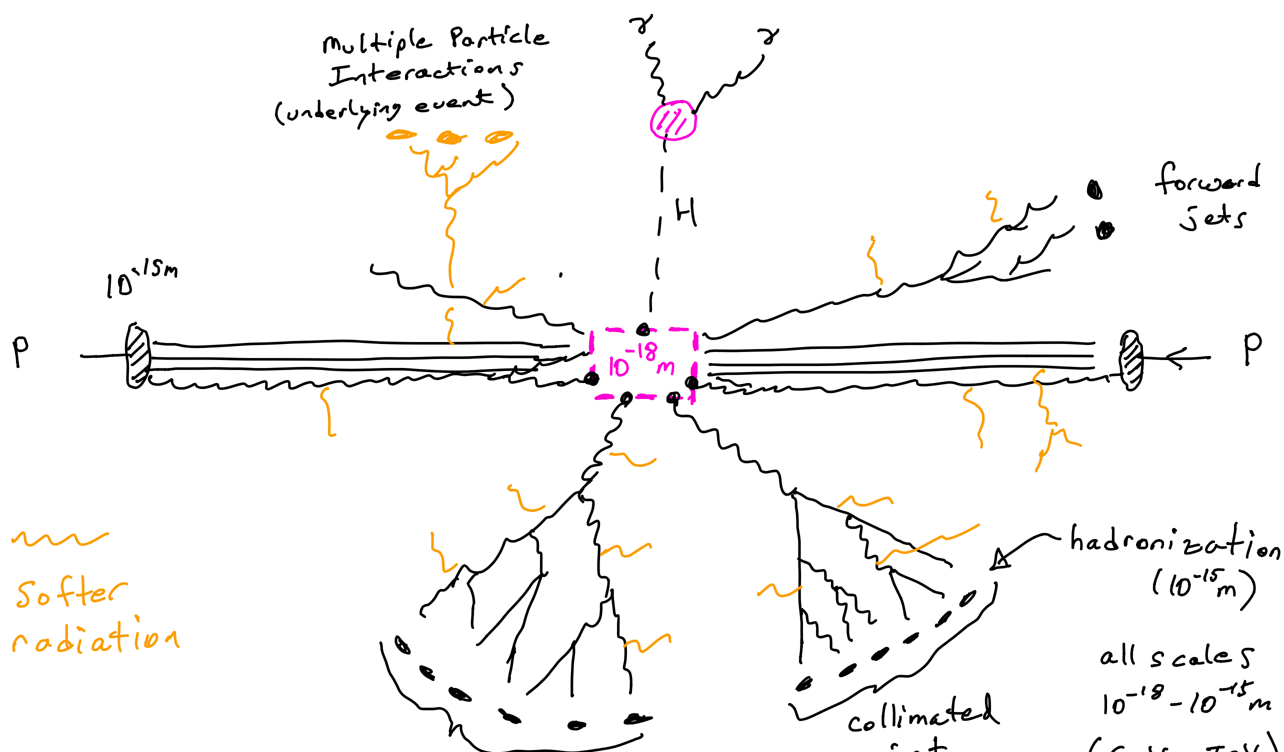
high Temperature
Heavy Ions

ν - Physics
(int. w Nuclei)

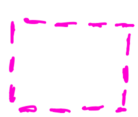
...

QFT
(prototypical non-abelian gauge thy)

LHC collision of two protons



all scales $10^{-18} - 10^{-15}$m (GeV - TeV) are probed

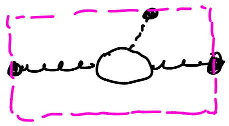


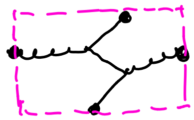
short distance process

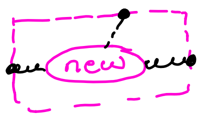
=

sum all diagrams

$gg \to H + gg$ $\quad (pp \to H + 2\text{ jets})$

or  $gg \to H$ $\quad (pp \to H + 0 \text{ jets})$

or  $gg \to q\bar{q}$ $\quad (pp \to 2 \text{ jets})$
$> 10^5$ more likely

or 

Concepts to Explore:

- Perturbative (& Non-Perturbative) interactions
  collide protons not q or g, pQCD applies at high E = short dist.
  but observe hadrons in jets
- Duality: quarks & gluons vs. hadrons
- Soft & Collinear Singularities in pQCD
- Jets & Shower of Radiation
- Factorization (very successful!)

$\boxed{\text{QCD}}$ SU(3) gauge theory # colors = $N_c = 3$

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \sum_{i = u,d,s,c,b,t} \bar{\psi}^\alpha_i (i\not{D} - m_i)^{\alpha\beta} \psi^\beta_i$$

quark field

$$+ \mathcal{L}_{\text{gauge fix}}$$

$\beta = 1,2,3$ fundamental rep.

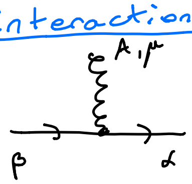
$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g f^{ABC} A^B_\mu A^C_\nu$$ gluon field

$A = 1, \dots, 8$ adjoint rep. $\qquad [T^A, T^B] = i f^{ABC} T^C$

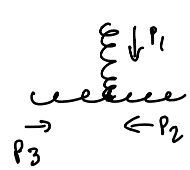
$$iD_\mu = i\partial_\mu - g A^A_\mu T^A$$

universal coupling

interactions:



$= -i g T^A_{\alpha\beta} \gamma^\mu$



$= -g f^{ABC} [g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3}$
$+ (1 \to 2 \to 3 \to 1)$
$+ (1 \to 3 \to 2 \to 1)]$

I. QCD SUMMARY

The $SU(N_c)$ QCD Lagrangian without gauge fixing

$$\mathcal{L} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}G^A_{\mu\nu}G^{\mu\nu A}\,, \qquad G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - gf^{ABC}A^B_\mu A^C_\nu \tag{1}$$
$$D_\mu = \partial_\mu + igA^A_\mu T^A\,, \qquad [D_\mu, D_\nu] = igG^A_{\mu\nu}T^A\,.$$

The equations of motion and Bianchi

$$(i\not{D} - m)\psi = 0\,, \qquad \partial^\mu G^A_{\mu\nu} = gf^{ABC}A^{B\mu}G^C_{\mu\nu} + g\bar\psi\gamma_\nu T^A\psi\,, \qquad \epsilon^{\mu\nu\lambda\sigma}(D_\nu G_{\lambda\sigma})^A = 0. \tag{2}$$

Color identites

$$[T^A, T^B] = if^{ABC}T^C\,, \qquad \mathrm{Tr}\big[T^A T^B\big] = T_F\delta^{AB}\,, \qquad \bar T^A = -T^{A*} = -(T^A)^T\,,$$
$$T^A T^A = C_F\mathbf{1}\,, \qquad f^{ACD}f^{BCD} = C_A\delta^{AB}\,, \quad f^{ABC}T^B T^C = \frac{i}{2}C_A T^A\,,$$
$$T^A T^B T^A = \Big(C_F - \frac{C_A}{2}\Big)T^B\,, \quad d^{ABC}d^{ABC} = \frac{40}{3}\,, \qquad d^{ABC}d^{A'BC} = \frac{5}{3}\delta^{AA'}\,, \tag{3}$$

where $C_F = (N_c^2-1)/(2N_c)$, $C_A = N_c$, $T_F = 1/2$, and $C_F - C_A/2 = -1/(2N_c)$. The color reduction formula and Fierz formula are

$$T^A T^B = \frac{\delta^{AB}}{2N_c}\mathbf{1} + \frac{1}{2}d^{ABC}T^C + \frac{i}{2}f^{ABC}T^C\,, \quad (T^A)_{ij}(T^A)_{k\ell} = \frac{1}{2}\,\delta_{i\ell}\delta_{kj} - \frac{1}{2N_c}\,\delta_{ij}\delta_{k\ell}\,. \tag{4}$$

Feynman gauge rules, fermion, gluon, ghost propagators, and Fermion-gluon vertex

$$\frac{i(\not{p}+m)}{p^2-m^2+i0}\,, \qquad \frac{-ig^{\mu\nu}\delta^{AB}}{k^2+i0}\,, \qquad \frac{i}{k^2+i0}\,, \qquad -ig\gamma^\mu T^A\,. \tag{5}$$

Triple gluon and Ghost Feynman rules in covariant gauge for $\{A^A_\mu(k), A^B_\nu(p), A^C_\rho(q)\}$ all with incoming momenta, and $\bar c^A(p)A^B_\mu c^C$ with outgoing momenta p:

$$-gf^{ABC}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big]\,, \qquad gf^{ABC}p^\mu\,. \tag{6}$$

