Effective Field Theory

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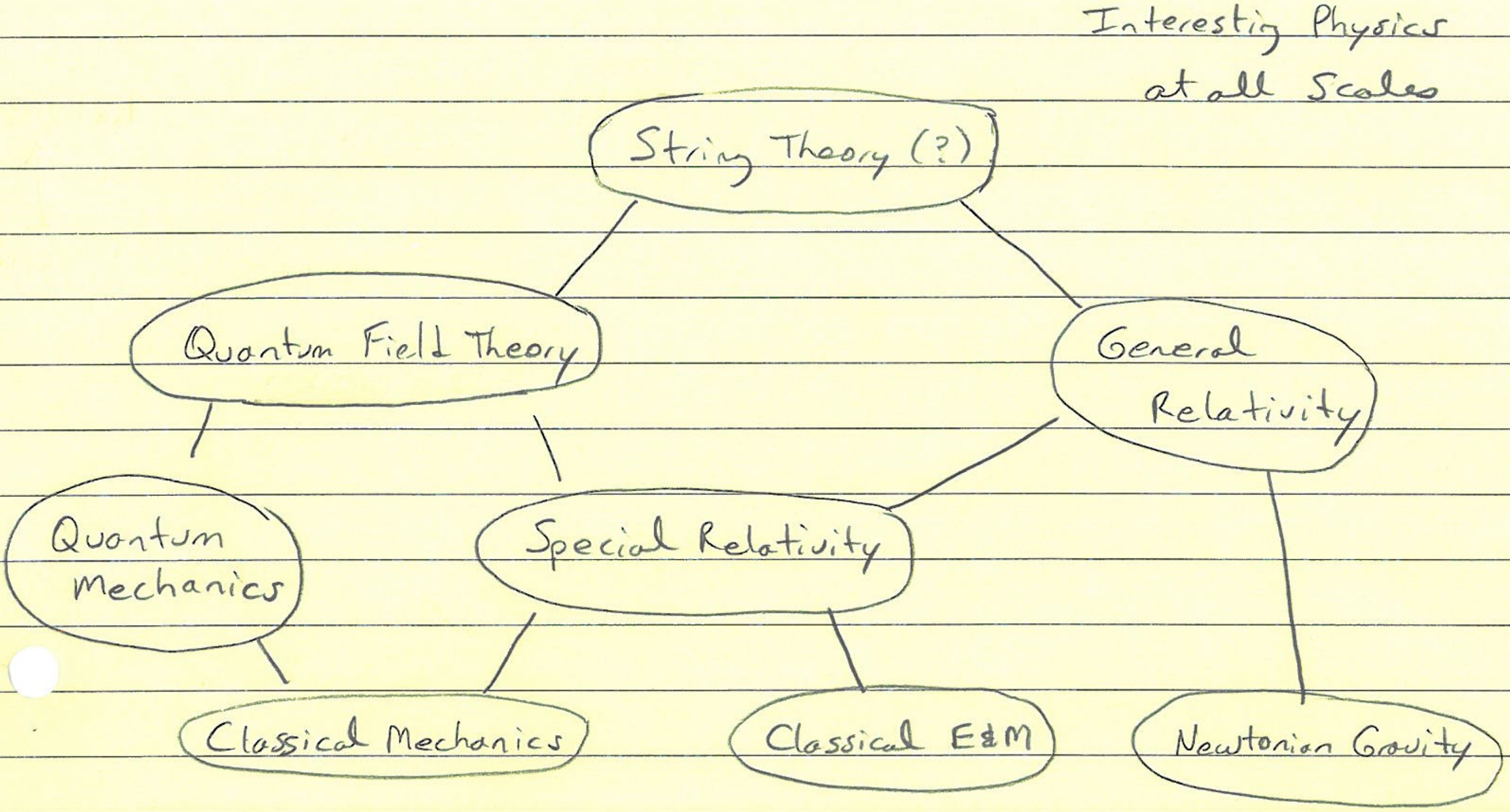
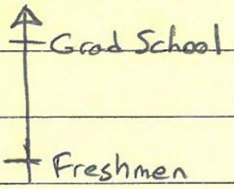
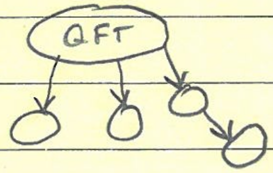
The Big Picture





For most of your physics career you've been moving up, towards more and more general theories

In this class we'll be going the other direction



Why? • As we move up it becomes harder to compute
eg. Hydrogen Energy Levels, Elliptic Orbit of planets

- Want the <u>simplest</u> framework that captures the essential physics, but in a manner that can be corrected to arbitrary precision

eg. $v/c \sim 1/10$ Non.Rel. Expn, Post-Newtonian Expansion

$\boxed{\text{QFT}}$

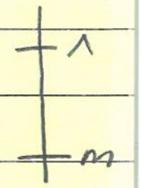
To Describe a Physical System

- determine relevant degrees of freedom ← what fields
- symmetries ← what interactions
- expansion parameters, leading order description ← what power counting

⇓

Effective Field Theory

Note: In an EFT the power counting is just as important as something like gauge symmetry



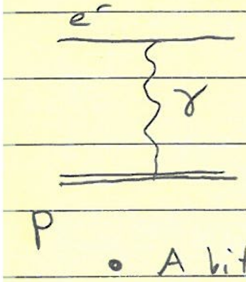
Key Principle: To describe the physics at some scale m^2 we do not need to know the detailed dynamics of what is going on at scales $\Lambda^2 \gg m^2$.

[Good: focus on relevent dof, interactions, simplify calc ; Bad: tells us we must work harder to probe short dist.

Question: is power counting always in mass scales? an example?

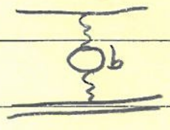
Discuss: examples , even 2-dim

eg. Do not need to know about bottom quarks to describe Hydrogen



$$E_0 = \frac{1}{2} m_e \alpha^2 \Big[1 + \underbrace{\mathcal{O}\Big(\frac{m_e^2}{m_b^2}\Big)}_{10^{-8}} \Big]$$

- A bit subtle. m_b does effect the coupling α in $\overline{MS}$ since it runs $\alpha(m_W) \simeq \frac{1}{128}$, $\alpha(0) \simeq \frac{1}{137.036}$



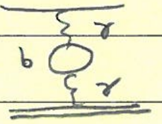
More precisely, if α is a parameter of the Standard Model which is fixed at high energy then α for Hydrogen depends on m_b. BUT if we simply extract $\alpha(0)$ from low energy atomic physics then this value can be used for other experiments at these energies.

$$\mathcal{L}(p, e^-, \gamma, b; \alpha, m_b) = \mathcal{L}(p, e^-, \gamma; \alpha') + \mathcal{O}(1/m_b^2)$$

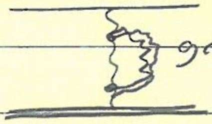
<u>Also</u>
- insensitive to quarks in proton since
  $m_e \alpha \ll (\text{proton size})^{-1} \sim 200\,\text{meV}$
- insensitive to proton mass, $m_e \alpha \ll M_p \sim 1\,\text{GeV}$
  (proton acts like static charge source)
- $m_e \alpha \ll m_e$, non-relativistic $\mathcal{L}$ for e^- suffices

These conclusions hold despite UV divergences

un regulated



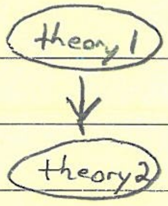
$= \infty$



$= \infty$

In general EFT's are used in two distinct ways:

① "top-down" High energy theory is understood but we find it useful to have a simpler theory at low energies



- So <u>integrate out</u> (remove) heavier particles and <u>match</u> onto a low energy theory
  → find new operators & new low energy constants

$$\mathcal{L}_{High} \rightarrow \sum_n \mathcal{L}_{Low}^{(n)} \leftarrow \text{expn. in decreasing relevance}$$

- $\mathcal{L}_{Low}$ & $\mathcal{L}_{High}$ <u>agree</u> in infrared, <u>differ</u> in ultraviolet
- desired precision tells us when to stop (what n)

[• in QCD can be used to distinguish perturbative & non-perturbative effects $m_b \gg \Lambda_{QCD}$]

egs.
- Integrate out heavy top, W, Z for Weak Int.
- Heavy Quark Effective theory for charm & bottom
- Non-Relativistic QCD, QED for bound states
- Soft-Collinear EFT, processes with energetic hadrons or jets

(ii) "bottom-up" Underlying theory is unknown <u>or</u> matching is too difficult to carry out. (eg. non-perturbative)

- Construct $\sum_n \mathcal{L}_{low}^{(n)}$ by writing down most general set of possible interactions consistent with all symmetries

theory1?

theory 2

- Couplings are unknown, but can be fit to experiment (like $\alpha(0)$) [or calculated from numerical matching, eg. Lattice QCD]

- desired precision tells us when to stop $\sum_n$

egs.
Chiral Perturbation Theory - low energy π, K interactions
Standard Model
Eistein Gravity made Quantum

Comment: $\sum_n$ expansion is in powers, but there are also logs renormalization of $\mathcal{L}_{low}^{(n)}$ allows us to sum large logs $\ln\left(\frac{m_1}{m_2}\right)$, $m_2 \ll m_1$. [True even when m_1 & m_2 are not masses of particles. I have not met logs in QFT that could not be summed with some EFT.]

Standard Model as EFT

$$\sum_n \mathcal{L}^{(n)}_{LOW} = \underset{\substack{\uparrow \\ \text{standard model} \\ \mathcal{L}\ (\text{QFT}-3)}}{\mathcal{L}^{(0)}} + \underset{\substack{\uparrow \\ \text{what's} \\ \text{this?}}}{\mathcal{L}^{(1)}} + \cdots$$

SM degrees of freedom (http://pdg.lbl.gov)

- Gauge theory $SU(3)_{color} \times SU(2)_{weak} \times U(1)_Y$

A^μ_A	W^μ_a	B^μ
8 gluons	3 weak boson	1 U(1) boson

- Fermions

		mass
quarks	u_L, u_R	1.5 - 3.3 MeV
	d_L, d_R	3.5 - 6.0
	s_L, s_R	100 ± 30
	c_L, c_R	1.27 ± .09
	b_L, b_R	4.20 ± .12
	t_L, t_R	171200 ± 1000
leptons	e_L, e_R	0.511
	μ_L, μ_R	105.66
	τ_L, τ_R	1777

$$\left.\begin{array}{l}\nu_{eL}\\ \nu_{\mu L}\\ \nu_{\tau L}\end{array}\right\}\begin{array}{ll}\Delta m^2_\odot \simeq 8\times10^{-5}\,eV^2 & \left[\nu_e \leftrightarrow \begin{pmatrix}\nu_\mu\\ \nu_\tau\end{pmatrix}\right]\\ \Delta m^2_{atm} \simeq 2\times10^{-3}\,eV^2 & \left[\nu_\mu \to \nu_\tau\right]\end{array}$$

sterile neutrinos N_R ?

$$\mathcal{L}^{(0)} = \mathcal{L}_{gauge} + \mathcal{L}_{Fermi} + \mathcal{L}_{Higgs} + \mathcal{L}_{N_R}$$

Other Masses:

$M_\gamma = 0$

$M_{gluon} = 0$

$M_W = 80.42$

$M_Z = 91.19$

$M_H = 125$ GeV (!)

$\mathcal{L}_{Higgs}$, $\mathcal{L}_{NR}$
more symmetries $\rightarrow$ 8.325

$$\mathcal{L}_{Gauge} = -\frac{1}{4} \underset{\substack{\uparrow \\ \text{field strength} \\ \text{for } B^\mu}}{B^{\mu\nu}} B_{\mu\nu} - \frac{1}{4} W_a^{\mu\nu} W_{a\mu\nu} - \frac{1}{4} G^{\mu\nu}_A G_{A\mu\nu}$$

$$\mathcal{L}_{Fermi} = \sum_{\psi_L} \bar{\psi}_L i \not{D} \psi_L + \sum_{\psi_R} \bar{\psi}_R i \not{D} \psi_R$$

$$iD_\mu = i\partial_\mu + g_1 B_\mu \underset{\substack{\uparrow \\ U(1) \\ \text{charge}}}{Y} + g_2 W_\mu^a \underset{\substack{\uparrow \\ SU(2) \\ \text{charge/rep}}}{T^a} + g A_\mu^A \underset{\substack{\uparrow \\ SU(3) \text{ chg/rep}}}{T^A}$$

[eg. $\sigma^a/2$ on $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$]

[eg 0 on e^-
$\frac{\lambda^a}{2}$ on quark]

Power Counting?

Related to what we've left out

$\rightarrow$ $\epsilon = \dfrac{M_{SM}}{\Lambda_{new}}$ $\leftarrow$ $m_W, m_Z, m_t, \ldots$

$\Lambda_{new} \leftarrow M_{plank}, M_{GUT}$, ~~$M_{SUSY}$~~

a new mass scale at higher energy

$\rightarrow$ higher dimension operators (dimension > 4)
built out of SM fields

[we'll see why these are related momentarily]

What does "Renormalizable" mean in EFT?

Traditional Defn Theory is renormalizable if at any order in perturbation theory divergences from loop integrals can be absorbed into a finite set of parameters

<u>EFT Defn</u> A theory must be renormalizable order by order in its expansion parameters.

- this allows an ∞ # of parameters, <u>but</u> only a finite # at any given order in ϵ

- If an $\mathcal{L}^{(0)}$ is renormalizable in traditional sense, it means it does not know directly about Λ_{new} [1 at LO, 5 at NLO ...]

  [we'll meet examples where $\mathcal{L}^{(0)}$ does know later on]

<u>Marginal, Irrelevant, Relevant Operators</u> $\downarrow_{L2}$

case where <u>mass</u> <u>dimension</u> determines power counting (p.c.)