Triple gluon Feynman rule in bkgnd Field covariant gauge $\mathcal{L}_{gf} = -(D^A_\mu Q^A_\mu)^2/(2\xi)$ for $\{A^A_\mu(k), Q^B_\nu(p), Q^C_\rho(q)\}$ with A^A_μ a bkgnd field:

$$-gf^{ABC}\Big[g^{\mu\nu}\Big(k-p-\frac{q}{\xi}\Big)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}\Big(q-k+\frac{p}{\xi}\Big)^\nu\Big]\,. \tag{7}$$

Lorentz gauge:

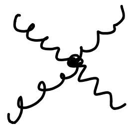
$$\mathcal{L} = -\frac{(\partial_\mu A^\mu)^2}{2\xi}\,, \qquad D^{\mu\nu}(k) = \frac{-i}{k^2+i0}\Big(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\Big)\,, \tag{8}$$

where Landau gauge is $\xi \to 0$. Coulomb gauge:

$$\vec\nabla\cdot\vec A = 0\,, \qquad D^{\mu\nu}(k) = \frac{-i}{k^2+i0}\left(g^{\mu\nu} - \frac{[g^{\nu0}k^0k^\mu + g^{\mu0}k^0k^\nu - k^\mu k^\nu]}{\vec k^2}\right),$$
$$D^{00}(k) = \frac{i}{\vec k^2 - i0}\,, \qquad D^{ij}(k) = \frac{i}{k^2+i0}\Big(\delta^{ij} - \frac{k^ik^j}{\vec k^2}\Big). \tag{9}$$

Running coupling with $\beta_0 = 11C_A/3 - 4T_Fn_f/3 = 11 - 2n_f/3$:

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_0)\ln\frac{\mu}{\mu_0}} = \frac{2\pi}{\beta_0\ln\frac{\mu}{\Lambda_{\rm QCD}}}\,, \qquad \frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_s(\mu_0)} + \frac{\beta_0}{2\pi}\ln\frac{\mu}{\mu_0}\,. \tag{10}$$

 $= \cdots$

$T^A T^A = C_F \mathbb{1}$ $\quad C_F = 4/3$ "quark color charge"

$f^{ACD} f^{BCD} = C_A \delta^{AB}$ $\quad C_A = 3$ "adj. color charge"

covariant gauge (Fadeev-Popov)

$$\mathcal{L}_{\substack{\text{gauge}\\\text{fix}}} = -\frac{1}{2\xi}\left(\partial_\mu A^{\mu A}\right)^2 + \bar{c}^A\left(-\partial_\mu D^\mu_{AB}\right) c^B$$

ghosts, anti-commuting Lorentz scalars

needed to invert $\mathcal{L}$ to obtain gluon propagator

path integral integrates over gauge inv. configurations (2 gluon polarizations), turned into $\mathcal{L}_{\substack{\text{gauge}\\\text{fix}}}$ by Fadeev-Popov procedure

- we'll use Feynman gauge $\xi = 1$,  $= \dfrac{-i g^{\mu\nu}\delta^{AB}}{p^2 + i0}$

Running Coupling

In QCD resolution scale μ of a process is very important

$$\alpha_s = \frac{g^2}{4\pi} = \alpha_s(\mu)$$

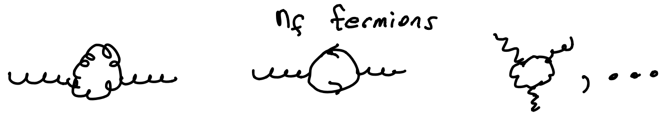
parameters in QFT defined by $\overline{MS}$ renormalization scheme here

scheme parameter μ $\qquad \alpha_s^{\text{bare}} = Z_\alpha \mu^{2\epsilon} \alpha_s(\mu)$

$1/\epsilon$ poles

(4 indep. ways to compute β)

n_f fermions



$$\mu \frac{d}{d\mu} \alpha_s^{(n_f)}(\mu) = -\frac{\beta_0^{(n_f)}}{2\pi}\left[\alpha_s^{(n_f)}(\mu)\right]^2 + \cdots$$

β-function $\qquad \beta_0^{(n_f)} = \frac{11}{3} C_A - \frac{2}{3} n_f$

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_0)\ln\frac{\mu}{\mu_0}} = \frac{2\pi}{\beta_0 \ln\left(\frac{\mu}{\Lambda_{QCD}}\right)},$$

$\Lambda_{QCD} \sim 250\,\text{MeV}$

dimensional transmutation

- processes with physical scale s will involve $\alpha_s(\mu)\ln\frac{\mu}{s}$ terms, so we pick $\mu \simeq s$ to avoid large logs

$\Rightarrow$ more than one scale s_i $\Rightarrow$ more than one relevant $\alpha_s(\mu_i)$

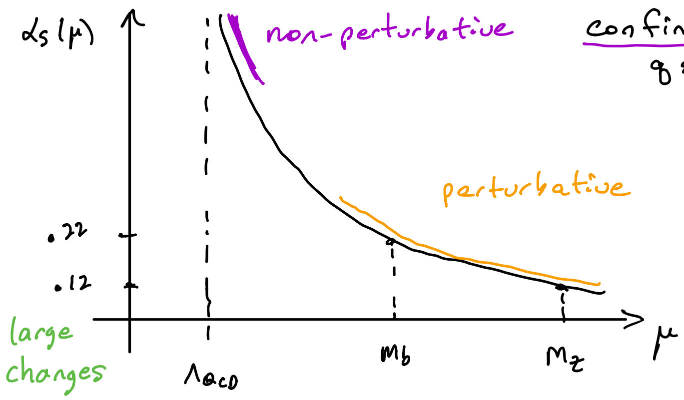
precisely what happens if both pert & non-pert physics is involved

• heavy particles decouple

$$m_t \;\Big|\; \begin{matrix}\alpha_s^{(6)}(\mu) \\ \alpha_s^{(5)}(\mu)\end{matrix}$$

continuous at $\mu = m_t$ (at this order)

If this is unfamiliar see [Hmwk] http:www2.lns.mit.edu/~iains/registerEFTx Chapter 4



confinement, α_s so strong
q & g are confined in hadrons

qqq baryons $\qquad q\bar{q}$ mesons

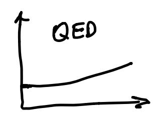
$M_{proton} \propto \Lambda_{QCD}$

$(\text{size}) \propto \Lambda_{QCD}^{-1}$

Lattice (Giusti)

$\alpha_s(\mu \to \infty) = 0$ Asymptotic freedom
free quarks at short dist.

Physical Picture: large magnetic moments of charged spin-1 gluons make vacuum paramagnetic, screen mag. charge, antiscreen electric chg.



Factorization

key tool to calculate cross sections is the ability to independently consider different parts of the process

$$d\sigma \sim \begin{pmatrix}\text{Prob. for} \\ \text{gluons taken} \\ \text{from protons}\end{pmatrix} \begin{pmatrix}\hat{\sigma}(gg \to H), \\ \hat{\sigma}(gg \to Hg), \\ \ldots\end{pmatrix} \begin{pmatrix}\text{Prob. for gluons} \\ \text{to produce} \\ \text{jets}\end{pmatrix}$$

Another key idea is to exploit inclusive observables

$e^+e^- \to X$ (any hadrons)

$e^- p \to e^- X$ DIS

eg. Higgs Production via gluon fusion

$pp \to H + X_{g}$ (any hadrons or 0+1+2+... jets)

$$\sigma = \int dx_a\, dx_b\, \underbrace{f_g(x_a,\mu)}\, f_g(x_b,\mu)\, \boxed{\hat{\sigma}_{gg\to H+X}(x_a,x_b,\mu,m_H)} \;\; * \; (1)$$

universal   parton dist'n function (PDF)   discuss μ later (distinct time scales)

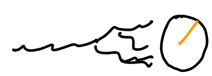
f_g = Prob. of finding g in proton with momentum fraction x_a = Probability Density (proton snapshot)

$(1) = \sum_i \text{Prob}(i)$ sum over everything that can happen to final state quarks & gluons, so we are not sensitive to this dynamics (jets etc)

Practical Limits on $\sum_i$ → cuts on jets to control background or enhance signals ($\geq N$ jets SUSY)

→ need for more exclusive events to determine expt. efficiencies etc.

Still sum over dynamics inside the jet & characterize it by a few variables: jet momentum $p_J^\mu = \sum_{i\in J} p_i^\mu$

angular size  R

$e^+e^- \to X$ (hadrons)

$e^+e^- \xrightarrow[\gamma^*]{} q\bar{q},\ q\bar{q}g, \ldots$

massless quarks
ignore Z exchange

Tree Level: 

Often normalize to $e^+e^- \to \mu^+\mu^-$ "R-ratio"

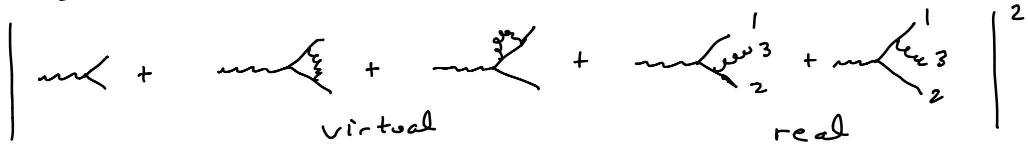
$$\sigma_0 = \frac{4\pi\alpha_{em}^2}{3q^2} N_c \sum_i Q_i^2$$

Q_i = active quark E&M charge $= \frac{2}{3}, -\frac{1}{3}$

$q^2 \geq m_i^2$

→ $\sum_i$ over active/massless quarks, transitions at quark thresholds

$\mathcal{O}(\alpha_s)$:

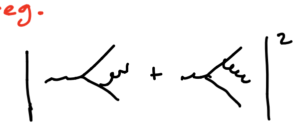


$$\sigma = \sigma_0 \left(1 + \hat{\sigma}_V + \hat{\sigma}_R \right)$$

Real first:

$d-1$ $\quad$ d $\quad$ for dim. reg.

$$\int d\Phi_3 \left| A^{real} \right|^2 = \int \prod_{i=1}^{3} \frac{d^3 p_i}{2 p_i^0} (2\pi)^4 \, \delta^{(4)}(q - p_1 - p_2 - p_3)$$



$\uparrow \; p_i^2 = 0 \;, \; p_i^0 = |\vec{p}_i|$

cm frame: $\quad q = (Q, \vec{0}) \qquad x_i \equiv \dfrac{2 q \cdot p_i}{q^2} = \dfrac{2}{Q} p_i^0 \qquad$ energy fractions $0 \le x_i \le 1$

$$x_1 + x_2 + x_3 = 2$$

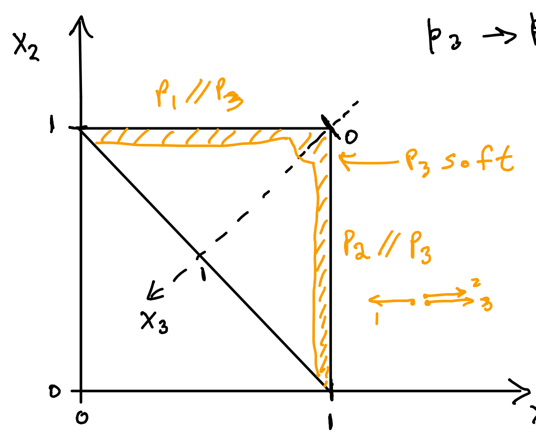
$$p_1^2 = 0 = (q - p_2 - p_3)^2 \implies 2 p_2 \cdot p_3 = Q^2 (x_2 + x_3 - 1) = Q^2 (1 - x_1)$$
$$= 2 E_2 E_3 (1 - \cos\theta_{23})$$

get
$$\int_0^1 dx_1 \, dx_2 \, dx_3 \; \frac{\delta(2 - x_1 - x_2 - x_3)}{(1-x_1)^\epsilon (1-x_2)^\epsilon (1-x_3)^\epsilon} \left[\frac{x_1^2 + x_2^2 - \epsilon x_3^2}{(1-x_1)(1-x_2)} \right]$$

IR divergences : $\quad p_3 \to 0 \quad$ soft gluon $\quad x_3 \to 0 \;$ so $\; x_1 \,\&\, x_2 \to 1$

$p_3 \to p_1 \quad g$ collinear $q \qquad p_1 \cdot p_3 = 0 \,, \; x_2 \to 1 \; (\theta_{13} \to 0)$

$p_3 \to p_2 \quad g$ collinear $\bar{q} \qquad p_2 \cdot p_3 = 0 \,, \; x_1 \to 1 \; (\theta_{23} \to 0)$



IR singularities at edges of phase space

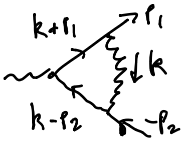
Regulate with dimensional regularization $\quad d = 4 - 2\epsilon$

These are limits where we can't resolve the partons like 2-jets , rest 3-jets

KLN Thm: singularities cancel if we sum over degenerate states

IR divergenes cancel with virtual graphs

eg.

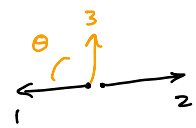


most IR singular integral

$$\int \frac{d^dk \; p_1\cdot p_2}{k^2 (k+p_1)^2 (k-p_2)^2} \underset{\text{soft}}{\overset{k\to 0}{\sim}} \int \frac{d^4k \; p_1\cdot p_2}{k^2 \, p_1\cdot k \; p_2\cdot k} \sim \frac{d^4k}{k^4}$$

IR singular

also collinear limits: $k \to p_1$, $k \to p_2$ IR singular



eg soft integrand

$$\int \frac{dk^0 \, d^{d-1}k \qquad E_1 E_2}{(k^0 - |\vec{k}| + i0)(k^0 + |\vec{k}| - i0) \, E_1(k^0 - |\vec{k}|\cos\theta) \, E_2(k^0 + |\vec{k}|\cos\theta)}$$

$$\sim \int \frac{d^{d-1}k}{|\vec{k}|^3 (1-\cos^2\theta)} \sim \underbrace{\int_0 \frac{dk}{k} k^{-2\epsilon}}_{\substack{\text{soft IR} \\ 1/\epsilon}} \underbrace{\int_{-1}^{1} \frac{d\cos\theta \; (\sin\theta)^{-2\epsilon}}{(1-\cos^2\theta)}}_{\substack{\text{collinear IR} \\ 1/\epsilon}}$$

<u>Full results</u>

$$\hat{\sigma}_V = \frac{\alpha_s(\mu) C_F}{\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \frac{\cos(\pi\epsilon) e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - 4\right)$$

$$\hat{\sigma}_R = \frac{\alpha_s(\mu) C_F}{\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \frac{\cos(\pi\epsilon) e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4}\right)$$

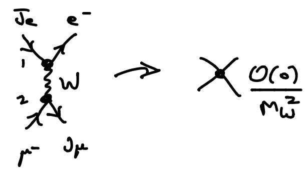
Here: $\sigma_B \to \sigma_B$

$$\sigma = \sigma_0 \left(1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi}\right) \quad \text{IR finite}$$

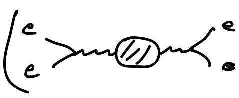
- At next order we find $\alpha_s^2 \ln(\mu^2/Q^2)$ so $\mu^2 \simeq Q^2$ is good scale choice
- Can we really compare q & g calculation with hadronic cross-section? (eg. # particles ~ 30 not 2 or 3)
- What happens if we restrict real radiation?

Operator Product Expansion for $e^+e^- \to$ hadrons

OPE : $\quad J_1(x)\, J_2(0) \xrightarrow{x\to 0} \sum_n C_n(x)\, O_n(0)$



unitarity (optical Thm)

$$\sigma = \frac{1}{2q^2}\, Im\Big($$  $$\Big)$$

$$= -\frac{(4\pi\alpha)^2}{q^2}\, Im\, \Pi_h(q^2)$$

$Q_i\, \bar\psi_i \gamma_\mu \psi_i$ EM quark current

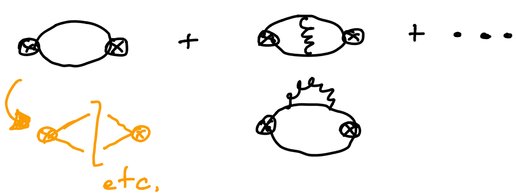
$$\Pi_h(q^2) = \underset{-2}{\frac{i}{3q^2}} \underset{-4}{\int d^4x}\; e^{iq\cdot x} \langle 0 |\, T\; \underset{+3}{J^\mu(x)}\; \underset{+3}{J_\mu(0)}\, | 0 \rangle$$