Consider

$$S[\phi] = \int \overset{\text{d dimensions}}{d^d x} \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 - \frac{\tau}{6!} \phi^6 \right)$$

$[\phi] = \frac{d-2}{2}$ $\quad [m^2] = 2$, $\quad [\lambda] = 4-d$, $\quad [\tau] = 6-2d$

$[d^d x] = -d$

Say we want to study $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ at long distance $x^\mu = s\, x'^\mu$ $\quad s \to \infty$, x'^μ fixed

Let $\phi(x) = s^{(2-d)/2} \phi'(x')$ (to normalize kinetic term)

$$S'[\phi'] = \int d^d x' \left[\frac{1}{2} \partial^\mu \phi' \partial_\mu \phi' - \frac{1}{2} m^2 s^2 \phi'^2 - \frac{\lambda}{4!} s^{4-d} \phi'^4 - \tau/6!\, s^{6-2d} \phi'^6 \right]$$

$$\langle \phi(s x_1') \cdots \phi(s x_n') \rangle = s^{n(2-d)/2} \langle \phi'(x_1') \cdots \phi'(x_n') \rangle$$

Take $d=4$

As $s \to \infty$
- m^2 term becomes more and more important
- τ term is less important
- λ term is equally important at all scales

ϕ^2 is relevant	dimension $< d$	$[m^2] > 0$
ϕ^4 is marginal	" $= d$	$[\lambda] = 0$
ϕ^6 is irrelevant	" $> d$	$[\tau] < 0$

Tipped One Way or the other by renormalization

For finite (large) s the dimension of parameters (or operators) tells us their importance

Use mass scale of parameters

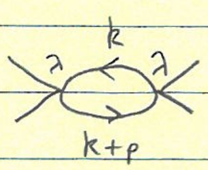
$m^2 \sim (\Lambda^{new})^2$, $\lambda \sim (\Lambda^{new})^0$, $\tau \sim (\Lambda^{new})^{-2}$

large distance sx' means small momenta

$p \ll \Lambda^{new}$

Note: relevant operators can upset p.c. set by Kinetic term [Higgs fine tuning, ...]

Divergences take $m=0$ or small, $m^2 s^2 \sim 1$

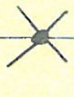


$$\sim \lambda^2 \int \frac{d^dk \quad 1}{(k^2-m^2+i0)((k+p)^2-m^2+i0)}$$

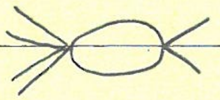
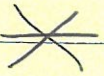
diverges as Λ^{d-4}, $d-4$ = degree of divergence

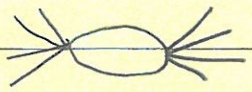
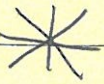
$d=4$ is $\sim \int \frac{d^4k}{k^4} \sim \int \frac{dk}{k} \sim \ln \Lambda$

renormalizes $\lambda \phi^4$



$\delta\lambda$

 $\sim \lambda\tau \int \frac{d^dk}{()()}$ renormalites  $\tau\phi^6$

 $\sim \tau^2 \int \frac{d^dk}{()()}$ renormalizes  ie ϕ^8

Since ϕ^8 is not in $S[\phi]$ the theory is not renormalizable in the traditional sense

But if $\tau \sim \frac{1}{m_{new}^2}$ is small, $p^2\tau \ll 1$, then the theory can be renormalited order by order in $1/m_{new}$

At order $\frac{1}{Mew^4} \sim \tau^2$ we must add a ϕ^8 operator

To include all corrections up to $\frac{1}{(\Lambda_{new})^r}$ or $\frac{1}{s^r}$.

we include all operators with dimension $[O] \leq d+r$

Here power counting $\leftrightarrow$ dimensions

[Seems generic, can anyone see what assumption might change that would lead to non-dimensional p.c.? (homegeneity $s x'^{\mu}$)]

For standard model $\mathcal{L}^{(0)}$ we have all operators with $[O] \leq 4$ ($\rightarrow$ and renormalizable in traditional sense)

For SM correction $\mathcal{L}^{(1)}$ we add $\mathcal{L}^{(1)} = \frac{c}{\Lambda^{new}} \underset{\uparrow \text{ dim-5}}{O_5}$

using coefficient $[c]=0$, $c \sim 1$, and making Λ^{new} explicit.

Since nothing in $\mathcal{L}^{(0)}$ constrains Λ_{new} we're free to take $\Lambda^{new} \gg m_t, m_W$ by as much as we want.

$\mathcal{L}^{(1)}$ gives small corrections.

Corrections to $\mathcal{L}^{(0)} = \mathcal{L}^{SM}$

$$\mathcal{L} = \mathcal{L}^{(0)} + \underset{\sim \frac{1}{\Lambda_{new}}}{\mathcal{L}^{(1)}} + \underset{\sim \frac{1}{\Lambda_{new}^2}}{\mathcal{L}^{(2)}} + \cdots \qquad \text{for } p^2 \sim M_{top}^2 \text{ (say)}$$

- Assume Lorentz invariance and gauge invariance are unbroken: each $\mathcal{L}^{(i)}$ is $SU(3)\times SU(2)\times U(1)$ invariant, Lorentz invariant
- construct $\mathcal{L}^{(i)}$ from <u>same</u> d.o.f. as $\mathcal{L}^{(0)}$ & assume Higgs vacuum expectation value, $v = 246$ GeV (for $E \gg v$ we see full gauge symmetry)
- Assume no new particles produced at p

$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$ doublet Higgs

$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ lepton doublet

eg 1 $$\mathcal{L}^{(1)} = \frac{C_5}{\Lambda_{new}} \epsilon_{ij} \underset{\overset{\uparrow}{(L_L^i)^T C}}{\bar{L}_L^{c\,i}} H^j \epsilon_{kl} L_L^k H^l$$

is <u>only</u> dimension-5 operator consistent with symmetries

Replacing $H \to \begin{pmatrix} 0 \\ v \end{pmatrix}$ gives Majorana mass term for observed ν

$$\frac{1}{2} m_\nu \nu_L^a \nu_L^b \epsilon_{ab} + h.c. \quad , \quad m_\nu = \frac{C_5 v^2}{2\Lambda_{new}}$$

[Pset]

$$m_\nu \lesssim 0.5 \text{ eV} \quad \Rightarrow \quad \Lambda_{new} \gtrsim 6 \times 10^{14} \text{ GeV} \quad \text{for } C_5 \sim 1$$

[Note: majorana mass term violates Lepton number]

eg 2 dimension-6 operators exist that violate baryon number

[Pset]

eg 3 with Lepton # & Baryon # imposed there are 80 dimension 6 operators $\mathcal{L}^{(2)} = \sum_{i=1}^{80} C_i O_i^{(6)}$

- for any observable only a few contribute
- for any new theory at Λ^{new} a particular pattern of C_i's is predicted

For easy reference, here is a reminder of SM charges

Fields	rep SU(3)	rep SU(2)	chg. U(1)	Lorentz	
$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$	3	2	$1/6$	$(\frac{1}{2}, 0)$	spinors 4
u_R^i	3	1	$2/3$	$(0, 1/2)$	
d_R^i	3	1	$-1/3$	$(0, 1/2)$	
$L_L^i = \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix}$	1	2	$-1/2$	$(1/2, 0)$	
e_R^i	1	1	-1	$(0, 1/2)$	
N_R^i	1	1	0	$(0, 1/2)$	
$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$	1	2	$1/2$	$(0, 0)$	scalar
A_A^μ	8	1	0	$(1/2, 1/2)$	vectors
W_a^μ	1	3	0	$(1/2, 1/2)$	
B^μ	1	1	0	$(1/2, 1/2)$	

$i = 1, 2, 3$ family index $\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$

$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$

$\mathcal{L}^{(1)}$ is a singlet under SU(3), SU(2) & carries zero U(1) charge.

$$\mathcal{O}_G = f_{ABC}\, G_\mu^{A\nu}\, G_\nu^{B\lambda}\, G_\lambda^{C\mu}$$

$$\mathcal{O}_{LQ} = (\bar{L}_L \gamma^\mu \sigma^a L_L)(\bar{Q}_L \gamma_\mu \underset{\uparrow SU(2)}{\sigma^a} Q_L) \qquad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

magnetic $\mathcal{O}_W = \bar{L}_L \sigma^{\mu\nu} \sigma^a e_R H W^a_{\mu\nu}$

$\mathcal{O}_B = \bar{L}_L \sigma^{\mu\nu} e_R H B_{\mu\nu}$ which contribute to muon anomalous magnetic moment

$$(g-2)_\mu = \begin{pmatrix} \text{SM contributions} \\ \text{from } \mathcal{L}^{(0)} \end{pmatrix} + c\, \frac{4 m_\mu v}{\underset{\uparrow \Lambda_{new} > 100\ \text{TeV}}{\Lambda_{new}^2}}$$

For the remaining 76 operators see

W. Buchmüller, D. Wyler, Nucl. Phys B268 (1986) p 621-653

A caveat? Their analysis used the tree level equations of motion derived from $\mathcal{L}^{(0)}$ to reduce the # of operators

eg. $$i\not{\partial} e_R^i = \underbrace{g_e^{ij} H^+}_{\sim mass} L_L^j \qquad (\not{p} u = m u)$$

This is obviously fine at lowest order since external lines are put on-shell in Feynman rules

eg $(H^+H)(\bar{e}_R i\not{\partial} e_R)$

$= \bar{u} \not{p} u = \bar{u} m u$

What about loops or propagators?

Representation Independence Thm

Let $\phi = \chi F(\chi)$, $F(0) = 1$, then calculations of observables with $\underset{\text{quantized } \phi}{\mathcal{L}(\phi)}$ give *same* results as with $\underset{\text{quantized } \chi}{\mathcal{L}'(\chi)} = \mathcal{L}(\chi F(\chi))$

eg. $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 + \eta g_1 \phi^6 + \eta g_2 \phi^3 \partial^2 \phi$

$\eta \ll 1$

can "use eom" to drop the last term via $\partial^2\phi = -m^2\phi$

Pf: make field redefinition $\phi \to \phi + \eta g_2 \phi^3$

$\frac{1}{2}(\partial_\mu\phi)^2 \to \frac{1}{2}(\partial_\mu\phi)^2 - \eta g_2 \phi^3 \partial^2\phi + \dots$

$m^2\phi \to m^2\phi^2 + 2\eta g_2 m^2 \phi^4 + m^2\eta^2 g_2^2 \phi^6$

etc.

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda'\phi^4 + \eta g_1'\phi^6 + \mathcal{O}(\eta^2)$ [more on Pset]

Generalized Theorem

Field redefinitions that preserve symmetries & have same 1-particle states allow classical equations of motion to be used to simplify a local $\mathcal{L}_{EFT}$ without changing observables

Refs: C. Arzt, hep-ph/9304230

H. Georgi, Nucl. Phys. B361, p 339-350 (1991)

"On-shell Effective Field Theory"

Pf

$\mathcal{L}_{EFT} = \sum_{n=0}^{\infty} \eta^n \mathcal{L}^{(n)}$, consider removing

$\eta\, T[\varphi]\, D^2\phi$ from $\mathcal{L}^{(1)}$, ϕ complex scalar

$\uparrow$ some local function of various fields "φ"

Gen. Function

$$Z[j] = \int \prod_i \mathcal{D}\varphi_i \exp i \int d^dx \Big[\mathcal{L}^{(0)} + \eta \overbrace{\left[\mathcal{L}^{(1)} - T D^2\phi\right]}^{\text{what we want}} + \eta T D^2\phi + \sum_k j_k \varphi_k + \mathcal{O}(\eta^2) \Big]$$

where Green's functions are obtained by functional derivatives with respect to sources j_k

[dim.reg as regulator for convenience]

Let $\phi^+ = \phi'^+ + \eta T$ in path integral

$$Z = \int \prod_i \mathcal{D}\varphi_i' \left[\frac{\delta\phi^+}{\delta\phi'^+}\right] \exp i \int d^dx \left\{ \mathcal{L}^{(0)} + \eta T \left[\frac{\delta\mathcal{L}^{(0)}}{\delta\phi^+} - \partial_\mu \frac{\delta\mathcal{L}^{(0)}}{\delta\partial_\mu\phi^+}\right]\right.$$

$$\left. + \eta\left(\mathcal{L}^{(1)} - T D^2\phi'\right) + \eta T D^2\phi' + \sum_k j_k \varphi_k + j_{\phi^+}\eta T + \cdots \right\}$$

Changes: i) $\delta\mathcal{L}$ ii) Jacobian , iii) Source term j_{ϕ^+}

Claim is that without changing S-matrix we can remove ii) and iii) , so only need change of variable in $\mathcal{L}$.

ⓘ $\delta\mathcal{L}$: needs $\phi'^+ + \eta T$ to transform as ϕ^+ does [to respect symmetries of theory]

$$\mathcal{L}^{(0)} = (D^\mu\phi)^+(D_\mu\phi) - m^2\phi^+\phi + \cdots$$

$$= (D^\mu\phi')^+(D_\mu\phi') - m^2\phi'^+\phi' + \eta T\left[\underset{\uparrow}{-D^2\phi'} - m^2\phi'\right] + (\cdots)'$$

$\delta\mathcal{L}$ removes desired term

Since $\mathcal{L}^{(1)}$ contains <u>all</u> terms allowed by symmetries the terms in $(\cdots)'$ are already present in $(\cdots)$ Thus couplings are simply redefined

↓ L3

ⓘⓘ Jacobian

recall $\det(\partial^\mu D_\mu) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\, \exp i \int d^4x\, \bar{c}\left[-\partial^\mu D_\mu\right] c$

↑ ghost of Fadeev-Popov procedure

write $\dfrac{\delta\phi^+}{\delta\phi'^+} = 1 + \eta \dfrac{\delta T}{\delta\phi'^+}$ as $\bar{c}c + \eta\, \bar{c}\, \dfrac{\delta T}{\delta\phi'^+}\, c$

term in Lagrangian

Now recall EFT is valid for $p^2 \ll \Lambda_{new}^2$ where $\Lambda_{new} = \frac{1}{\sqrt{\eta}}$. The ghosts will have mass $\sim \Lambda_{new}$ and hence decouple, just like other particles at the mass scale Λ_{new} that we left out

[dropping ghosts can change couplings]

e.g. $T = \partial^2 \phi^\dagger + \lambda \phi^\dagger(\phi^\dagger \phi) \rightarrow \bar{c}(1 + \eta\partial^2 + 2\eta\lambda\phi^\dagger\phi)c$

rescale $c \rightarrow c/\sqrt{\eta}$ $\rightarrow \bar{c}\,(\underset{\substack{\uparrow \\ \text{mass} \\ \Lambda_{new}}}{\Lambda_{new}^2} + \underset{\substack{\uparrow \\ \text{canonical} \\ \text{kinetic term}}}{\partial^2} + 2\lambda\phi^\dagger\phi)\,c$

[Note: we needed $\phi'^\dagger$ term in field redefinition for this arguement, $\phi^\dagger \rightarrow \partial^2\phi^\dagger + \lambda\phi^\dagger\phi^\dagger\phi$ would not be OK]

Ghosts always appear in loops, can be removed like a heavy particle (more later)

(iii) Source $\delta/\delta j_{\phi^\dagger}$

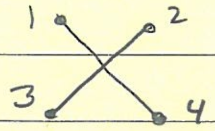
Consider $G^{(n)} = \langle 0|\, T\, \underset{\substack{\uparrow \\ \text{real } \phi \text{ here} \\ \text{for notational} \\ \text{simplicity}}}{\phi(x_1) \cdots \phi(x_n)} \underset{\substack{\uparrow \\ \text{other} \\ \text{fields}}}{\cdots} |0\rangle$

$G^{(n)} = \langle 0|\, T\, (\phi(x_1) + \eta T^{x_1}) \cdots (\phi(x_n) + \eta T^{x_n}) \cdots |0\rangle$

recall LSZ

$$\int d^4x_i\, e^{\pm i p_i x_i} \langle 0|\, T\, \phi(x_1) \cdots \phi(x_n)|0\rangle \underset{p_i^0 \rightarrow \sqrt{\vec{p}_i^{\,2} + m_i^2}}{\sim} \left(\prod_i \frac{\sqrt{Z}\, i}{(p_i^2 - m_i^2 + i0)}\right) \langle p_1 p_2 \cdots |\underset{\substack{\uparrow \\ \text{observables}}}{S}| p_j p_{j+1} \cdots \rangle$$

change to source will drop out for $\langle S \rangle$



eg. $\phi = \phi + \eta \phi = (1+\eta)\phi$

$T[\phi]\, \partial^2 \phi = \phi\, \partial^2 \phi$

get $(1+\eta)^4 \langle 0 | T\, \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) | 0 \rangle$

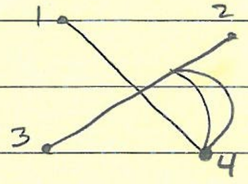
↑ cancelled by $\sqrt{Z} = 1+\eta$

eg. $\phi = \phi + \eta g_2 \phi^3$ ie $T[\phi] = g_2 \phi^3$

get

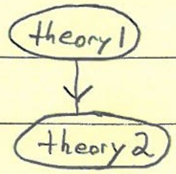
$\eta g_2 \langle 0 | T\, \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi^3(x_4) | 0 \rangle$ etc

↑ less singular, no pole, no contribution for the scattering



eg. $\phi = \phi + \partial^2 \phi = \phi + \underbrace{(\partial^2 + m^2)\phi}_{\frac{(p^2-m^2)}{(p^2-m^2)} \text{ no pole}} - \underbrace{m^2 \phi}_{\text{same as } (1+\eta)\phi \text{ eg. above}}$

Section II : Loops, Renormalization, & Matching



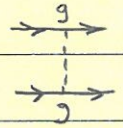
Lets take a theory with a heavy scalar ϕ of mass M and light fermion ψ of mass m

$$\mathcal{L}_{\text{theory}1} = \bar\psi(i\not\partial - m)\psi + \frac{1}{2}\left[(\partial^\mu\phi)^2 - M^2\right] + g\phi\bar\psi\psi$$

$m^2 \ll M^2$, $\dim \leq 4$

To describe ψ's at low energy, $p^2 \ll M^2$ we can remove (integrate out) ϕ

$$\mathcal{L}_{\text{theory}2} = \bar\psi(i\not\partial - m)\psi + \frac{a}{M^2}(\bar\psi\psi)^2 + \dots$$

tree level 

$= \frac{(ig)^2(+i)}{q^2-M^2} = \frac{ig^2}{M^2} + \dots$ so $\underline{a = g^2}$

Matching with Loops requires more care

Regularization : cutoff ultraviolet (UV) divergences to obtain finite results. Introduces cutoff parameters eg. $p^2_{\text{Euclidean}} \leq \Lambda^2_{UV}$, $d = 4-2\epsilon$, lattice spacing

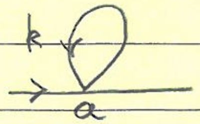
Renormalization : pick a scheme to give definite meaning to each coefficient / operator of the EFT.
May also introduce parameters, egs: μ in $\overline{MS}$, $p^2 = -\mu_R^2$ for offshell momentum subtraction scheme, Λ for Wilsonian

Bare, Renormalized, Counterterm Coefficients:

$$a^{\text{bare}}(\Lambda_{UV}) = a^{\text{ren}}(\Lambda) + \delta a(\Lambda_{UV}, \Lambda)$$
$$a^{\text{bare}}(\epsilon) = a^{\text{ren}}(\mu) + \delta a(\epsilon, \mu)$$

↑ counterterms

Regularization & Power Counting

Consider 

which corrects the ψ's mass by

$$\Delta m \sim \frac{ia}{M^2}\int \frac{d^4k\,(\cancelto{0}{\not k}+m)}{k^2-m^2} = \frac{am}{M^2}\int\frac{d^4k_E}{k_E^2+m^2} \qquad k^0 = ik_E^0$$

If integral is dominated by $k_E \sim m$, $\int \frac{d^4k_E}{k_E^2+m^2} \sim m^2$, and
$\Delta m \sim \frac{am^3}{M^2}$ is a small correction

(as expected for this higher dim. operator)

[What can go wrong?]

Choice:

ⓐ Seems natural to take UV cutoff $\Lambda_{uv} \sim M$

$$\frac{am}{M^2}\int\frac{d^4k_E}{k_E^2+m^2} = \frac{am}{M^2}\frac{2}{(4\pi)^2}\int_0^{\Lambda_{uv}}\frac{dk_E\,k_E^3}{k_E^2+m^2} = \frac{am}{M^2(4\pi)^2}\left[\Lambda_{uv}^2 - m^2\ln\left(1+\frac{\Lambda_{uv}^2}{m^2}\right)\right]$$

$$= \frac{am}{(4\pi)^2}\left[\frac{\Lambda_{uv}^2}{M^2} + \frac{m^2}{M^2}\ln\left(\frac{m^2}{\Lambda_{uv}^2}\right) - \frac{m^4}{M^2\Lambda_{uv}^2} + \cdots\right]$$

$\uparrow$ $\mathcal{O}(1)$ not a correction (here $k\sim\Lambda_{uv}$ dominates)

Can absorb a piece $\int_\Lambda^{\Lambda_{uv}} \frac{d^4k_E}{k_E^2+m^2}$ into $\delta m(\Lambda_{uv},\Lambda)$ $\rightarrow\!\times\!\rightarrow$

to improve things, leaving $\frac{\Lambda^2}{M^2}$ & $\ln\left(\frac{m^2}{\Lambda^2}\right)$ in $\langle(\bar\psi\psi)^2\rangle_{ren}(\Lambda)$

ⓑ $\overline{MS}$ with dim. reg.

$$\frac{\overset{\text{bare}}{a}\,m}{M^2}\int\frac{d^dk_E}{k_E^2+m^2} = \frac{am}{M^2}\frac{2\left(\mu^2e^{\gamma_E}/4\pi\right)^\epsilon}{(4\pi)^{d/2}\,\Gamma(d/2)}\int_0^\infty\frac{dk_E\,k_E^{d-1}}{k_E^2+m^2} = -\frac{am}{M^2}\frac{\pi\csc(\pi\epsilon)e^{\epsilon\gamma_E}}{\Gamma(2-\epsilon)(4\pi)^2}\left(\frac{\mu^2}{m^2}\right)^\epsilon$$

$$= \frac{am}{(4\pi)^2}\left[\frac{m^2}{M^2}\left(-\frac{1}{\epsilon} + \ln\left(\frac{m^2}{\mu^2}\right) - 1\right) + \mathcal{O}(\epsilon)\right]$$

$\uparrow$ as expected for $k^\mu \sim m$ same log as ⓐ with $\mu \leftrightarrow \Lambda$

$\overline{MS}$ counterterm $\overset{\delta m}{-\!\!\times\!\!-} \sim \frac{am^3}{(4\pi)^2M^2}\frac{1}{\epsilon}$

In ⓑ the regularization does not "break" the power counting. We can p.c. regularized graph with $k^\mu \sim M$.

In ⓐ we say p.c. only applies to renormalized couplings & operators order by order [we add counterterms to restore p.c.; could do the same if we broke a symmetry like gauge inv. that we wanted to keep]

In principal any regulator is fine, but we make computations easier if our regulator preserves symmetries [Gauge Inv, Lorentz, Chiral Symmetry, ...]
and
if it preserves power counting by not yielding a mixing of terms of different orders in the expansion.

For dimensional p.c. this corresponds to using "mass independent regulators"

In general operators will always mix with other operators of same dimension & same quantum numbers

$$\mathcal{O}_i^{bare} = \underset{\uparrow}{Z_{ij}}\ \mathcal{O}_j^{ren}$$

matrix of counterterms

(more later)

Dimensional Regularization (type ⓑ)

$$\int \text{đ}^d p = \int \frac{d^d p}{(2\pi)^d}$$

Axioms

- Linearity $\int \text{đ}^d p \,[a f(p) + b g(p)] = a \int \text{đ}^d p\, f + b \int \text{đ}^d p\, g$
- Translation $\int \text{đ}^d p\, f(q+p) = \int \text{đ}^d p\, f(p)$
  & Rotation
- Scaling $\int \text{đ}^d p\, f(s p) = s^{-d} \int \text{đ}^d p\, f(p)$ $\quad p \to s^{-1} p$

gives unique integration def'n up to normalization ⇒ dim. reg.

[Steps of proofs, Collins pg 65, expand f in basis fns $f_{s,q}(p) = e^{-s(p+q)^2}$
let $\int d^d p_E\, e^{-p_E^2} = \pi^{d/2}$]

Euclidean $d^d p = dp\, p^{d-1}\, d\Omega_d = dp\, p^{d-1}\, d(\cos\theta)\,(\sin\theta)^{d-3}\, d\Omega_{d-1}$

UV div. occurs in 1-dim integration

$$\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

$\text{đ}^n p = \dfrac{2}{(4\pi)^{n/2}\,\Gamma(n/2)}\, dp\, p^{n-1}$ for spherical symm. integration

$$\int \text{đ}^d p\, \frac{(p^2)^\alpha}{(p^2+A)^\beta} = \frac{1}{(4\pi)^{d/2}}\, A^{d/2+\alpha-\beta}\, \frac{\Gamma(\beta-\alpha-d/2)}{\Gamma(\beta)}\, \frac{\Gamma(\alpha+d/2)}{\Gamma(d/2)}$$

[integral tricks on next page ...]

$d = 4 - 2\epsilon$ $\quad \epsilon > 0$ tames UV , $\epsilon < 0$ tames IR

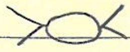
FACTS

① $\int \text{đ}^d p\, (p^2)^\alpha = 0$ [Pf. see Collins pg 71] eg $\int đ^d p \ (p+q_0)^{2K}$ $K \in \mathbb{Z}$, $K>0$

$= \int đ^d p \ [p^{2K} + \underset{\uparrow > 0}{(\#)} p^{2K-2} q_0^2 + \underset{\uparrow > 0}{(\#)} p^{2K-4} q_0^4 + \dots]$

$= \int đ^d p \ p^{2K}$ any k, any q_0

→ proves FACT ① for $\alpha = K$ integers

eg. (Be Careful)  zero momenta mass $\int \frac{đ^d p}{p^4} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right) = 0$

$\epsilon_{UV} = \epsilon_{IR}$

but still needs counterterm

 $= \frac{-i}{16\pi^2}\frac{1}{\epsilon_{UV}}$ so  +  $= \frac{-i}{16\pi^2} 1/\epsilon_{IR}$

② Dim. Reg. is well defined even with both UV & IR divergences (where no value of d yields convergent integral)

Use analytic continuation

eg. Suppose $\int đ^d p \ f(p^2) = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dp \ p^{d-1} f(p^2)$ exists for $0 < d < d_{max}$

To obtain range $-2 < d < d_{max}$

$$\int đ^d p \ f(p^2) = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\left\{\underbrace{\int_C^\infty dp \ p^{d-1} f(p^2)}_{UV\ part} + \underbrace{\int_0^C dp \ p^{d-1}[f(p^2) - f(0)] + \frac{f(0)\, C^d}{d}}_{IR\ part}\right\}$$

which is independent of C, so taking $C \to \infty$ when $-2 < d < 0$

gives $\int đ^d p \ f(p^2) = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dp \ p^{d-1}[f(p^2) - f(0)]$

[Also $\int đ^0 p \ f(p^2) = f(0)$]

etc.

<u>MS scheme</u> Introduce μ to keep any renormalized couplings dimensionless

$g^{bare}\, \bar\Psi \not{A} \Psi$ $[g^{bare}] = \frac{(4-d)}{2} = \epsilon$

$\underset{g^{bare}}{\frac{(4-d)}{2}} + \underset{\bar\Psi, \Psi}{\frac{(d-1)}{2} 2} + \underset{A}{\frac{(d-2)}{2}} = d$

$g^{bare} = Z_g \, \mu^\epsilon \, g(\mu)$

$$\frac{a^{bare}}{M^2}\,(\bar\psi\psi)^2 \;,\quad [a^{bare}] = 4-d = 2\epsilon \;,\quad \underset{a^{bare}}{(4-d)} + \underset{M^{-2}}{(-2)} + \underset{\psi\text{'s}}{4\frac{(d-1)}{2}} = d$$

$$a^{bare} = Z_a\, \mu^{2\epsilon}\, a(\mu)$$

[Note: μ^ϵ factors are not associated with loop measure]

$\overline{MS}$ scheme $\mu^2 \to \mu^2 e^{\gamma_E}/4\pi$ (remove large universal constants)

good: - preserves symmetries [Gauge, Lorentz, ...]
- easy to calculate [multiloop, ...]
- often gives manifest p.c.

disadvantages:
- physical picture less clear, can loose positive definiteness for renormalized quantities
- does not satisfy decoupling theorem
- can introduce "renormalons", ie poor convergence at large orders in perturbation theory (more later)

Decoupling Thm (Appelquist & Carazone) If remaining low energy theory is renormalizable, and we use a physical renormalization scheme, then all effects due to heavy particles appear as change in coupling constants or are suppressed by $1/M$

[physical scheme like offshell mom. subt.]

→ $\overline{MS}$ not "physical", its mass independent & does not see mass thresholds. Must implement decoupling "by hand" by removing particle of Mass M for $\mu \le M$

e.g. QCD in $\overline{MS}$, $\beta(g) = \mu \frac{d}{d\mu} g(\mu) = -\frac{g^3}{16\pi^2} \underbrace{\left(\frac{11}{3} C_A - \frac{4}{3} T_F n_f\right)}_{\equiv b_0} < 0$

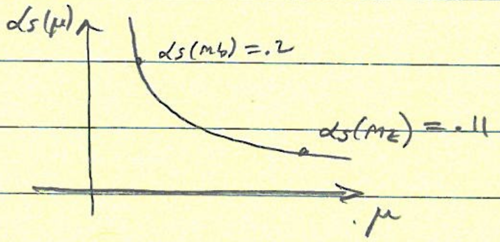
$+ \mathcal{O}(g^5)$

$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi}$

Asymptotically Free

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0) \frac{b_0}{2\pi} \ln\left(\frac{\mu}{\mu_0}\right)}$$

↑ lowest order solution



$$\Lambda_{QCD}^{\overline{MS}} \equiv \mu \exp\left[-\frac{2\pi}{b_0\,\alpha_s(\mu)}\right] , \quad \alpha_s(\mu) = \frac{2\pi}{b_0 \ln(\mu/\Lambda_{QCD})}$$

↑ indep of μ, ~ scale where QCD becomes non-pert ~200 MeV

depends on ① order of loop expansion for $\beta(g)$ ⓘⓘ n_f ⓘⓘⓘ scheme (beyond 2-loops)

Apriori top & up quarks contribute to b_0 for any μ & decoupling thm doesn't apply

Solution Implement decoupling by hand by integrating out fermion at $\mu \sim m$. An example of matching.