$\boxed{IF}$ dominated by short distances $x \to 0$

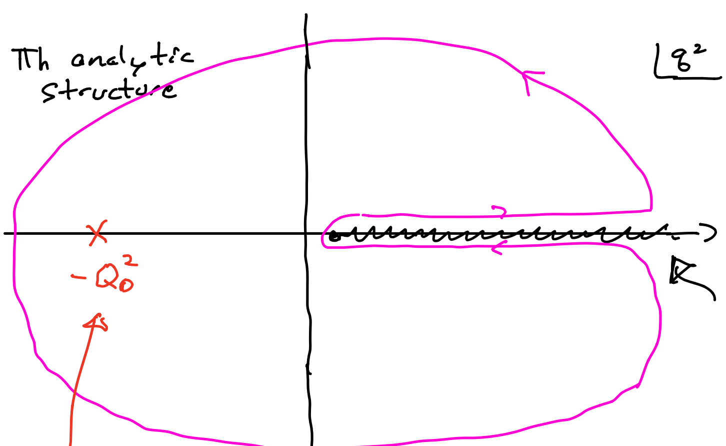
$$= -\Big[\, \underset{0}{C^1(q^2)}\, \mathbb{1} + \underset{-4}{C^{q\bar q}(q^2)}\, \underbrace{m\bar\psi\psi}_{m\Lambda_{QCD}^3} + \underset{-4}{C^{G^2}(q^2)}\, \underbrace{G^{\mu\nu}G_{\mu\nu}}_{\Lambda_{QCD}^4} + \cdots \Big]$$

$\langle 0 | \mathbb{1} | 0 \rangle = 1$ $\qquad$ suppressed

$Im\, C^1(q^2)$: $\quad Im$



reproduces q & g pQCD calculation



Want $\Pi_h(q^2)$ for large time like q^2 where dominated by high E int. states with many hadrons

OPE valid for spacelike points $x_E \to 0$

$$\oint \frac{dq^2}{2\pi i}\, \frac{\Pi_h(q^2)}{(q^2+Q_0^2)^2} = \frac{d\Pi_h}{dq^2}\Big|_{q^2=-Q_0^2}$$

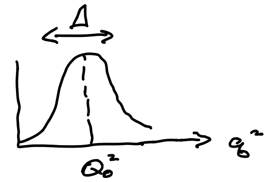
result we can calculate with OPE & pQCD

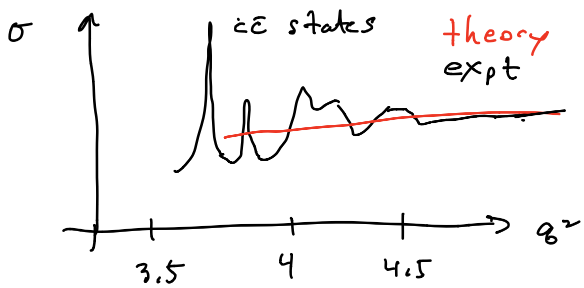
$$= \int \frac{dq^2}{2\pi i}\, \frac{1}{(q^2+Q_0^2)^2}\, \underbrace{\mathrm{Disc}}_{2i\, Im\, \Pi_h}\, \Pi_h(q^2)$$

$$= \frac{-1}{(4\pi\alpha)^2} \int_{4m_\pi^2}^{\infty} \frac{dq^2}{\pi} \frac{q^2}{(q^2+Q_0^2)^2}\, \sigma(q^2) \leftarrow \text{smeared hadronic cross-section}$$

<u>Moral</u>: Need to average over enough states to get agreement between hadronic & pQCD results "quark-hadron duality"

Other smearing functions are possible

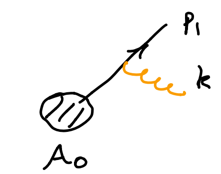




At large q^2 more states in given Δq^2 & can consider unaveraged comparison "local duality"

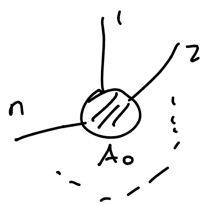
If we put cuts on phase space: $\frac{1}{\epsilon}$ poles in $\hat{\sigma}_R$ accompanied by logs of cutoff parameters.

Soft Approximation (Eikonal)



$$\underbrace{\bar{u}_1 \left(-ig \not\epsilon^a T^a\right) \frac{i(\not p_1 + \not k)}{(p_1+k)^2}} A_0 \qquad \bar{u}_1 \not p_1 = 0 \,,\; p_1^2 = 0$$

$$\simeq g \, \frac{p_1 \cdot \epsilon^a}{p_1 \cdot k} T^a \qquad \text{Eikonal amplitude}$$



$$= \sum_i \frac{g \, p_i \cdot \epsilon^a \, \hat{T}_i^a}{p_i \cdot k} A_N$$

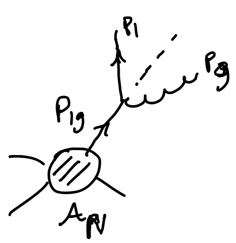
Factorizes with non-trivial color correlation

Applied to $\hat{\sigma}_R$ gives $\frac{p_1 \cdot p_2}{p_1 \cdot p_3 \; p_2 \cdot p_3}$ integrand, using $\int_0^\delta dx_3$, $\delta \ll 1$

$$\hat{\sigma}_R^{soft} = \frac{C_F \alpha_s}{\pi} \left[\frac{1}{\epsilon^2} - \frac{2}{\epsilon} \ln \delta + 2 \ln^2 \delta + \text{finite} \right]$$

↑ reproduces $1/\epsilon^2$

Collinear Approximation



$p_{1g} = p_1 + p_g$

$p^\mu = \left(|\vec{p}_{1g}|, \vec{p}_{1g}\right)$

$\bar{n}^\mu = \left(1, -\frac{\vec{p}_{1g}}{|\vec{p}_{1g}|}\right)$

Sudakov decomposition:

fixed by $p_1^2 = 0$

$$p_1^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{\bar{n}^\mu}{2 \bar{n} \cdot p}$$

$p_g^2 = 0$

$$p_g^\mu = (1-z) p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z} \frac{\bar{n}^\mu}{2 \bar{n} \cdot p}$$

p^μ & $\bar{n}^\mu$ light like $p^2 = \bar{n}^2 = 0$

$$z = \frac{p_1^0}{p_{1g}^0} \quad \text{energy fraction} \,, \qquad p_{1g}^2 = \frac{-k_\perp^2}{z(1-z)}$$

consider $k_\perp \to 0$: $\quad p_{13}^2 \to 0$ approx. on-shell

$$\left|A_{N+1}(p_g, p_1, \dots)\right|^2 \simeq \left[\frac{2 C_F}{p_{1g}^2} g^2 P_{qq}(z, \epsilon) \right] \left| A_N(p_1 + p_g, \dots) \right|^2$$

Amplitude factorizes into lower-pt amplitude times splitting function

$$P_{qq}(z,\epsilon) = \left(\frac{1+z^2}{1-z} - \epsilon(1-z)\right)$$

Applied to $\hat{\sigma}_R$ for $p_1 \parallel p_3$, $z = 1-x_3$ with

$$\int_{1-\delta_c}^{1} dx_2 \int_{\delta}^{1} dx_3 \qquad \text{gives}$$

collinear region   avoid soft region

$$\hat{\sigma}_R^{p_1 \parallel p_3} = \frac{C_F \alpha_s}{\pi} \frac{1}{2} \left[\frac{3}{2\epsilon} + \frac{2\ln\delta}{\epsilon} - \ln^2\delta - \frac{3}{2}\ln\delta_c - 2\ln\delta\ln\delta_c + \text{finite}\right]$$

will be doubled by adding $p_2 \parallel p_3$ reproduces $1/\epsilon$ Cancels with $\hat{\sigma}_R^{soft}$

Gives us an idea what a jet cross-section looks like

$$\sigma_{1\text{-loop}}^{total} = \sigma_{2\text{-jet}} + \sigma_{3\text{-jet}}$$

$\frac{1}{\epsilon_{IR}}$ cancel

$$\sigma_{2\text{-jet}} = \sigma_0 \left(1 + \hat{\sigma}_V + \hat{\sigma}_R^{soft} + \hat{\sigma}_R^{p_1 \parallel p_3} + \hat{\sigma}_R^{p_2 \parallel p_3} \right)$$

$$= \sigma_0 \left(1 + \frac{\alpha_s C_F}{\pi} \left[-2\ln\delta\ln\delta_c + \ln^2\delta - \frac{3}{2}\ln\delta_c + \text{finite}\right]\right)$$

$$\sigma_{3\text{-jet}} = \sigma_{1\text{-loop}}^{total} - \sigma_{2\text{-jet}}$$

"Exclusive Jets"