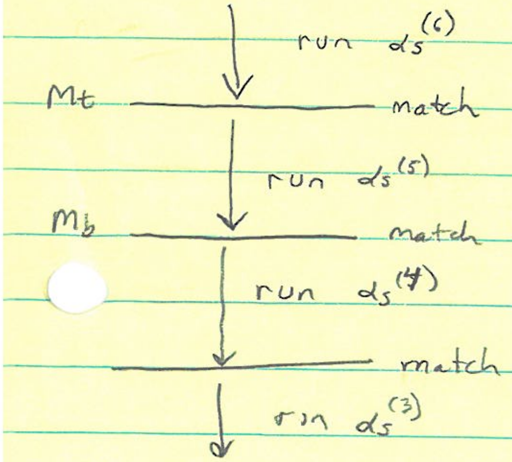
← what's allowed?

$$b_0 = \begin{cases} \frac{11}{3} C_A - \frac{4}{3} T_F \overset{n_f}{(6)} & \text{for } m_t < \mu \\ \frac{11}{3} C_A - \frac{4}{3} T_F \,(5) & \text{for } m_b < \mu < m_t \\ \vdots & \end{cases}$$

[Can do matching in other schemes too, but in $\overline{MS}$ we're forced to do it.]

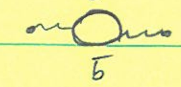
Matching Condition : At a scale $\mu = \mu_m \sim m$ we demand that S-matrix elements with light external particles agree between theories I & II.

LO condition $\alpha_s^{I}(\mu_0) = \alpha_s^{II}(\mu_0)$



$\alpha_s^{(5)}(\mu_b) = \alpha_s^{(4)}(\mu_b)$, LO condition makes coupling continuous at threshold

This is not true at higher orders



$$\alpha_s^{(4)}(\mu_0) = \alpha_s^{(5)}(\mu_0)\left[1 + \frac{\alpha_s^{(5)}}{\pi}\left(\frac{-1}{6}\ln\frac{\mu_0^2}{m_b^2}\right) + \left(\frac{\alpha_s^{(5)}}{\pi}\right)^2\left(\frac{11}{72} - \frac{11}{24}\ln\frac{\mu_0^2}{m_b^2} + \frac{1}{36}\ln^2\frac{\mu_0^2}{m_b^2}\right) + \dots\right]$$

$\triangle$

There are <u>no</u> <u>large</u> <u>logs</u> as long as we match for $\mu_0 = \mu_b \sim m_b$

<u>General Procedure</u> for massive particles (any operator/couplings)

$$m_1 \gg m_2 \gg m_3 \gg \dots \gg m_n$$
$$\mathcal{L}_1 \to \mathcal{L}_2 \to \mathcal{L}_3 \to \dots \quad \mathcal{L}_n$$

① Match theory $\mathcal{L}_1$ at $\mu_1 \sim m_1$ onto $\mathcal{L}_2$

② Compute β-functions and anom. dim in theory $\mathcal{L}_2$ (which does not have particle 1), evolve couplings down

③ Match $\mathcal{L}_2$ at $\mu_2 \sim m_2$ onto $\mathcal{L}_3$

④ ...

⋮

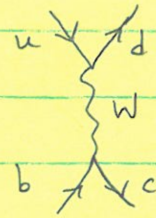
Ⓚ Say we're interested in dynamics at scale $\mu \sim m_n$, then we compute final matrix elements in $\mathcal{L}_n$

$\boxed{\text{Massive SM Particles}}$ t, H, W, Z remove them simultaneously

Other possibility: $m_t \gg m_W, m_Z$? Integrating out top breaks $SU(2) \times U(1)$ gauge inv since we remove t from $\begin{pmatrix} t_L \\ b_L \end{pmatrix}$ doublet [Wess-Zumino terms]. Also $\frac{m_Z}{m_t} \sim \frac{1}{2}$ is not a great expansion.

By removing them simultaneously we "miss" running $m_t \to m_W$, ie we treat $\alpha_s(m_W)\ln(m_W^2/m_t^2)$ perturbatively eg. $b \to c \bar{u} d$ $\qquad \mathcal{L}_{sm} = \frac{g_2}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu V_{ckm} d_L$

<u>tree Level</u>



$$= \left(\frac{i g_2}{\sqrt{2}}\right)^{2} \overset{V_{cb}V_{ud}^*}{\downarrow} (-i) \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{M_W^2}\right) \frac{1}{k^2 - M_W^2} \left[\bar{u}^c \gamma_\mu P_L u^b\right]\left[\bar{u}^d \gamma_\nu P_L v^u\right]$$

spinors

$k^\mu = p_b - p_c = p_u + p_d$ $\qquad$ momenta ~ masses

$\not{p}_b u^b = m_b u^b$ etc

Expand propagator $\quad \frac{-i g^{\mu\nu}}{(-M_W^2)} + \mathcal{O}\left(\frac{m_b^2}{M_W^4}\right)$

$$= -\frac{i 4 G_F}{\sqrt{2}} \left[\bar{u}^c \gamma_\mu P_L u_b\right]\left[\bar{u}_d \gamma^\mu P_L v^u\right] V_{cb} V_{ud}^*$$

$$G_F = \frac{\sqrt{2}\, g_2^2}{8 M_W^2}$$ [measured by muon decay]

EFT removing t, W, Z, H is called "Electroweak Hamiltonian"
often work to first order in G_F

$$H_W = -\mathcal{L}_W = \frac{4 G_F}{\sqrt{2}} \left[\bar{c} \gamma_\mu P_L b\right]\left[\bar{d} \gamma^\mu P_L u\right] V_{cb} V_{ud}^*$$

Most General Basis of Operators (this flavor) under QCD

- At $\mu = m_W$ can treat b, c, d, u as massless to get Coeff.
  & QCD does not change chirality

  $\bar{c}\, \gamma_\mu P_L b$

  ↑ need odd # γ's, but three reduces to one

  $[\gamma_\alpha \gamma_\beta \gamma_\nu = g_{\alpha\beta}\gamma_\nu + g_{\beta\nu}\gamma_\alpha - g_{\alpha\nu}\gamma_\beta - i\epsilon_{\alpha\beta\nu\tau}\gamma^\tau\gamma_5]$

  $\epsilon_{0123} = 1$

- Spin Fierz

$$(\bar{\Psi}_1 \gamma_\mu P_L \Psi_2)(\bar{\Psi}_3 \gamma^\mu P_L \Psi_4) = (-1)^2 (\bar{\Psi}_1 \gamma_\mu P_L \Psi_4)(\bar{\Psi}_3 \gamma^\mu P_L \Psi_2)$$

↑ one from Fierz, one from fermion field interchange

write $(\bar{c}\,\Gamma b)(\bar{d}\,\Gamma u)$

Color Fierz remove $(T^A)_{\alpha\beta}(T^A)_{\gamma\delta}$

$$H_W = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \left[C_1(\mu)\, O_1(\mu) + C_2(\mu)\, O_2(\mu) \right]$$

$$O_1(\mu) = \bar{c}^\alpha \gamma_\mu P_L b_\alpha \ \bar{d}_\beta \gamma^\mu P_L u^\beta \qquad C_i(\mu) = C_i\left(\frac{\mu}{m_W}, \alpha_s(\mu)\right)$$

$$O_2(\mu) = \bar{c}^\beta \gamma_\mu P_L b_\alpha \ \bar{d}^\alpha \gamma^\mu P_L u_\beta$$

Match at $\mu = m_W$:

tree level

$$C_1(1, \alpha_s(m_W)) = 1 + O(\alpha_s(m_W))$$

$$C_2(1, \alpha_s(m_W)) = 0 + \text{"}$$

<u>Key Fact</u>: Matching <u>indep.</u> of choice of states & IR regulator as long as same choice is made in <u>both</u> theories 1 & 2

Used free quark states, but <u>result is valid</u> for use with bound states $B, D, \pi, \ldots$

Renormalize EFT at 1-loop in $\overline{MS}$ $(d=4-2\epsilon)$

$\psi = Z_\psi^{-1/2}\,\psi^{(0)}$

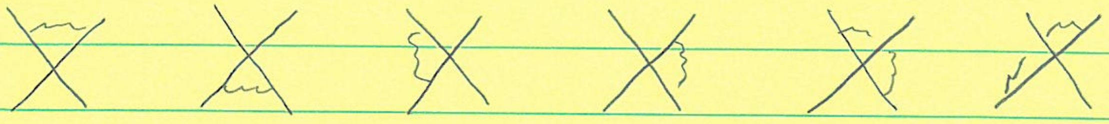
$\overline{MS}$ $\quad Z_\psi = 1 - \frac{\alpha_s C_F}{4\pi\epsilon}$, $C_F = \frac{4}{3}$, suppress prefactor $\frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^*$

Use Feyn. Gauge

Let $\langle \bar{u} d c | O_1 | b\rangle = S_1 = (\bar{c}^\alpha \gamma_\mu P_L b_\alpha)(\bar{d}_\beta \gamma^\mu P_L u^\beta)$

$\langle \bar{u} d c | O_2 | b\rangle = S_2$

↑ tree level matrix elements



regulate IR with offshell momenta p (masses 0)

bare

cancelled by Z_ψ

$$\langle O_1\rangle^{(0)} = \left[1 + 2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2}\right)\right] S_1$$

c.t. for O_2

c.t. for O_1

$$+ \frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2}\right) S_1 - 3\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2}\right) S_2 + \ldots$$

$\langle O_2\rangle^{(0)}$ = (same with $S_1 \leftrightarrow S_2$) "O_1 mixes into O_2"

Two methods to renormalize:

① Composite Operator Renormalization $\quad O_i^{(0)} = Z_{ij}\, O_j(\psi^{(0)})$

$[\psi^{(0)} = Z_\psi^{1/2}\psi]$ ↓ if your confused about why its -2 see reading!

$$\langle O_i\rangle^{(0)} = Z_\psi^{-2} Z_{ij}\langle O_j\rangle$$

↑ renormalized amputated green's fn

$$\langle O_j\rangle = Z_\psi^{2} (Z^{-1})_{ji}\langle O_i\rangle^{(0)}$$

② Renormalize Coefficients

$$\mathcal{H} = C_i^{(0)} O_i(\psi^{(0)}) = Z_\psi^2 (Z_{ij}^c C_j) O_i = C_i O_i + \underbrace{(Z_\psi^2 Z_{ij}^c - \delta_{ij})}_{\text{counterterm}} C_j O_i$$

$$= (C_j O_j)^{ren}$$ renormalized

$$Z_\psi^2 \, Z_{ij}^c \, C_j \langle O_i \rangle^{(0)} = C_j \langle O_j \rangle = C_j \, Z_{ji}^{-1} Z_\psi^2 \langle O_i \rangle^{(0)}$$

$$\Rightarrow \qquad Z_{ij}^c = Z_{ji}^{-1}$$

same info either way

Need $Z_{ij} = \mathbb{1}_{ij} + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \begin{pmatrix} 3/N_c & -3 \\ -3 & 3/N_c \end{pmatrix}_{ij}$

$\uparrow \overline{MS}$

explain
- mixing
- diagonal

Anom. Dim. for Operators

$$0 = \mu \, d/d\mu \, O_i^{(0)} = (\mu \, d/d\mu \, Z_{ij'}) O_{j'} + Z_{ij} (\mu \, d/d\mu \, O_j)$$

rearrange:

$$\underset{\uparrow \text{anom. dim matrix}}{\mu \, d/d\mu \, O_j \equiv -\gamma_{ji} O_i} \quad , \quad \gamma_{ji} = \underset{\uparrow \text{drop @ 1-loop}}{Z_{jk}^{-1}} (\mu \, d/d\mu \, Z_{ki})$$

Use: $\mu \, d/d\mu \, \alpha_s(\mu) = -2\epsilon \alpha_s + \underset{\uparrow \text{drop @ 1-loop}}{\beta[\alpha_s]}$ in d-dimensions

$$\gamma_{ji} = -\frac{\alpha_s}{2\pi} \begin{pmatrix} 3/N_c & -3 \\ -3 & 3/N_c \end{pmatrix}$$

Solve Diagonalize $O_\pm = O_1 \pm O_2$, coeff $C_\pm$

$\mu \, d/d\mu \, O_+ = \gamma_+ O_+$ $\qquad \gamma_+ = -\frac{\alpha_s}{2\pi}\left(3/N_c - 3\right) = \alpha_s/\pi$

$\mu \, d/d\mu \, O_- = \gamma_- O_-$ $\qquad \gamma_- = -\frac{\alpha_s}{2\pi}\left(\frac{3}{N_c} + 3\right) = -\frac{2\alpha_s}{\pi}$