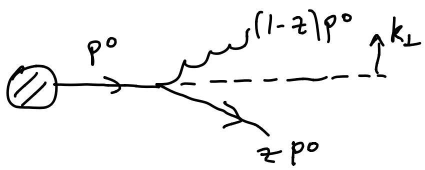
(analogous to 1st jet defn, Sterman-Weinberg Jet, 1977)

Inclusive Jets: ask for 1 jet in region away from edge of phase space, then $\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$ fine as leading result without considering δ, δ_c

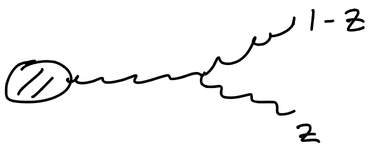
[Weinberg Story]

Jets   Why does QCD produce Jets?

log enhancement from collinear singularities

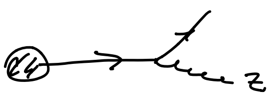


$$\sigma \propto \frac{\alpha_s C_F}{\pi} \frac{dk_\perp}{k_\perp} dz\, P_{qq}(z)$$

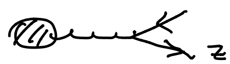


$$\sigma \propto \frac{\alpha_s C_A}{\pi} \frac{dk_\perp}{k_\perp} dz\, P_{gg}(z)$$

$$P_{gg}(z) = \frac{z}{1-z} + \frac{1-z}{z} + z(1-z)$$



$C_F P_{gq}(z)$ ← $\frac{1+(1-z)^2}{z}$



$\frac{1}{2} P_{qg}(z)$ ← $z^2+(1-z)^2$

prefer to split in collimated manner

Soft singularity also plays a role. Here the fact that soft gluons are prefferentially emitted within cone of collinear emissions (angular ordering) plays a role.

Leading contribution is strongly ordered

$$k_{1\perp} \gg k_{2\perp} \gg k_{3\perp} \cdots \gg k_{n\perp} \sim \Lambda_{QCD}$$

If $\alpha_s \ln\left(\frac{k_{i\perp}}{k_{i+1\perp}}\right) \sim 1$ no perturbative suppression

How do we define a Jet?

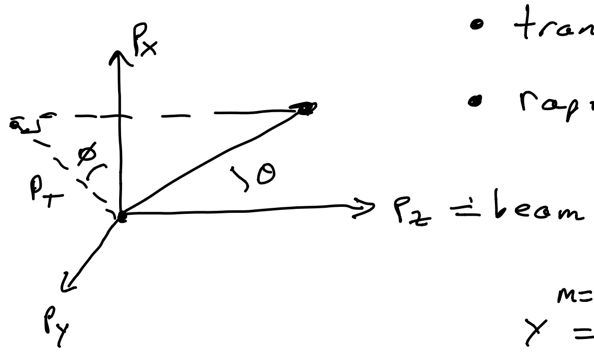
final state is collection of hadrons

which particles do we group together? (not unique)

need IR safe algorithm: invariant under $p_i \to p_j + p_k$
if $p_j \parallel p_k$ or $p_j \to 0$

Hadron Collider Vars

know proton CM, not partonic collision's CM



- transverse momentum p_T
- rapidity $y = \frac{1}{2} \ln\left(\frac{E+p_z}{E-p_z}\right)$

$\Delta y = y_1 - y_2$ is invariant under $\hat{z}$ boosts

$$y \overset{m=0}{=} \ln \cot \frac{\theta}{2}$$

$\Delta R = \left[(\Delta y)^2 + (\Delta\phi)^2\right]^{1/2}$ is boost invariant angular distance

Recombination Algorithms:

consider set of particles L (hadrons, partons, calorimeter cells)

$$d_{ij} = \min\left(p_{Ti}^{2r}, p_{Tj}^{2r}\right) \frac{\Delta R_{ij}^2}{R^2} = \text{distance}(i,j)$$

$$d_{iB} = p_{Ti}^{2r} = \text{distance}(i,\text{beam})$$

Find $\min_{i,j \in L} \left(\{d_{ij}\}, \{d_{iB}\} \right)$

join i & j into new particle in L & repeat

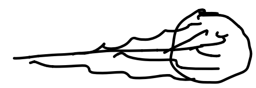
call i a jet and remove it, & repeat

Stop when L is empty

- $r=1$ k_T algorithm, clusters soft particles first (jet regions not circular)
- $r=0$ Cambridge/Aachen, geometric

- $r = -1$ Anti-k_T, clusters harder particles first (circular jet regions)
  <u>default</u> ATLAS & CMS

R = jet radius parameter



eg. $R = 0.5$, demand 1-jet with $p_T > 30$ GeV & all remaining jets having $p_T \le 30$ GeV
"1-jet event"

eg. $H + 0$-jets (used in Higgs coupling measurements)
all jets have $p_T \le 30 \text{ GeV} = p_T^{cut}$

$$\sigma \sim \sigma_{incl} \left[1 - \frac{2 \alpha_s C_A}{\pi} \ln^2\left(\frac{p_T^{cut}}{m_H}\right) + \cdots \right]$$

← large log series that must be summed to all orders

Leading Logs: $\sim 1 + \alpha_s L^2 + \alpha_s^2 L^4 + \cdots$ exponentiate

$$\sigma \sim \sigma_{incl} \exp\left[- \frac{2 \alpha_s C_A}{\pi} \ln^2\left(\frac{p_T^{cut}}{m_H}\right) \right]$$

example of Sudakov form factor from restricting radiation

Also Cone Algorithms 

which are no longer popular

Parton Shower

- construct an exclusive description of events at hadron level (needed for experimental analyses)
- Monte Carlo program to iterate collinear approximation
- LL shower + large N_c + model for hadronization
  simplify interference, planar color flow

(• improvements MC@NLO, POWHEG, ...)

Probability for parton i to branch between

$$q^2 \ \& \ q^2 + dq^2 = dP_i = \frac{\alpha_s}{2\pi} \frac{dq^2}{q^2} \int_{Q_0^2/q^2}^{1-Q_0^2/q^2} dz \ P_{ji}(z)$$

evolution var.

absorbed color factors

partons no longer resolved if $\Delta q^2 \leq Q_0^2$, cuts off z gives finite probability

Probability for no branching between Q^2 & q^2 is

$$\equiv \Delta_i(Q^2, q^2)$$

then

$$\frac{d\Delta_i(Q^2,q^2)}{dq^2} = \lim_{dq^2 \to 0} \frac{\Delta_i(Q^2, q^2+dq^2) - \Delta_i(Q^2, q^2)}{dq^2}$$

$$= \Delta_i(Q^2, q^2) \frac{dP_i}{dq^2}$$

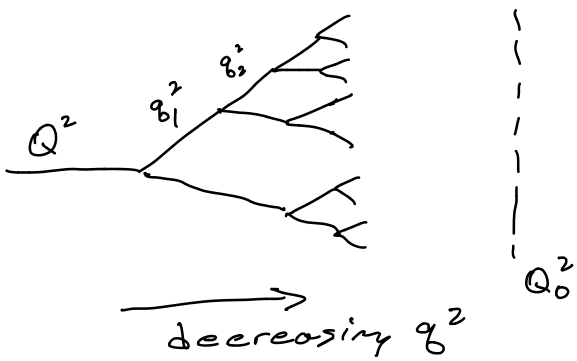
no branching to q^2

branching btwn q^2 & $q^2 + dq^2$

Solution:

$$\Delta_i(Q^2, q^2) = \exp\left[- \int_{q^2}^{Q^2} \frac{dk^2}{k^2} \frac{\alpha_s}{2\pi} \int_{Q_0^2/k^2}^{1-Q_0^2/k^2} dz \, P_{ji}(z) \right]$$

note: $\Delta_i(Q^2, Q_0^2) \sim \exp\left[-C_F \frac{\alpha_s}{2\pi} \ln^2 \frac{Q^2}{Q_0^2} \right]$ Sudakov Form Factor



Implementation

- random number $\rho \in [0,1]$, solve $\Delta_i(Q^2, q_1^2) = \rho$

  if $q_1^2 > Q_0^2$ choose z-value with $P_{ji}(z)$

  if $q_1^2 < Q_0^2$ stop

- repeat on daughter branches with $\Delta_i(q_1^2, q_2^2)$

Pythia, Herwig, Sherpa, ...

Lecture 3 Outline

- DIS : Parton Dist'ns & factorization

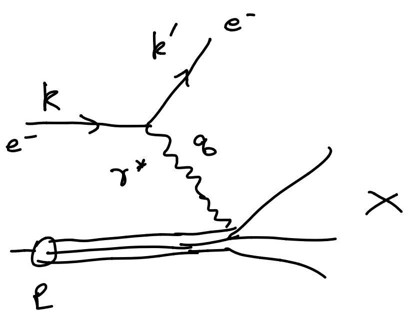
  $pp \to H + X$

- $e^+e^- \to$ 2 jets, event shapes, factorization

- $pp \to H + 0$-jets

Deep Inelastic Scattering (DIS) $e^- p \to e^- X$

– key process for foundations of QCD (quarks, asym. freedom)



$$s = (k+P)^2$$

$$q^2 = -Q^2 \qquad Q^2 > 0 \qquad y = \frac{Q^2}{xs}$$

$$x = \frac{Q^2}{2P\cdot q} \qquad 0 < x < 1$$

$$y = \frac{q\cdot P}{k\cdot P} \qquad 0 < y < 1$$

$\left(y = 1 - \frac{k^{0\prime}}{k^0} \right.$ e^- energy loss in proton rest frame $\left.\right)$

measurable with leptons

$Q^2 \gg \Lambda_{QCD}^2$ $\qquad p_X^2 = (q+P)^2 = \frac{Q^2(1-x)}{x} \sim Q^2$ large

(proton blown apart)

$$\frac{d\sigma}{dx\, dQ^2} = \frac{8\pi\alpha^2}{Q^4}\left[(1+(1-y)^2)\, \underline{F_1(x,Q^2)} + \frac{(1-y)}{x}\left\{ \underline{F_2(x,Q^2)} - 2x\, F_1(x,Q^2) \right\} \right]$$

QCD/hadronic dependence in dimensionless structure functions

hadronic tensor

$$W^{\mu\nu} = \frac{1}{4\pi} \sum_X (2\pi)^4 \delta^4(q+P-p_X) \langle P | J^\mu(0) | X \rangle \langle X | J^\nu(0) | P \rangle$$

$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x,Q^2) + \left(P^\mu + \frac{q^\mu}{2x} \right)\left(P^\nu + \frac{q^\nu}{2x} \right) \frac{F_2(x,Q^2)}{P\cdot q}$$

- uses current conservation $\partial^\mu J_\mu = 0 \Rightarrow q^\mu W_{\mu\nu} = 0$
- Parity & Time Reversal & hermiticity $J^\dagger = J$

actually $\quad F_i = F_i\left(x, \frac{Q^2}{\Lambda_{QCD}^2}\right)$

Factorization Theorem

$$F_1\left(x, \frac{Q^2}{\Lambda_{QCD}^2}\right) = \sum_j \int_x^1 \frac{d\xi}{\xi}\, C_j\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right) f_j\left(\xi, \frac{\mu}{\Lambda_{QCD}}\right) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

similar for F_2

depend on quark flavor u, d, $\bar{u}, \bar{d}, \ldots$

parton distribution functions f_j: f_{q_i} & f_g

take snapshot of proton on short time scale $t \sim \frac{1}{Q}$

x = mom. fraction of struck quark, ξ = mom. fraction of parton j in proton

Proof:
- OPE (long), twist-2 operators
- IR structure of QCD eg. with Soft-Collinear Effective Theory (SCET) $\to$ extra reading

$$f_{q_i}\left(\xi, \frac{\mu}{\Lambda}\right) = \int \frac{dy}{2\pi}\, e^{-2i(\xi \bar{n}\cdot p)y} \langle p|\, \bar{\psi}_i(\bar{n}y)\, W(\bar{n}y, -\bar{n}y)\, \bar{n}\!\!\!/\, \psi_i(-\bar{n}y)\, |p\rangle \qquad (*)$$

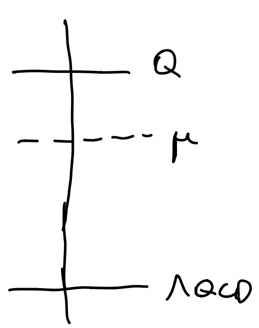
- $\bar{n}^2 = 0$ light cone matrix element ($\to$ twist 2, symmetric & traceless, $\bar{n}^{\mu} \ldots \bar{n}^{\mu_K}$)
- $W = P\exp \int_{-y}^{y} ds\, \bar{n}\cdot A(\bar{n}s)$ for gauge invariance

  Wilson Line

a fundamental mom. distribution of proton

Scale Separation

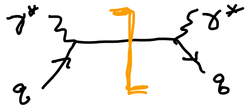
- μ divides long & short distance physics



$$\ln\left(\frac{Q}{\Lambda_{QCD}}\right) = \underbrace{\ln\left(\frac{Q}{\mu}\right)}_{\text{in } C_j} + \underbrace{\ln\left(\frac{\mu}{\Lambda_{QCD}}\right)}_{\text{in } f_j}$$

$f_j\left(\xi, \frac{\mu}{\Lambda}\right)$ depending on scale "μ" where we probe the parton j the distribution changes (more later)

Tree Level



$$C_j\left(\frac{x}{\xi}, \frac{Q^2}{\mu}\right) = \frac{Q_j^2}{2}\,\delta\left(1-\frac{x}{\xi}\right)$$

$$\therefore\ F_1\left(x, \frac{Q^2}{\Lambda^2}\right) = \sum_j \frac{Q_j^2}{2} f_{q_j}\left(x, \frac{\mu}{\Lambda}\right)$$

$\Rightarrow$ measurements of universal f_{q_j}'s

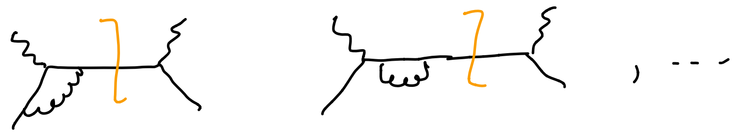
"parton model", independent of Q $\rightarrow$ scaling

Also $F_2 = 2x F_1$ $\Rightarrow$ spin-½ quarks

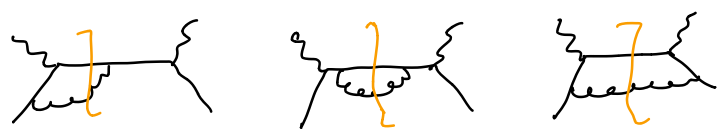
scaling violation from $\ln Q$ dependence at higher orders (excellent agreement w data)

IR divergences

virtual



real



$$4F_1^V = \frac{\alpha_s C_F}{\pi} Q_f^2 \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(-\frac{1}{\epsilon^2} - \frac{1}{2\epsilon} + \cdots\right)\delta(1-x)$$

$$4F_1^R = \frac{\alpha_s C_F}{\pi} Q_f^2 \left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left[-\frac{(1+x^2)}{2\epsilon\,(1-x)^{1+\epsilon}} + \frac{1}{4(1-x)^{1+\epsilon}} + \cdots\right]$$

Treat $(1-x)^{-1-\epsilon}$ as distribution, test fn $g(x)$

$$\int_0^1 dx\, \frac{g(x)}{(1-x)^{1+\epsilon}} = \int dx\, \frac{g(x)-g(1)+g(1)}{(1-x)^{1+\epsilon}} = -\frac{g(1)}{\epsilon} + \int dx\, \frac{g(x)-g(1)}{1-x}$$

$$\therefore\ \frac{1}{(1-x)^{1+\epsilon}} = -\frac{1}{\epsilon}\,\delta(1-x) + \frac{1}{(1-x)_+} + \cdots$$

Now $1/\epsilon^2$ cancels $\quad -\frac{1}{\epsilon^2} + \frac{1}{\epsilon^2} = 0$

$$\text{sum} = \frac{\alpha_s C_F}{2\pi} Q_f^2 C_F \left[-\frac{1}{\epsilon} P_{qq}(x) - \ln\frac{\mu^2}{Q^2} P_{qq}(x) + \dots \right]$$

where $P_{qq}(x) = \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$ splitting function with distn s

$\int_0^1 dx\, P_{qq}(x) = 0 \Rightarrow$ # quarks conserved

- left over $1/\epsilon$ collinear divergence $P_g \parallel P_{In}$

  which is part of $f_q(\xi)$

$$f_q(\xi,\mu)^{\text{partonic}} = \delta(1-\xi) - \frac{\alpha_s(\mu)}{2\pi\epsilon} P_{qq}(\xi) \qquad \overline{MS} \text{ defn}$$

Then

$$C_1\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right) = \frac{Q_f^2}{2} \left[\delta\left(1-\frac{x}{\xi}\right) - \frac{\alpha_s}{2\pi} \ln\frac{\mu^2}{Q^2} P_{qq}\left(\frac{x}{\xi}\right) + \dots \right]$$

indeed direct calculation from def'n above (*) :

$$f_q(\xi)^{\text{bare}} = \delta(1-\xi) + \frac{\alpha_s}{2\pi}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right) P_{qq}(\xi)$$