↑ $O_\pm$ don't mix @ 1-loop

$$H_W = C_1 O_1 + C_2 O_2 = C_+ O_+ + C_- O_-$$

$$C_+ = \frac{C_1 + C_2}{2} , \quad C_- = \frac{C_1 - C_2}{2}$$

Tree Level matching $C_\pm(\mu = M_W) = \frac{1}{2} + O(\alpha_s(M_W))$

↓L5

Anom. Dim for Coeffs $\quad 0 = \mu \frac{d}{d\mu} H = (\mu \, d/d\mu \, C_i) O_i - C_j (\gamma_{ji} O_i)$

$$\mu \, d/d\mu \, C_i = C_j \gamma_{ji} = (\gamma^T)_{ij} C_j$$

Solution for $\mu \gg \Lambda_{QCD}$

$C = C_+$ or C_- $\qquad \mu \, d/d\mu \ln C(\mu) = \gamma_\pm[\alpha_s(\mu)]$

$$\mu \, d/d\mu \, \alpha_s(\mu) = \beta[\alpha_s(\mu)] = -2\beta_0 \alpha_s^2$$

Swap variables $\mu \to \alpha_s$: $\quad \dfrac{d\mu}{\mu} = \dfrac{d\alpha_s}{\beta[\alpha_s]} = \dfrac{-1}{2\beta_0} \dfrac{d\alpha_s}{\alpha_s^2}$

$$\ln \frac{C^\pm(\mu)}{C^\pm(\mu_W)} = \int_{\mu_W}^{\mu} \frac{d\mu}{\mu} \gamma^\pm = \frac{-1}{2\beta_0} \int_{\alpha_s(\mu_W)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\alpha_s^2} \gamma^\pm$$

$$= a_\pm \ln\left(\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)}\right) \qquad a_+ = \frac{1}{2\pi\beta_0}$$

$$a_- = \frac{-1}{\pi\beta_0}$$

Here boundary condition is $C^\pm(\mu_W)$ where $\mu_W \sim M_W$

typically $\mu_W = M_W$, $2M_W$, or $M_W/2$

You should think of $C^\pm(\mu_W)$ as fixed order series in $\alpha_s(\mu_W)$

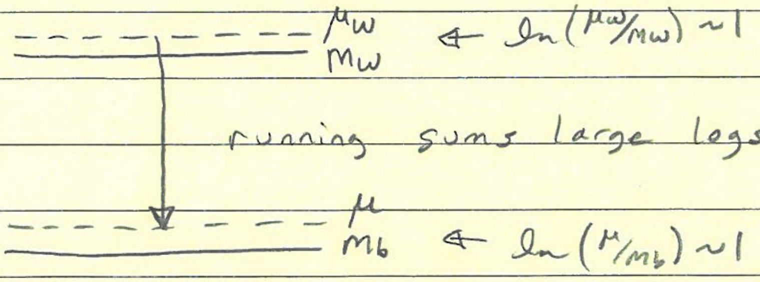
Solution $\quad C^\pm(\mu) = C^\pm(\mu_W) \left[\dfrac{\alpha_s(\mu_W)}{\alpha_s(\mu)}\right]^{a_\pm} = C^\pm(\mu_W)\, e^{a_\pm \ln\left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)}\right]}$

Say $\mu \sim m_b$ for $b \to c\bar{u}d$ transition

Result for $C_\pm(\mu)$ sums the leading logs (LL)

$\sim \frac{1}{2} + \alpha_s(M_W) \ln\left(\frac{M_W}{m_b}\right) + \alpha_s^2 \ln^2 + \cdots$ $\quad$ as if $\alpha_s \ln \sim 1$

Physical Picture:



General Form $C^i(\mu) = C^j(\mu_W) \underbrace{U^{ji}(\mu_W,\mu)}_{\text{evolution}}$ ← valid to higher orders

$$H_W = \left(\frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cb}\right) \sum_i \underbrace{C_j(\mu_W)}_{\text{fixed order}} U_{ji}(\mu_W,\mu_b) \underbrace{O_i(\mu_b)}_{\text{matrix elts at } \mu_b \sim m_b}$$

What do we need to compute to get:

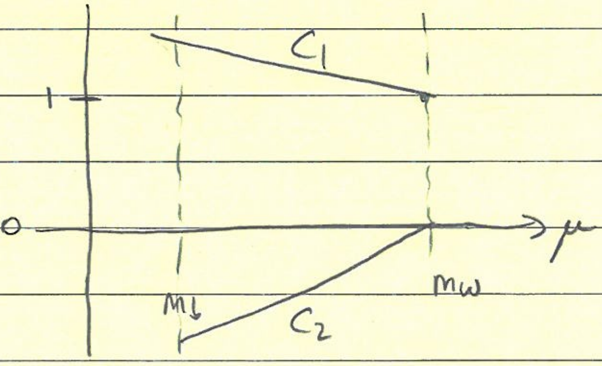
$$C_i(\mu) \sim \underbrace{\sum_K [\alpha_s \ln]^K}_{LL} + \underbrace{\alpha_s \sum_K [\alpha_s \ln K]}_{NLL} + \overbrace{\alpha_s^2 \sum_K [\alpha_s \ln]^K}^{NNLL} + \dots$$

RGE improved perturbation theory

NLL = next-to-LL, etc

	Matching $C(\mu_W)$	Running γ
LL	tree-level	1-loop
NLL	1-loop	2-loop
⋮	⋮	⋮

Note: even though O_2 has $C_2 = 0$ (zero tree-level matching) it is induced by renormalization group flow



$C_1(m_b) \simeq 1.12$

$C_2(m_b) \simeq -.28$

at LL

Physical application: $b \to c\,\bar{u}\,d$ gives $\bar{B} \to D\pi$

$(\bar{u}b)\ (\bar{u}c)(\bar{u}d)$

$\langle D\pi \,|\, H_w \,|\, B\rangle = C_i(M_W)\,\langle D\pi|\, O_i(M_W)\,|B\rangle$ $\underline{\text{at } \mu = M_W}$

↑ has large logs, $\ln\left(M_W^2/-p_{IR}^2\right)$,

a problem for dim. analysis or lattice calculations

$\underline{\mu = M_b}$ $= C_i(M_b)\,\langle D\pi|\, O_i(M_b)\,|B\rangle$

↑ no $\ln(M_W/M_b)$ now

↑ has large logs, but they are summed by our renormalization group expression

Physically $C_i(M_b)$ are the right couplings to use.

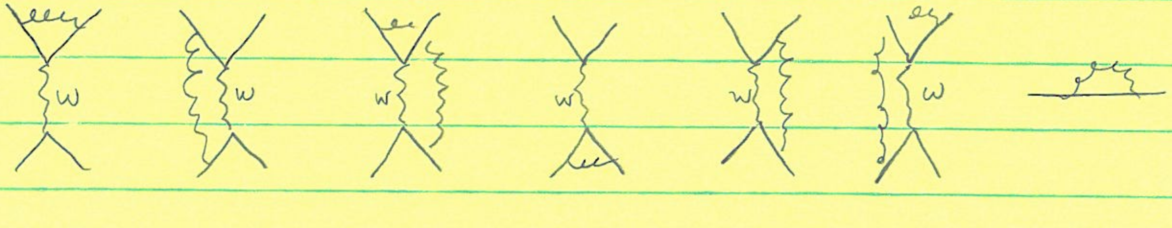
Compare with full theory

We already renormalized EFT in $\overline{MS}$ (theory 2)

"Full theory" calculation involves conserved currents, so UV divergences in vertex & wavefunction graphs cancel to leave UV finite result

↗ theory 1

full theory



$\boxed{\text{Consider Logs}}$ first,

$$iA^{\text{1-loop}} = \left[1 + 2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right]S_1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\ln\frac{M_W^2}{-p^2}\,S_1 + \cdots$$

↑ non-log & S_2 terms ↓

EFT, set $C_1 = 1$, $C_2 = 0$

$$\langle O_1\rangle^{\text{1-loop}} = \left[1 + 2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right]S_1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\,S_1 + \cdots$$

comments:

① EFT computation involves only triangles , much easier. And $1/N_c$ term is all that's required to compute anom. dim, and is even easier.

② In EFT $M_W \to \infty$, so M_W's become μ's (cutoff's)

③ $\ln(-p^2)$ terms match. ie IR agrees. This is a check that our EFT has the right low energy degrees of freedom. [Kind of obvious here, but in other theories can be non-trivial]

④ Difference of $\mathcal{O}(\alpha_s)$ renormalized calc's gives <u>one-loop matching</u>
↑ logs & non-logs

$$0 = iA - \left[C_1\langle O_1\rangle + C_2\langle O_2\rangle\right]$$

$$= S_1 - (1)\,S_1 \qquad \text{tree-level} \checkmark$$

$$+\ iA^{\mathcal{O}(\alpha_s)} - C_1^{(1)}S_1 - (1)\langle O_1\rangle^{\mathcal{O}(\alpha_s)} - \underset{\uparrow\text{ starts at }\mathcal{O}(\alpha_s)}{C_2^{(1)}S_2}$$

wrote $C_1 = 1 + \underset{\uparrow\,\mathcal{O}(\alpha_s)}{C_1^{(1)}} + \cdots$, $C_2 = C_2^{(1)} + \cdots$

For S_1 * Logs:

$$i A^{\mathcal{O}(\alpha_s S_1)} - \langle O_1 \rangle^{\mathcal{O}(\alpha_s S_1)} = C_1^{(1)} S_1$$

$$\frac{3}{N_c}\frac{\alpha_s C_F}{4\pi} \ln\left(\frac{M_W^2}{-p^2}\right) - \frac{3}{N_c}\frac{\alpha_s C_F}{4\pi} \ln\left(\frac{\mu^2}{-p^2}\right) = C_1^{(1)}$$

$$\left[\begin{array}{l}\text{analog for } S_2 \text{ gives} \\ C_2^{(1)} = \frac{3\alpha_s}{4\pi}\ln\frac{\mu^2}{M_W^2}\end{array}\right]$$

$$C_1^{(1)} = -\frac{3}{N_c}\frac{\alpha_s C_F}{4\pi} \ln\left(\mu^2/M_W^2\right)$$ ← indep. of IR regulator

Full theory = Large Momenta * Small Momenta

$$ C \qquad \langle O \rangle$$

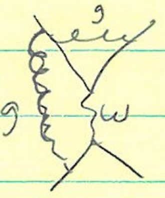
$$\ln\frac{M_W^2}{-p^2} = \ln\frac{M_W^2}{\mu^2} + \ln \mu^2/_{-p^2}$$

μ^2 seperates large M_W^2 from small $-p^2$

$$\left(1 + \alpha_s \ln\frac{M_W^2}{-p^2}\right) = \left(1 + \alpha_s \ln\frac{M_W^2}{\mu^2}\right)\left(1 + \alpha_s \ln\frac{\mu^2}{-p^2}\right)$$

⑤ order by order in $\alpha_s(\mu)$ the $\ln \mu$'s in $C(\mu)\langle O\rangle(\mu)$ cancel between C and $\langle O\rangle$. The result is μ-independent at the order in α_s we're working

⑥ Note that at level of log's full theory has less information. To compute 2-loop $\alpha_s^2 \ln^2(M_W^2/_{-p^2})$ we'd need



etc. In EFT just need 1-loop anom-dim of all operators.

⑦ The one-loop anomalous dimensions γ_{ij} are scheme independent (in mass indep. renormalization schemes)

Next Logs & Constants

- full 1-loop matching (enters at NLL)
- coefficients, matrix elements, anom. dim at NLO are <u>scheme dependent</u>, but $C(\mu)\langle \mathcal{O}(\mu)\rangle$ is independent of EFT scheme (much like its indep. of μ)
- ignore mixing for simplicity

$$A^{EFT} = C(\mu)\ \langle \mathcal{O}(\mu)\rangle$$

a constant like 3

<u>Match</u>
$$A^{full} = 1 + \frac{\alpha_s(\mu)}{4\pi}\left[-\frac{\gamma^{(0)}}{2}\ln\frac{M_W^2}{-p^2} + A^{(1)}\right]$$

a constant

$$A^{EFT} = C(\mu)\left[1 + \frac{\alpha_s(\mu)}{4\pi}\left(-\frac{\gamma^{(0)}}{2}\ln\frac{\mu^2}{-p^2} + B^{(1)}\right)\right]$$

$$C(\mu_W \sim M_W) = 1 + \frac{\alpha_s(\mu_W)}{4\pi}\left[+\frac{\gamma^{(0)}}{2}\ln\frac{\mu_W^2}{m_W^2} + A^{(1)} - B^{(1)}\right]$$

$$C(\mu = M_W) = 1 + \frac{\alpha_s(M_W)}{4\pi}\left[A^{(1)} - B^{(1)}\right]$$

constant from two-loop EFT calc

<u>Run @ NLL</u>
$$\mu\, d/d\mu\ C(\mu) = \gamma[\alpha_s]\ C(\mu)$$
$$\mu\, d/d\mu\ \ln C = \gamma[\alpha_s] = \gamma^{(0)}\frac{\alpha_s(\mu)}{4\pi} + \gamma^{(1)}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2$$

this RGE is a coupled diff. equation, need RGE for $\alpha_s(\mu)$ at NLL too

$$\mu\, d/d\mu\ \alpha_s(\mu) = -2\alpha_s(\mu)\left[\underset{\uparrow \atop \frac{11}{3}C_A - \frac{2}{3}n_f}{\beta_0\frac{\alpha_s(\mu)}{4\pi}} + \underset{\uparrow \atop \frac{34}{3}C_A^2 - \frac{10}{3}C_A n_f - 2C_F n_f}{\beta_1\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2}\right] = \beta[\alpha_s]$$

$$\frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta[\alpha_s]}$$

[sorry, I switched conventions here]

$$\begin{tikzcd} \mu_0 \arrow[d] \\ \mu \end{tikzcd}$$

All orders solution $\ln \dfrac{C(\mu)}{C(\mu_0)} = \displaystyle\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \, \frac{\gamma[\alpha_s]}{\beta[\alpha_s]}$

← expand to 2nd order for NLL

With notation

$$C(\mu) = U(\mu,\mu_0)\, C(\mu_0)$$

$$U(\mu,\mu_0) = \exp\left[\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \, \frac{\gamma}{\beta}\right]$$

- take $\mu_0 = m_W$
- expand / integrate to find

$$U^{NLL}(\mu,\mu_0) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J\right] \left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{\frac{\gamma^{(0)}}{2\beta_0}} \left[1 - \frac{\alpha_s(m_W)}{4\pi} J\right]$$

$$J = \frac{\gamma^{(0)}\beta_1}{2\beta_0^2} - \frac{\gamma^{(1)}}{2\beta_0}$$

Combine ^{NLO}Matching & ^{NLL}Running

$$C(\mu) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J\right] \left(\frac{\alpha_s(m_W)}{\alpha_s(\mu)}\right)^{\frac{\gamma^{(0)}}{2\beta_0}} \left[1 + \frac{\alpha_s(m_W)}{4\pi}\left(A^{(1)} - B^{(1)} - J\right)\right]$$

Claim:
- $B^{(1)}, \gamma^{(1)}, J, C, \langle O \rangle$ are scheme dependent
- $\beta_0, \beta_1, \gamma^{(0)}, A^{(1)}, B^{(1)} + J, C\langle O \rangle$ are independent of EFT scheme choice

Sketch Proof: Consider $\langle O \rangle' = \left(1 + \frac{\alpha_s}{4\pi} S\right) \langle O \rangle$

← some constant, specifying scheme

from $Z' = Z\left(1 - \frac{\alpha_s}{4\pi} S\right)$

← scheme indep. $\langle O \rangle^{(0)} = Z_\psi^{-2} Z \langle O$

then $B^{(1)\prime} = B^{(1)} + S$, $C' = \left(1 - \frac{\alpha_s}{4\pi} S\right) C$

Now $\gamma^{(1)\prime} = \gamma^{(1)} + 2\beta_0 S$

$$\langle O(\mu) \rangle\, U(\mu, m_W)\, C(m_W) = \langle O' \rangle\, U'(\mu, m_W)\, C'(m_W)$$

← observables scheme indep.

$$U'(\mu, m_W) = \left(1 - \frac{\alpha_s(\mu)}{4\pi} S\right) U(\mu, m_W) \left(1 + \frac{\alpha_s(m_W)}{4\pi} S\right)$$

so $J' = J - S$

$$J' + B^{(1)\prime} = J + B^{(1)}$$

Conclude

① $A^{(1)} - B^{(1)} - J$ is scheme indep. Cancels between $\gamma^{(1)}$ and $B^{(1)}$ at upper end of evolution ($\mu = M_W$) (matching & anom. dim. scheme dependencies cancel)

② LL result $\left(\alpha_s / \alpha_s\right)^{\gamma^{(0)}/2\beta_0}$ scheme indep.