UV renormalization gives RGE equation

$$\mu\frac{d}{d\mu} f_j(\xi,\mu) = \int_\xi^1 \frac{d\xi'}{\xi'} P_{jk}\left(\frac{\xi}{\xi'}\right) f_k(\xi',\mu)$$

DGLAP equations

Exercise (next page): Explore PDFs

- dist'n terms in splitting functions.
- action of this RGE

Lectures on Perturbative QCD

Iain Stewart, ICTP Summer School 2017

Problem: Splitting Functions

Infrared enhancements in the quark and gluon branching processes $q \to qg$, $g \to gg$, and $g \to q\bar{q}$ are key ingredient in the formation of jets. The structure of collinear enhancements is described by splitting functions P_{ab}, which to first order in the strong coupling α_s are:

$$\begin{aligned} P_{qq}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} C_F \Big[\frac{1+x^2}{(1-x)_+} + a_q\, \delta(1-x) \Big] \,, \\ P_{qg}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} T_R \Big[x^2 + (1-x)^2 \Big] \,, \\ P_{gq}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} C_F \Big[\frac{1+(1-x)^2}{x} \Big] \,, \\ P_{gg}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} 2C_A \Big[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \Big] + a_g\, \delta(1-x) \,. \end{aligned} \tag{1}$$

Here the color factors are $C_F = 4/3$, $T_R = 1/2$, and $C_A = 3$, and you will determine the constants a_q and a_g below. Each $P_{ab}^{(0)}(x)$ should be thought of as the probability of finding a parton of type a inside an initial parton b, with a having a fraction x of the parent b's momentum. These expressions include the familiar Dirac δ-function, and the less familiar $+$-function. The latter is defined by $1/(1-x)_+ = 1/(1-x)$ for any $x < 1$, and by the fact that the singularity at $x = 1$ is regulated such that

$$\int_0^1 dx \frac{1}{(1-x)_+} g(x) = \int_0^1 dx \frac{1}{(1-x)} \big[g(x) - g(1) \big] \tag{2}$$

for any function $g(x)$.

a) Derive results for the constants a_q and a_g such that quark number is conserved:

$$\int_0^1 dx\, P_{qq}^{(0)}(x) = 0 \,, \tag{3}$$

and momentum is conserved by the quark and gluon splittings:

$$\int_0^1 dx\, x \left[P_{qq}^{(0)}(x) + P_{gq}^{(0)} \right] = 0 \,, \qquad \int_0^1 dx\, x \left[P_{gg}^{(0)}(x) + 2n_f P_{qg}^{(0)} \right] = 0 \,. \tag{4}$$

Here n_f is the number of light quarks. Show that you can rewrite $P_{qq}^{(0)}$ as $P_{qq}^{(0)}(x) = (\alpha_s(\mu) C_F/2\pi) \left[(1+x^2)/(1-x) \right]_+$.

Given an initial distribution of quarks $q(\xi, \mu_0)$ and gluons $g(\xi, \mu_0)$ at a momentum scale μ_0, the distribution of quarks at a scale μ_1 is given by

$$q(x, \mu_1) = q(x, \mu_0) + \int_{\mu_0}^{\mu_1} \frac{2\, d\mu}{\mu} \int_x^1 \frac{d\xi}{\xi} \Big[P_{qq}^{(0)}\Big(\frac{x}{\xi}\Big) q(\xi, \mu) + P_{qg}^{(0)}\Big(\frac{x}{\xi}\Big) g(\xi, \mu) \Big] \,, \tag{5}$$

where the terms in the integral account for the possibility that the quark we observe came from a splitting rather than the initital distribution.

b) By iterative use of Eq. (5) derive a series in α_s that writes $q(x,\mu_1)$ in terms of terms only involving q's and g's at $\mu = \mu_0$. Draw Feynman diagrams to describe physically what is happening with the various terms in your infinite series.

The subtraction term from the plus function in $P_{qq}^{(0)}$ in Eq. (5) sets $\xi = x$, and is related to evolution to the scale μ_1 without branching, so strictly speaking Eq. (5) does not yet have a clean separation between branching and no-branching. To better distinguish the two possibilites we will rewrite this equation in a different way. To simplify the formulas below, we'll set $P_{qg}^{(0)} = 0$. The probability that a quark does not split when it evolves from μ_0 to μ_1 is then given solely by the quark Sudakov form factor:

$$\Delta_{qq}(\mu_1,\mu_0) \;=\; \exp\Big[-\int_{\mu_0}^{\mu_1}\frac{2\,d\mu}{\mu}\int dx\,\hat{P}_{qq}^{(0)}(x)\Big]\,. \tag{6}$$

Here $\hat{P}_{qq}^{(0)}(x) = (\alpha_s(\mu)C_F/2\pi)\,(1+x^2)/(1-x)$ and we will assume that the limits on the x integration keep us away from the singularity at $x=1$ (more on this in part d).

c) Taking $\mu_1 d/d\mu_1$ derive differential equations for $q(x,\mu_1)$ and $\Delta_{qq}(\mu_1,\mu_0)$. Next derive an equation for $\mu_1 d/d\mu_1(q/\Delta_{qq})$ and show that its solution yields

$$q(x,\mu_1) = \Delta_{qq}(\mu_1,\mu_0)\,q(x,\mu_0) + \int_{\mu_0}^{\mu_1}\frac{2\,d\mu}{\mu}\frac{\Delta_{qq}(\mu_1,\mu_0)}{\Delta_{qq}(\mu,\mu_0)}\int\frac{d\xi}{\xi}\hat{P}_{qq}^{(0)}\Big(\frac{x}{\xi}\Big)q(\xi,\mu)\,. \tag{7}$$

Since this result does not involve the +-function we can interpret the second term as the probability from splitting, and the first term as the probability of having no splitting. Thus the Sudakov form factor in the first term gives the no-splitting probability when we evolve from μ_0 to μ_1. Can you provide an interpretation for the presence of the ratio of Δ_{qq}'s in the second term? This result with its probabilistic interpretation is used in parton shower Monte Carlo programs that describe parton branching and QCD jets.

Next you will calculate the form of the exponent in $\Delta_{qq}(\mu_1,\mu_0)$. The result can be thought of as an infinite series in $\alpha_s(\mu_0)$, but to keep things simple for this calculation we'll freeze $\alpha_s(\mu) = \alpha_s(\mu_0)$ and approximate $P_{qq}^{(0)}(x) \simeq (\alpha_s(\mu_0)C_F/\pi)/(1-x)$ which will allow us to determine the dominant term for $\mu_1 \gg \mu_0$.

d) Lets identify the evolution scale parameter as the parton's virtual mass squared, $\mu^2 = p^2 \equiv t'$, and hence impose the corresponding kinematic limits on the x-integral: $\mu_0^2/\mu^2 < x < 1-\mu_0^2/\mu^2$ (obtained for particles with large energy and expanding $\mu_0 \ll \mu$). With the approximations above and these limits perform the double integral in Eq. (6), and show that your result involves a $\ln^2(\mu_1/\mu_0)$. This double log is related to the presence in the branching and no-branching probabilities of the soft ($x \to 1$) singularity and the collinear singularity described by the splitting function equations.

$$F_1\left(x, \frac{Q^2}{\Lambda_{QCD}^2}\right) = \sum_j \int_x^1 \frac{d\xi}{\xi}\, \underbrace{C_j\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right)}_{\substack{\text{wants}\\ \mu \simeq Q}}\, \underbrace{f_j\left(\xi, \frac{\mu}{\Lambda_{QCD}}\right)}_{\substack{\text{wants}\\ \mu \sim \Lambda_{QCD}}}$$

Evolve PDF to appropriate scale:

$$\underset{\mu \simeq Q}{f_j(\xi, \mu)} = \underbrace{\int_\xi^1 \frac{d\xi'}{\xi'}\, U_{jk}\left(\frac{\xi}{\xi'}, \mu, \mu_0\right)}\, \underset{\mu_0 \simeq \Lambda_{QCD}}{f_k(\xi', \mu_0)}$$

perturbative evolution of PDFs

sums ∞ series of large

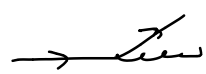
$L = \ln\left(\frac{\mu}{\mu_0}\right)$'s: $1 + \alpha_s L + \alpha_s^2 L^2 + \dots$

(numerical solution here)