③ scheme dependence of $\left(1 + \frac{\alpha_s(\mu)}{4\pi} J\right)^{\text{in } C(\mu)}$ is cancelled by scheme dependence of $\langle O(\mu)\rangle$ at lower end of evolution (at μ)

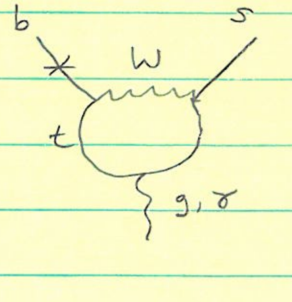
Remarks

- $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is inherently 4-dimensional, and must be treated carefully in dim. reg. (HV scheme see Collins)
- Evanescent Operators
  $\{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}\}$ is not a complete basis in d-dim. Additional ops called evanescent (vanish as $\epsilon \to 0$)

Phenomenology from Weak Hamiltonian

e.g. $b \to s\gamma$ flavor changing neutral current process

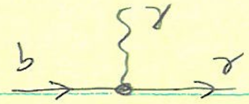
$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s} \sigma^{\mu\nu} (1+\gamma_5) b F_{\mu\nu}$$



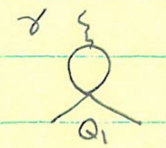
$$Q_{8G} = \frac{g}{8\pi^2} m_b \bar{s} T^a \sigma^{\mu\nu} (1+\gamma_5) b G^a_{\mu\nu}$$

$$Q_1 = [\bar{s} \gamma^\mu (1-\gamma_5) u] [\bar{u} \gamma_\mu (1-\gamma_5) d]$$

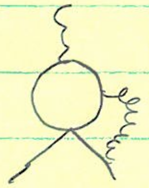
[also $Q_2, \dots, Q_{10}$
4-quark ops]

LO  $C_{7\gamma}^{LO} = C_{7\gamma}^{LO}\left(\frac{m_W}{m_t}\right)$ −0.195

$O_{7\gamma}$



$= 0$ in HV scheme for γ_5



diverges, 2-loop calc givs $\gamma^{(0)}$ at order α_s

LL solution

$$C_{7\gamma}(\mu) = \eta^{16/23} C_{7\gamma}^{LO} + \frac{8}{3}\left(\eta^{14/23} - \eta^{16/23}\right) C_{8G}^{LO} + \sum_{i=1}^{8} h_i \eta^{a_i} C_1^{LO}$$

$\mu = m_b$ $(.7)\,(-.195)$ $(.085)\,(-.096)$ $(-.158)\,(1)$

$= -0.300$

↑ 50% bigger

$\eta = \frac{\alpha_s(m_W)}{\alpha_s(\mu)}$

ie enhances $Br(b \to s\gamma)$ by $(1.5)^2 \sim 2.3$!

[QCD corrections are crucial for using $b \to s\gamma$ to constrain new physics

Ch 3. Chiral Lagrangians

- example of bottom-up EFT
- non-linear symmetry representations
- loops are not coupling expansion
- non-trivial power counting, p.c. theorem

QCD Chiral Symmetry for light quarks (Read 8.325 notes pg 70-77)

$$\mathcal{L}_{QCD} = \bar\psi i \not{D} \psi = \bar\psi_L i \not{D} \psi_L + \bar\psi_R i \not{D} \psi_R \,, \quad \psi_L \to L \psi_L$$
$$\psi_R \to R \psi_R$$

$G = (L, R) \to H$	ψ	Goldstones	Expansion
$SU(3)_L \times SU(3)_R \to SU(3)_V$ 8 gen. 8 gen. 8 gen	$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$	8 π, K, η	$\frac{m_{u,d,s}}{\Lambda_{QCD}} \sim \frac{1}{3}$
$SU(2) \times SU(2)_R \to SU(2)_V$ 3 gen. 3 gen. 3 gen.	$\begin{pmatrix} u \\ d \end{pmatrix}$	3 π	$\frac{m_{u,d}}{\Lambda_{QCD}} \sim \frac{1}{50}$

Matching at "Λ_{QCD}" is non-perturbative, construt EFT $\mathcal{L}_\chi$ for Goltstones (lightest d.o.f.) based <u>just</u> on symmetry breaking pattern. Fix coefficients C_i with data (or lattice).

Note: any other theory with same symmetry breaking will give <u>some</u> $\mathcal{L}_\chi$. Just different C_i.
We'll use an example of such a theory to see why non-linear representations are useful

eg. Linear σ-model $\qquad \Pi = \sigma \mathbb{1} + i \vec{\tau} \cdot \vec{\pi}$

$\uparrow SU(2)$

full theory

$$\mathcal{L}_\sigma = \frac{1}{4} \mathrm{tr}(\partial_\mu \Pi \, \partial^\mu \Pi) + \frac{\mu^2}{4} \mathrm{tr}(\Pi^+ \Pi) - \frac{\lambda}{16} \left[\mathrm{tr}(\Pi^+ \Pi) \right]^2$$

$$+ \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R - g (\bar{\psi}_L \Pi \psi_R + \bar{\psi}_R \Pi^+ \psi_L)$$

$SU(2)_L \times SU(2)_R$: $\quad \psi_L \to L \psi_L$, $\psi_R \to R \psi_R$, $\Pi \to L \Pi R^+$

$\uparrow$ linear trnsfm, mixes $\sigma, \vec{\pi}$

$$L = \exp\left(\frac{i \vec{\alpha}_L \cdot \vec{\tau}}{2} \right), \quad R = \exp\left(\frac{i \vec{\alpha}_R \cdot \vec{\tau}}{2} \right)$$

Spont. Breaking: $\quad V = -\frac{\mu^2}{2} (\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2)^2 = \frac{\lambda}{4} \left[\sigma^2 + \vec{\pi}^2 - \frac{\mu^2}{\lambda} \right]^2 + \text{const.}$

take $\langle 0 | \sigma | 0 \rangle = v = \sqrt{\mu^2/\lambda}$, $\quad \langle \vec{\pi} \rangle = 0$, $\quad \tilde{\sigma} \equiv \sigma - v$

Linear

$$\mathcal{L} = \frac{1}{2} \left[\partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma} - 2\mu^2 \tilde{\sigma}^2 \right] + \frac{1}{2} (\partial^\mu \vec{\pi}) \cdot (\partial_\mu \vec{\pi})$$

$$- \lambda v \tilde{\sigma} (\tilde{\sigma}^2 + \vec{\pi}^2) - \frac{\lambda}{4} \left[(\tilde{\sigma}^2 + \vec{\pi}^2)^2 \right] + (\psi \text{ terms})$$

$SU(2)_V$: $\quad \tilde{\sigma} \to \tilde{\sigma} \qquad \vec{\pi} \to V \vec{\pi} V^+$

Masses: $\quad m_{\tilde{\sigma}}^2 = 2\mu^2 = 2\lambda v^2$, $\quad m_\psi = g v \qquad$ take them large

$m_\pi = 0$

Field Redefinitions (can be useful to find a nice formulation for EFT)

Square Root

$(s, \vec{\varphi})$

$$s = \sqrt{(\tilde{\sigma} + v)^2 + \vec{\pi}^2} - v = \tilde{\sigma} + \dots, \qquad \vec{\varphi} = \frac{v \vec{\pi}}{\sqrt{(\hat{\sigma} + v)^2 + \vec{\pi}^2}} = \vec{\pi} + \dots$$

$$\mathcal{L} = \frac{1}{2} \left[(\partial^\mu s)^2 - 2\mu^2 s^2 \right] + \frac{1}{2} \left(\frac{v + s}{v} \right)^2 \left[(\partial_\mu \vec{\varphi})^2 + \frac{(\vec{\varphi} \cdot \partial_\mu \vec{\varphi})^2}{v^2 - \vec{\varphi}^2} \right]$$

$$- \lambda v s^3 - \frac{\lambda}{4} s^4 + \bar{\psi} i \not{\partial} \psi - g \left(\frac{v + s}{v} \right) \bar{\psi} \left[(v^2 - \vec{\varphi}^2)^{\frac{1}{2}} - i \vec{\varphi} \cdot \vec{\tau} \gamma_5 \right] \psi$$

Exponential $\sigma + i\vec{\tau}\cdot\vec{\pi} = (v+s)\,\Sigma$, $\Sigma = \exp\left(\frac{i\vec{\tau}\cdot\vec{\pi}}{v}\right)$

(s, Σ)

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu s)^2 - 2\mu^2 s^2\right] + \frac{(v+s)^2}{4}\,\mathrm{tr}\left(\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger\right)$$
$$- \lambda v s^3 - \frac{\lambda}{4} s^4 + \bar{\psi} i \partial\!\!\!/ \psi - g(v+s)\left(\bar{\psi}_L \Sigma \psi_R + \bar{\psi}_R \Sigma^\dagger \psi_L\right)$$

One More theory:

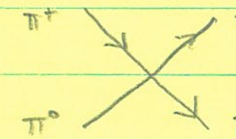
Non-Linear $\mathcal{L}_\chi$ just drop s, ψ in exponential rep

$$\mathcal{L}_\chi = \frac{v^2}{4}\,\mathrm{tr}\left(\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger\right)$$

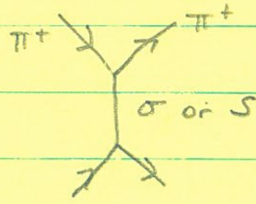
First 3 actions are equivalent to each other & for low energy pions to $\mathcal{L}_\chi$ as well

eg. $\pi^+\pi^0 \to \pi^+\pi^0$ $q = p_+' - p_+ = p_0 - p_0'$

ⓐ



ⓑ



expand in q^2/v^2

Linear $-2i\lambda$ $+$ $(-2i\lambda v)^2 \frac{i}{q^2 - m_\sigma^2}$ $= (-2i\lambda)\left(1 + \frac{2\lambda v^2}{q^2 - 2\lambda v^2}\right)$

$= i q^2/v^2 + \cdots$

Square Root $i q^2/v^2$ $\mathcal{O}(q^4)$

Exponential $i q^2/v^2$ $\mathcal{O}(q^4)$

Non-Linear $\mathcal{L}_\chi$ $i q^2/v^2$ 0

Note: • Linear is most inconvenient since derivative nature of interaction is only seen by cancellation btwn graphs

- Non-Linear $\mathcal{L}_\chi$ is most convenient, only has <u>low energy</u> Σ field, has deriv. coupling

$SU(2)_L \times SU(2)_R$: $S \to S$, $\Sigma \to L \Sigma R^+$ (from $\Pi \to L \Pi R^+$)

$$\Sigma = \exp\left(\frac{i \vec{\tau} \cdot \vec{\pi}}{v}\right)$$, $\vec{\pi}$ <u>transforms non-linearly</u>

inf. trnsfm

$$\pi^a \to \pi^a + \frac{v}{2}(\alpha_L^a - \alpha_R^a) + O(\pi^2)$$

↑ derivative needed to kill this term

<u>We can write down $\mathcal{L}_\chi$ from the start</u>

symmety breaking $G \to H$, parameterize coset G/H by Σ

$$g \in (L, R) \to (V, V) \ni h$$

$$g = (g_L, g_R) = \Xi(x)\, h ,$$

eg. $$(g_L \underset{\substack{\uparrow \\ g_R^+ g_R}}{} g_V, g_R g_V) = (\underbrace{g_L g_R^+}, 1)(\underbrace{g_R g_V, g_R g_V}_{\in H})$$

↑ $SU(N)$ matrix $\Sigma = g_L g_R^+$ parametrizes coset

$$\boxed{\Sigma \to L \Sigma R^+}$$

in terms of broken generators X

$$\Xi(x) = \exp\left[i X^a \left(\frac{\pi^a(x)}{v^0}\right)\right]$$, Callan-Coleman-Wess-Zumino prescription

↑ dimless, goldstone

can pick different choices for X^a

eg. $X^a = \tau_L^a$ $$\Xi(x) = (\underbrace{e^{i \vec{\tau} \cdot \vec{\pi}/v}}_{\Sigma(x)}, \underbrace{e^0}_{1})$$

See Ready in Review by Manohar

for QCD, common convention is $v = f/\sqrt{2}$

$$\Sigma = \exp\left(\frac{2iM}{f}\right), \quad M = \frac{\pi^a \tau^a}{\sqrt{2}} = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}$$

$$\mathcal{L}_\chi = \underset{\uparrow}{\frac{f^2}{8}} \operatorname{tr}(\partial^\mu \Sigma^\dagger \partial_\mu \Sigma)$$

gives standard Kinetic term for pion

$$\left[= \frac{1}{2} \operatorname{tr}(\partial_\mu M \partial^\mu M) + \frac{1}{6f^2} \operatorname{tr}([M, \partial_\mu M]^2) + \cdots \right]$$

Add Mass Spurion Analysis $\mathcal{L}_{QCD} = -\bar{\Psi}_L M_q \Psi_R - \bar{\Psi}_R M_q \Psi_L$

- $M_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$ pretend $M_q \rightarrow L M_q R^\dagger$ (spurion field)
- add invariant op $\mathcal{L}_\chi^{mass} = v_0 \operatorname{tr}(M_q^\dagger \Sigma + M_q \Sigma^\dagger)$
- fix M_q again

Expanding to quadratic order gives masses for goldstones

since M_q is explicit symmetry breaking

$$\mathcal{L} = \underbrace{-\frac{4v_0}{f^2}(m_u+m_d)}_{m_\pi^2}\left(\frac{1}{2}\pi^{0\,2} + \pi^+\pi^-\right)$$

Left Handed Current $J_{L\mu}^{\,a} = \bar{\Psi}\gamma_\mu P_L \tau^a \Psi$, $J_{L\mu}^{\,a} = -\frac{\delta\mathcal{L}}{\delta l_\mu^a(x)}$

chiral $J_{L\mu}^{\,a} = -\frac{if^2}{4}\operatorname{tr}(\tau^a \underset{\substack{\uparrow \\ L}}{\Sigma}\, \underset{\substack{\uparrow \\ R^\dagger R}}{\partial_\mu}\, \underset{\substack{\uparrow \\ L^\dagger}}{\Sigma^\dagger})$

Couple Spurion Current $l^\mu = l^\mu{}_a \tau_a$

$l^\mu \rightarrow L(x) l_\mu L^\dagger(x) + i\partial_\mu L(x) L^\dagger(x)$, like gauge field coupling to LH quarks

get $\partial^\mu \Sigma \rightarrow D^\mu \Sigma = \partial_\mu \Sigma + i l_\mu \Sigma$

Feyn. Rules, Power Counting, Loops

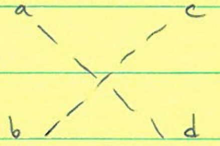
$$\mathcal{L}^{(0)} = \frac{f^2}{8} \, tr(\partial^\mu \Sigma^\dagger \partial_\mu \Sigma) + v_0 \, tr(m_q^\dagger \Sigma + m_q \Sigma^\dagger)$$

with $\partial^2 \sim v_0 m_q$

ie $p^2 \sim m_\pi^2$ expand in $\frac{p^2}{\Lambda_\chi^2}, \frac{m_\pi^2}{\Lambda_\chi^2} \ll 1$

- Derivative & m_q expansion

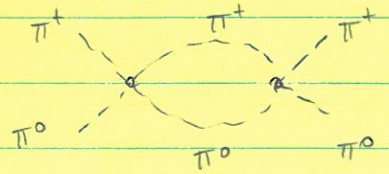
$\mathcal{L}^{(0)}$: $- - \rightarrow - - = \frac{i}{p^2 - m_\pi^2}$,



$\sim \frac{p^2}{f^2}$ or $\frac{m_\pi^2}{f^2}$, ...

from $\mathcal{L} = \frac{tr[M, \partial^\mu M][M, \partial^\mu M]}{6f^2} + \frac{4v_0}{3f^4} tr(m_q M^4)$

Loops & Λ_χ



$$\sim \int d^d l \, \frac{(p_+ - l)^2}{f^2} \, \frac{1}{l^2 - m_\pi^2} \, \frac{1}{[(l - p_+ - p_0)^2 - m_\pi^2]} \, \frac{(l - p'_+)^2}{f^2} + \dots$$

↑ preserves Chiral Symmetry & p.c. for $m_\pi \to 0$

$$\sim \underset{\uparrow\ \text{Loop factor}}{\frac{\{p^4, p^2 m_\pi^2, m_\pi^4\}}{(4\pi)^2 f^4}} \sim \underset{\uparrow\ \text{tree-level}}{\frac{\{p^2, m_\pi^2\}}{f^2}} * \frac{\{p^2 \text{ or } m_\pi^2\}}{(4\pi f)^2} \sim O(p^4)$$

loops are suppressed by p^2/Λ_χ^2 where $\Lambda_\chi = 4\pi f$

~ 1.6 GeV

or 1.16 GeV

On general grounds expect Λ is scale of lightest particle we "integrated out" $\Lambda \sim m_\rho \sim 770$ MeV use dim. reg. $[M] = 1-\epsilon$, $[f] = 1-\epsilon$

$$f^{\text{bare}} = \mu^{-\epsilon} f \quad , \quad v_0^{\text{bare}} = \mu^{-2\epsilon} v_0$$

recall $m_\pi^2 = \frac{4v_0}{f^2}(m_u+m_d)$ no μ's in m_π^2, nor in m_q

since $\frac{v_0^{\text{bare}}}{(f^{\text{bare}})^2} = \frac{v_0}{f^2}$

Loops have UV divergences $\frac{1}{\epsilon} + \ln\frac{\mu^2}{p^2}$, $\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m_\pi^2}\right)$

and enter at $\mathcal{O}(p^4)$, $\mathcal{O}(p^2 m_\pi^2)$, or $\mathcal{O}(m_\pi^4)$

let $\frac{f^2}{8}\chi = v_0 m_q$

At this order we also have new operators. For SU(2):

$$\mathcal{L} = L_1\left[\text{tr}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)\right]^2 + L_2\,\text{tr}(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger)\,\text{tr}(\partial^\mu\Sigma\partial^\nu\Sigma^\dagger)$$

$+ \dots \leftarrow$ terms involving one m_q & two ∂^μ's or two m_q's

and δL_i counterterms cancel UV divergences $1/\epsilon$.

> Note: eqtn of motion: $(\partial^2\Sigma)\Sigma^\dagger - \Sigma(\partial^2\Sigma^\dagger) - \chi\Sigma^\dagger + \Sigma\chi^\dagger + \frac{1}{2}\text{tr}(\chi\Sigma^\dagger - \Sigma\chi^\dagger) = 0$
>
> used to remove ∂^2 terms (see reading)
>
> SU(2) identities : $\text{tr}(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\partial_\nu\Sigma\partial^\nu\Sigma^\dagger) = \frac{1}{2}\left[\text{tr}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)\right]^2$
>
> remove more terms $\text{tr}(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\partial^\mu\Sigma\partial^\nu\Sigma^\dagger) = \dots$

At $\mathcal{O}(p^4)$ we include both loops with $p^4\ln\mu^2/p^2$ terms, and $p^4 L_{1,2}(\mu)$ terms $\underset{L_i}{\times} \sim L_i p^4$. μ dependence cancels between these two

Think of μ as cutoff between low energy physics (loops) and high energy physics, the $L_i(\mu)$'s.

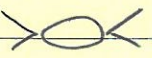
Expect $\frac{L_i(\mu)}{f^2} = \frac{1}{(4\pi f)^2}\left[a_i\ln\left(\frac{\mu}{\Lambda_\chi}\right) + b_i\right]$

"<u>Naive Dimensional Analysis</u>" changing μ moves pieces between loop graph & $L_i(\mu)$, so we expect them to be same order of magnitude ($a_i \sim b_i \sim 1$, this works!)

Typically pick $\mu \simeq m_\rho$ or $\mu \simeq \Lambda_\chi$ or $\mu \simeq 1\,\text{GeV}$, ie put large logs in matrix elts not coefficients (unlike QCD no ∞ log series at least in eg's done here)

- calculate matrix elements
- fit coefficients L_i

Could we use a cutoff? Λ



$\sim \dfrac{\Lambda^4}{\Lambda_\chi^4}$ breaks chiral symmetry (no c.t. in $\mathcal{L}_\chi$)

$\sim \dfrac{\Lambda^2 p^2}{\Lambda_\chi^4}$ "breaks" p.c., absorb in LO  $\sim p^2/f^2$

$\sim \dfrac{p^4 \ln\Lambda}{\Lambda_\chi^4}$ absorb in $L_i p^4$ ∞ in dim. reg.

Infrared divergences? ∂^μ couplings make graphs IR finite.
usually have good $m_\pi^2 \to 0$, $p^2 \to 0$ limits

eg Phenomenology $\pi\pi \to \pi\pi$ below inelastic thresholds (no $\pi\pi \to 4\pi$, etc)

S matrix $S_{\ell I} = e^{2\delta_{\ell I}}$ $\overset{\text{partial wave}}{\ell}$, $\overset{\text{isospin}}{I}$

effective range expansion describes scattering

$p^{2\ell+1} \cot\delta = -\dfrac{1}{a} + \dfrac{r_0 p^2}{2} + \cdots$ &  gives $a_{\ell I}$

ie parameter free predictions $a_{00} = \dfrac{7 m_\pi^2}{16\pi f_\pi^2}$, $a_{02} = -\dfrac{m_\pi^2}{8\pi f_\pi^2}$, ...

General Power Counting

Consider graph with
N_V vertices
N_I internal lines
N_E external lines
N_L loops

Let $N_V = \sum_n N_n$ where
N_n = # vertices $\propto p^n$ or m_π^n

Count

Mass dim, Λ_χ factors for matrix element $\mathcal{M}$ of N_E external bosons

$\swarrow$ dim of couplings

vertices $\quad (\Lambda_\chi)^{\sum_n N_n (4-n)}$ $\quad n=2 \to f^2$ in $\mathcal{L}^{(0)}$

$n=4 \to L_i$ dimless in $\mathcal{L}$

f's from Pion Lines $\quad (\Lambda_\chi)^{-2N_I - N_E}$ $\quad \frac{\pi}{f}$ each $f \to 4\pi f$

Euler Identity $\quad N_I = N_L + N_V - 1$ $\quad$ eliminate N_I

$\swarrow$ some power

$$\mathcal{M} \sim \Lambda_\chi^{\sum_n N_n (4-n) - N_E - 2N_L - 2\sum_n N_n + 2} \qquad E^D \; f(E/\mu)$$

$\uparrow$ $E = p$ or m_π $\qquad \nwarrow$ log's in dim. reg.

$$\sim (\text{mass})^{4-N_E}$$

so $\quad D = 2 + \sum_n N_n (n-2) + 2N_L$

$\uparrow$ $D \geq 2$ $\qquad \uparrow \qquad \uparrow$ adding vertices or loops always increases D

$\boxed{\text{just count loops \& vertices "p-counting"}}$

eg.

$D = 2$

$D = 4$

$D = 4$

theory organized by powers of p, m_π multiplied by $f(p/m_\pi)$

SU(3) case

octet of goldstone bosons

$$M_q = \begin{pmatrix} m_u & & 0 \\ & m_d & \\ 0 & & m_s \end{pmatrix}, \quad M = \frac{\pi^a \lambda^a}{\sqrt{2}} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

Expand $\mathrm{tr}[\Sigma m_q^\dagger + m_q \Sigma]$

$\Rightarrow$ get masses of mesons in terms of m_u, m_d, m_s

$$m^2_{K^0} = m^2_{\bar{K}^0} = \frac{4v}{f^2}(m_d + m_s) \quad \text{etc.}$$

isospin violation

$\eta - \pi^0$ mixing
$$(M^2)_{ij} = \frac{4v}{f^2}\begin{pmatrix} m_u + m_d & \frac{(m_u - m_d)}{\sqrt{3}} \\ \frac{m_u - m_d}{\sqrt{3}} & \frac{1}{3}(4m_s + m_u + m_d) \end{pmatrix}$$

Often treat isospin violation as perturbation

$$\hat{m} = \frac{1}{2}(m_u + m_d) \neq 0, \quad m_s \gg m_{u,d}, \quad m_{u,d} \simeq \hat{m}$$

$$D^\mu = \partial_\mu + i l_\mu$$

Higher Orders
$$\mathcal{L}^{(0)} = \frac{f^2}{8}\mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma + \chi^\dagger \Sigma + \chi \Sigma^\dagger]$$

pulled out common factor, let $v_0 m_q \equiv \frac{f^2}{8}\chi$

count p's

p.c. $\Sigma \sim 1$, $D_\mu \Sigma \sim p$, $\underset{\text{source}}{l_\mu} \sim p$, $\chi \sim m_q \sim p^2$ $(p \sim m_\pi \sim m_K)$

↑ behaves like scalar source too

← $\mathcal{O}(p^4)$

$$\mathcal{L}^{(1)} = L_1[\mathrm{tr}(D_\mu \Sigma D^\mu \Sigma^\dagger)]^2 + L_2\,\mathrm{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger)\,\mathrm{tr}[D^\mu \Sigma D^\nu \Sigma^\dagger]$$
$$+ L_3\,\mathrm{tr}(D_\mu \Sigma D^\mu \Sigma^\dagger D_\nu \Sigma D^\nu \Sigma^\dagger)$$

$$+ L_4\,\mathrm{tr}(D_\mu \Sigma D^\mu \Sigma^\dagger)\,\mathrm{tr}(\chi \Sigma^\dagger + \Sigma \chi^\dagger) + L_5\,\mathrm{tr}(D_\mu \Sigma D^\mu \Sigma^\dagger(\chi \Sigma^\dagger + \Sigma \chi^\dagger))$$

$$+ L_6[\mathrm{tr}(\chi \Sigma^\dagger + \Sigma \chi^\dagger)]^2 + L_7[\mathrm{tr}(\chi^\dagger \Sigma - \Sigma^\dagger \chi)]^2$$
$$+ L_8\,\mathrm{tr}(\chi \Sigma^\dagger \chi \Sigma^\dagger + \Sigma \chi^\dagger \Sigma \chi^\dagger)$$

$$+ L_9\,\mathrm{tr}[L_{\mu\nu} D^\mu \Sigma D^\nu \Sigma^\dagger]$$

9 operators when we couple l_μ, χ

[10 operators if we have right handed r_μ too from

$L_{10}\ tr(L_{\mu\nu}\Sigma R^{\mu\nu}\Sigma^+)$]

Here • $L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu + i[l_\mu, l_\nu]$

- e.o.m remove double derivatives $\partial^2\Sigma$

- SU(3) relation removes

$$tr[D_\mu\Sigma D_\nu\Sigma^+ D^\mu\Sigma D^\nu\Sigma^+] = \frac{1}{2}\left(tr(D_\mu\Sigma D^\mu\Sigma^+)\right)^2 + tr(\,)tr(\,) - 2\ tr(D_\mu\Sigma D^\mu\Sigma^+ D_\nu\Sigma D^\nu\Sigma^+)$$

In SU(2) 3 operators become redundant, coefficients $L_i^{(2)}$ which are independent of m_π, but now depend on m_s from integrating out kaon

eg. $$2L_1^{(2)} + L_3^{(2)} = 2L_1 + L_3 - \frac{1}{96(4\pi)^2}\left(1+\ln\frac{\mu^2}{M_K^2}\right)$$

Renormalization of L_i absorb divergences

$$L_i = \bar{L}_i + \delta L_i \quad , \quad \delta L_i = \frac{\gamma_i}{32\pi^2}\left(\frac{1}{\epsilon} - \ln 4\pi + \gamma - 1\right)$$

↑ convention

eg.

causes mass shift from

$$m_0^2 = \frac{4v}{f^2}(m_u+m_d) \equiv 2B_0\hat{m}$$

SU(2): $$m_\pi^2 = m_0^2\left[1 - \frac{16m_0^2}{f^2}\left(2\bar{L}_4^{(2)} + \bar{L}_5^{(2)} - 4\bar{L}_6^{(2)} - 2\bar{L}_8^{(2)}\right) + \frac{m_0^2}{(4\pi f)^2}\ln\left(\frac{m_0^2}{\mu^2}\right)\right]$$

↑ chiral log, long range pion force

eg. SU(3) [more on Hmwk]

$$f_\pi = f\left(1 - 2\mu_\pi - \mu_K + \frac{16\hat{m}B_0}{f^2}\bar{L}_5 + (m_s+2\hat{m})\frac{16B_0}{f^2}\bar{L}_4\right)$$

where $\mu_i = \frac{m_i^2}{(4\pi f)^2}\ln\frac{m_i^2}{\mu^2}$, $f_\pi = f$ in chiral limit

Ch 4. Heavy Quark Effective Theory

Goals here

- $\mathcal{L}$ with labelled fields, h_v
- Heavy Quark symmetry with covariant representations
- Anom. dim.'s that are functions
- Reparameterization Invariance
- Limitations of $\overline{MS}$ → scale separation for power law terms & renormalons

Ch.4

Heavy Quark Effective Theory

EFT for heavy particles that are not removed from the theory

[soures that can wiggle]

eg.