[like $\alpha_s(\mu)$]

non-pert. boundary condition

$j \neq k$ PDF mixing



P_{gq}

P_{qg}

Note: μ dependence cancels order-by-order in expansion between $C_j\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right)$ & $f_j(\xi, \mu)$

Often use residual μ dependence to estimate higher order terms: $\mu = \frac{Q}{2}, Q, 2Q$

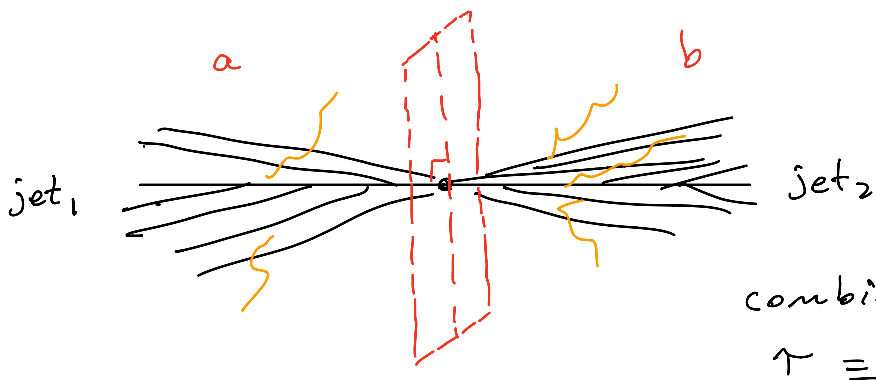
$\Rightarrow$ perturbative theory uncertainty

Same story for pp collisions:

$$\sigma = \sum_{i,j} \int dx_a\, dx_b\, \underbrace{f_i(x_a, \mu)\, f_j(x_b, \mu)}_{\substack{\text{evolve from}\\ \mu_0 \sim \Lambda_{QCD} \text{ to}\\ \mu \simeq m_H}}\, \hat{\sigma}_{ij \to H+X}(x_a, x_b, \underbrace{\mu}_{\mu \simeq m_H}, m_H)$$

$\boxed{e^+e^- \to 2\text{-jets}}$ $\qquad e^+e^- \to \gamma^*(q) \to q\bar{q}$

- factorization theorems can also be derived for processes involving jets



measure hemisphere masses

$$M_a^2 = \Big(\sum_{i\in a} p_i^\mu\Big)^2$$

$$M_b^2 = \cdots$$

combine

$$\tau \equiv \frac{M_a^2 + M_b^2}{Q^2}$$

[related to "thrust"]

demanding $\tau \ll 1$ ensures 2-jets "event shape"

$\boxed{\text{Collinear}}$ radiation with $p^0 \sim Q$ & $p_\perp \sim Q\sqrt{\tau}$
contributes $\to$ Jet Functions $\quad \sim p_\perp^2 \sim [Q\sqrt{\tau}]^2$

$\boxed{\text{Soft}}$ radiation with $k^\mu \sim Q\tau$ contributes
$\to$ Soft function $\quad (p+k)^2 \sim 2p\cdot k \sim (Q)(Q\tau)$

$$M^2 \simeq (p+k)^2 = p^2 + 2p\cdot k + \cancel{O(k^2)} = s + Qk^+ \Rightarrow Q^2\tau \simeq s + s' + Qk$$

$$\frac{d\sigma}{d\tau} = \sigma_0\, H(Q,\mu) \int ds\, ds'\, \underbrace{J(s,\mu)\, J(s',\mu)}_{\text{jet functions}}\, \underbrace{S\Big(Q\tau - \frac{s+s'}{Q}, \mu\Big)}_{\text{soft fn}}$$

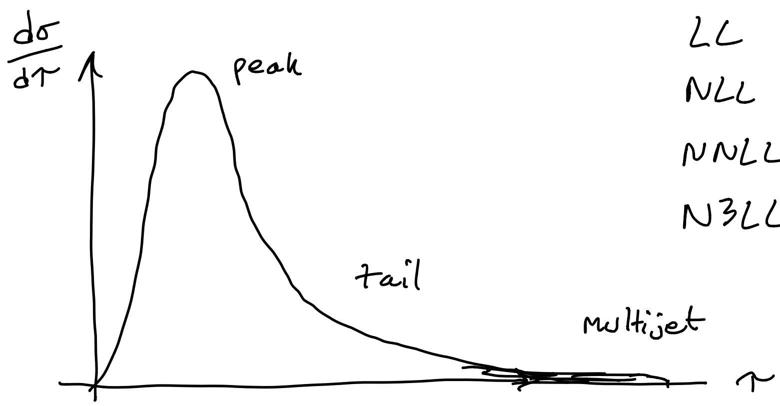
hard fn.
virtual corrections

$$\mu^2 \sim Q^2 \quad \gg \quad \mu^2 \sim Q^2\tau \quad \gg \quad \mu^2 \sim Q^2\tau^2$$

renormalization group evolution in μ
sums $\alpha_s \ln^2\tau$ factors

RGE for $H(Q,\mu)$: solution is Sudakov Form Factor

$\longrightarrow$ exercise in EFTx course, chapter 13



LL

NLL

NNLL

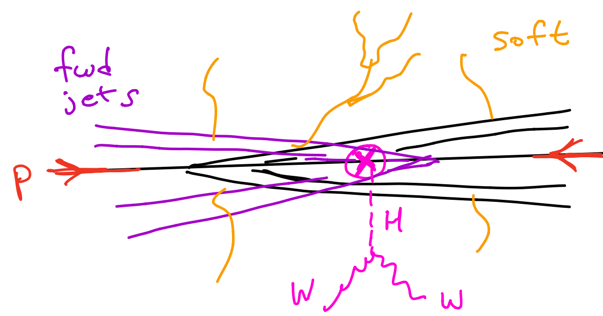
N3LL + $\mathcal{O}(\alpha_s^3)$ Known

(derived using SCET)

1% precision

$\Rightarrow$ extract $\alpha_s(m_Z)$

$pp \to H + 0\text{-jets}$



anti-k_T with R

not jets with $p_T > p_T^{cut}$

$$\sigma(p_T^{cut}) = \sigma_0 \, H_{gg}(m_t, m_H, \mu) \int dY \; B_g(m_H, p_T^{cut}, R, x_a, \mu, \nu)$$

$$* \, B_g(m_H, p_T^{cut}, R, x_b, \mu, \nu) \; S_{gg}(p_T^{cut}, R, \mu, \nu)$$

extra rapidity scale parameter

$$x_{a,b} = \frac{m_H}{E_{cm}} e^{\pm Y}$$

with

$$B_g(m_H, p_T^{cut}, R, x, \mu, \nu) = \sum_j \int_x^1 \frac{d\xi}{\xi} \, \mathcal{I}_{gj}\left(m_H, p_T^{cut}, R, \frac{x}{\xi}, \mu, \nu\right) f_j(\xi, \mu)$$

usual PDFs

all other functions perturbative

Sum $\alpha_s \ln^2\left(\frac{p_T^{cut}}{m_H}\right)$ to higher orders

NNLL gives $\sim 7\%$ precision

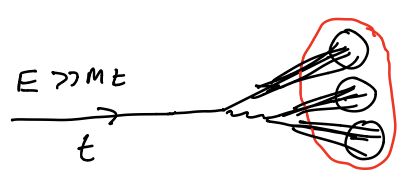
Active Areas in Collider Physics

- loop calculations, connection to Amplitudes, spinor/helicity techniques
- loops + legs, combining so that $1/\epsilon$'s cancel
  phase space slicing or subtractions
- Improving Parton Shower Monte Carlo
- Global Fits for determining PDFs
- Factorization (new formulas, new universal functions, factorization violation & MPI)
- Resummation, higher orders/precision, new types of logs (eg. $\log R$) & multiple variables
- Jet Substructure

* boosted particles that decay hadronically can be identified by substructure

2 prong substructure

find new observables



3 prong substructure

* also techniques to "groom" jets, remove soft contamination inside jets to better probe the hard mother particle

References for Further Reading

- Effective Field Theory, including Soft-Collinear EFT for collider physics, see EFTx course:

  http://www2.lns.mit.edu/~iains/registerEFTx

  (video lectures, SCET review notes, online problems)

- QCD Concepts (Renormalization Group, β-function, Fadeev-Popov, ...)

  http://www2.lns.mit.edu/~iains/talks/QFT3_Lectures_Stewart_2012.pdf

- Collider Physics: "QCD and Collider Physics" Book by Ellis, Stirling, and Webber

- Parton Shower Review, Buckley et al:

  https://arXiv.org/abs/1101.2599

- Review on Jets by Gavin Salam:

  https://arxiv.org/abs/0906.1833