$\bar{Q}q$ mesons like $\bar{b}d = B^0$

size $r^{-1} \sim \Lambda_{QCD} \ll M_Q$ two scales

Want to describe fluctuations of Q due to lighter d.o.f.

Take $\lim_{m_Q \to \infty} \mathcal{L}_{QCD} = \bar{Q}(i\slashed{\partial} - m_Q)Q$ (How?)

Consider propagator for Q, on-shell momentum $p = m_Q v$

$v^2 = 1$

In general $p^\mu = m_Q v^\mu + k^\mu$ where $k^\mu \sim \Lambda_{QCD}$

[↑ kicks from soft modes]

$$\frac{i(\slashed{p}+m_Q)}{p^2 - m_Q^2 + i\epsilon} = \frac{i m_Q \slashed{v} + m_Q + \slashed{k}}{2 m_Q v\cdot k + k^2 + i\epsilon} = i\left(\frac{1+\slashed{v}}{2}\right)\frac{1}{v\cdot k + i\epsilon} + \mathcal{O}\left(\frac{1}{m_Q}\right)$$

Vertex

$= -ig\gamma^\mu T^A$

Propagators on each side

$$\frac{(1+\slashed{v})}{2}\gamma^\mu\frac{(1+\slashed{v})}{2} = \frac{(1+\slashed{v})}{2}\frac{(1-\slashed{v})}{2}\gamma^\mu + \frac{(1+\slashed{v})}{2}v^\mu = v^\mu$$

$= -igT^A v^\mu$

Lagrangian $\mathcal{L}_{HQET} = \bar{Q}_v\, i v\cdot D\, Q_v$

where $\frac{(1+\slashed{v})}{2} Q_v = Q_v$

Direct Derivation

$Q(x) = e^{-im_Q v\cdot x}\left[Q_v(x) + B_v(x) \right]$, $\frac{(1+\not{v})}{2} Q_v = Q_v$

$\frac{(1-\not{v})}{2} B_v = B_v$

$i\not{D} = \not{v}\, i v\cdot D + i\not{D}_T$

$D_T^\mu \equiv D^\mu - v^\mu\, v\cdot D$

$\not{v}\, Q_v = Q_v$

$\not{v}\, B_v = -B_v$

So

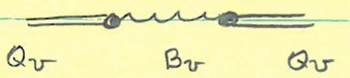
$$\mathcal{L}_{QCD} = \left[\bar{Q}_v + \bar{B}_v\right] e^{im_Q v\cdot x} \left\{ \not{v}\, i v\cdot D + i\not{D}_T - m_Q \right\} e^{-im_Q v\cdot x} (Q_v + B_v)$$

$$= \left[\bar{Q} + \bar{B}\right] e^{+i()} e^{-i()} \left\{ (\not{v}-1) m_Q + \not{v}\, i v\cdot D + i\not{D}_T \right\} (Q_v + B_v)$$

$$= \bar{Q}_v\, i v\cdot D\, Q_v - \bar{B}_v\,(i v\cdot D + 2m_Q)\, B_v$$

$$+ \bar{Q}_v\, i\not{D}_T\, B_v + \bar{B}_v\, i\not{D}_T\, Q_v$$

Consider only Q_v external particles, then as $m_Q \to \infty$
B_v decouples



$\sim \frac{1}{m_Q}$

Discussion

① Above field redefn. was <u>tree level</u>, and is valid to Leading order in $\frac{1}{m_Q}$ & $\alpha_s(m_Q)$ [hard gluons] (to be discussed later). We correctly describe couplings to $k^\mu \sim \Lambda_{QCD}$ gluons.

② Antiparticles are integrated out

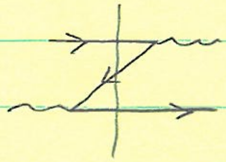
rest frame $v_r^\mu = (1,0,0,0)$

particles

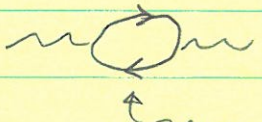
$$\frac{(1+\gamma_0)}{2}\, u_{\substack{\text{Dirac}\\ \text{Rep.}}} = \begin{pmatrix} \psi_v \\ 0 \end{pmatrix}$$

antiparticles

no pair creation



intermediate state
offshell by $2M_Q$



$2M_Q$, short distance

Thus number of Heavy Quarks is preserved: $U(1)$

③ Heavy Quark Symmetry for $\mathcal{L}_{HQET}$

- Flavour Symmetry $U(N_h)$, N_h = # heavy quarks
  since $\mathcal{L}_{HQET}$ indep of m_Q it doesn't
  see flavor of quark
- Spin Symmetry $SU(2)$ indep. of remaining two
  spin components

rest frame $S_Q^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \frac{1}{2}\gamma_5\gamma^0\gamma^i$

$Q_v' = (1 + i\vec{\epsilon}\cdot\vec{S}_Q)\, Q_v$, $\delta\mathcal{L} = \overline{Q}_v\,[\,iv\cdot D,\ i\vec{\epsilon}\cdot\vec{S}_Q\,]\,Q_v = 0$

$\not{v}\, Q_v' = Q_v'$ acts <u>within</u> two component subspace

- Together $U(2N_h)$ where Q_v is fundamental with
  N_h spin up, N_h spin down

spinors $\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \ldots$

④ Velocity v^μ is conserved by low energy QCD
interactions (only k changes)

v is label on fields

⑤ Power counting in $1/m_Q$

mode expn. $$Q(x) = \int \frac{d^3p}{\sqrt{2E}} \sum_s \left(a_p^s u^s e^{-ip\cdot x} + \dots\right)$$

$$Q_v(x) \sim e^{-ik\cdot x} \qquad p = m_Q v + k$$

$$i\partial^\mu Q_v(x) \sim k^\mu Q_v(x)$$

so coord. x now corresponds to variations over Λ_{QCD} scales

In subleading $\mathcal{L}$ & external operators all powers of $\frac{1}{m_Q}$ will be explicit. Easy to count.

[$\ln m_Q$ can occur in coefficients]

States (a hidden m_Q)

relativistic norm $$\langle H(p') | H(p) \rangle = 2E_p \, (2\pi)^3 \, \delta^3(\bar{p} - \bar{p}')$$

$\uparrow$ dim -1 $\qquad \uparrow$ $E_p = \sqrt{m_H^2 + \bar{p}^2}$

HQET states defined from $\mathcal{L}_{HQET}$ ($m_Q \to \infty$) differ by norm. & $1/m_Q$ corrections

$$|H(\vec{p})\rangle = \sqrt{m_H}\left[\,|H(v)\rangle + \mathcal{O}(1/m_Q)\right]$$

$$\langle H(v',k') | H(v,k) \rangle = 2v^0 \, \delta_{v,v'} \, (2\pi)^3 \, \delta^3(k-k')$$ indep m_Q

$\uparrow$ dim $-\frac{3}{2}$

[Similar for spinors $u^s(p) = \sqrt{m_Q}\, u^s(v)$

$\uparrow$ cancel $1/\sqrt{2E}$ in mode expn.]

Spectroscopy

Light quarks, gluons still described by $\mathcal{L}_{QCD}$

$m_Q \to \infty$

$Q\bar{q}$ has quantum #'s of Q, $\underbrace{\bar{q}\text{, any \# } q\bar{q}\text{, any \# gluons}}_{\text{light d.o.f.}}$

total $\vec{J}$ conserved $\qquad \vec{J}^2 \doteq J(J+1)$

heavy quark spin $\vec{S}_Q$ conserved

$\therefore \quad \vec{S}_\ell = \vec{J} - \vec{S}_Q$ conserved $\qquad \vec{S}_\ell^{\,2} \doteq S_\ell(S_\ell+1)$

Meson Symmetry <u>Doublets</u> $\qquad j_\pm = S_\ell \pm 1/2$ for each S_ℓ

S_ℓ^π	Mesons	
$\frac{1}{2}^-$	B, B^*	$j = 0, 1$
$\frac{1}{2}^+$	B_0^*, B_1^*	$j = 0, 1$
$\frac{3}{2}^+$	B_1, B_2^*	$j = 1, 2$

Can also add
SU(2) flavor u, d
SU(3) flavor u, d, s
eg. $B^+, \bar{B}^0, B_s$

Baryons

0^+	Λ_b	$j = 1/2$
1^+	Σ_b, Σ_b^*	$j = 1/2, 3/2$

Covariant Representation of Fields

encode HQS in objects with nice transformation properties

field $H_v^{(Q)}$ annihilates <u>meson</u> <u>doublet</u>

$$H_v^{(Q)} = \underset{\substack{\uparrow \\ \text{project} \\ \text{out anti}Q}}{\frac{(1+\not{v})}{2}} \Big[\underset{\substack{\uparrow \\ \text{vector } \epsilon^\mu \\ \epsilon^2=-1, \\ v\cdot\epsilon=0}}{P^{*(Q)}_{v\mu}}\, \gamma^\mu + i\, \underset{\substack{\uparrow \\ \text{pseudoscalar} \\ \text{meson}}}{P_v^{(Q)}}\, \gamma_5 \Big]$$

bispinor
$Q\bar{q}$

Lorentz Trnsfm $\quad H_{v'}'(x') = D(\Lambda)\, H_v(x)\, D(\Lambda)^{-1}$

$v' = \Lambda v$ $\qquad$ ↑ spinor L.T.

$x' = \Lambda x$

Parity ✓

$\underline{\not{v} H_v = H_v}$ (no anti Q), $\quad H_v \not{v} = -H_v$ since $v \cdot P_v^* = 0$

$\underline{S_Q \otimes S_\ell}$ its $(\frac{1}{2}, \frac{1}{2})$ $\qquad [S_Q^i, H_{v_r}] = \frac{1}{2} \sigma^i_{4\times4} H_{v_r}$

$v_r = (1,0,0,0)$ $\qquad [S_\ell^i, H_{v_r}] = -\frac{1}{2} H_{v_r} \sigma^i_{4\times4}$

$\sigma^i_{4\times4} = \frac{i\epsilon^{ijk}}{4}[\gamma^j, \gamma^k]$ $\qquad$ ↑ minus from $\bar{q}$

HQ Spin Symm $\quad H_v \to D(R)_Q H_v$, $\quad \delta H_v = i[\theta \cdot S_Q, H_v]$

find $\delta P_{v_r} = -\frac{1}{2} \vec{\theta} \cdot \vec{P}^*_{v_r}$, $\quad \delta \vec{P}^*_{v_r} = \frac{1}{2} \vec{\theta} \times \vec{P}^*_{v_r} - \frac{1}{2} \vec{\theta} P_{v_r}$

<u>mix</u> with each other

Use H_v to easily get HQS predictions

eg. Decay Constants

$\bar{B}, D$ $\quad \langle 0 | \bar{q} \gamma^\mu \gamma_5 Q | P(p) \rangle = -i f_P p^\mu = -i f_P m_P v^\mu$ $\quad$ ↓ dim 1

$\langle 0 | \bar{q} \gamma^\mu Q | P^*(p, \epsilon) \rangle = f_{P^*} \epsilon^\mu$

↑ dim 2

HQET current $\quad (\bar{q} \Gamma Q)(0) = (\bar{q} \Gamma Q_v)(0) + O(1/m_Q)$

$\langle 0 | \bar{q} \Gamma Q_v | H(v) \rangle$

under HQS $Q_v \to D(R) Q_v$. Encode in general object with H_v-field that transforms the same way.

Pretend $\Gamma \to \Gamma D(R)^{-1}$ $\qquad$ write invariant operators

ΓH_v invariant

Lorentz cov. requires $\text{tr}\ \overset{\overset{\text{QCD}}{\downarrow}}{X}\ \Gamma H_v$

$\leftarrow$ linear in # heavy quarks

general $X = \frac{a_0 + a_1 \not{v}}{2}$, $H_v \not{v} = -H_v$ so $X = \frac{a_0 - a_1}{2} = \frac{a}{2}$

<u>In</u> <u>Matrix</u> <u>Elt</u>: $\bar{q}\, \Gamma Q_v = \frac{a}{2}\, \text{tr}\ \Gamma H_v$

$$= a \begin{cases} -i v^\mu P_v & \Gamma = \gamma^\mu \gamma_5 \\ P_v^{*\mu} & \Gamma = \gamma^\mu \end{cases}$$

$$\left.\begin{array}{l} \langle 0 | \bar{q}\, \gamma^\mu \gamma_5\, Q_v | P(v) \rangle = -i a v^\mu \\ \langle 0 | \underset{3}{\bar{q}}\, \gamma^\mu Q_v\, \underset{-3/2}{| P^*(v) \rangle} = \underset{\uparrow\ \Lambda_{QCD}^{3/2}}{a}\, \epsilon^{*\mu} \end{array}\right\} \text{same } a! \text{ for } D, D^*, B, B^*$$

$f_P = \frac{a}{\sqrt{m_P}}$, $f_{P^*} = a \sqrt{m_{P^*}}$

$f_B \sim \frac{\Lambda_{QCD}^{3/2}}{m_b^{1/2}} \sim 180 \text{ MeV}$, $\frac{f_B}{f_D} \sim \sqrt{\frac{m_D}{m_B}} \sim 0.6$

eg 2. $\bar{B} \to D \ell \nu$, $B \to D^* \ell \nu$ weak decays

6 form factors in QCD, all reduce to

1 normalized form factor in HQET (Isgur-Wise function)

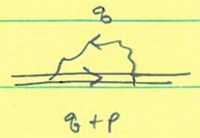
what impact do labels v^μ have?

HQET Radiative Corrections

$\alpha_s(\mu)$, $\mu \sim M_Q$

- renormalization $\mathcal{L}$, J^μ
- Matching $J^\mu_{QCD} = C(\mu/M_Q)\, J^\mu_{HQET} + \mathcal{O}(1/M_Q)$

Wave function ren. $Q_v^{(0)} = Z_h^{1/2} Q_v$



$$= -C_F \mu^{2\epsilon} g^2 \int \frac{d^d q\, v^2}{q^2\, v\cdot(q+p)}\, e^{\epsilon\gamma_E}(4\pi)^{-\epsilon}$$ Feyn. Gauge, $\overline{MS}$

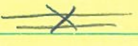
Trick: $$\frac{1}{ab} = 2\int_0^\infty \frac{d\lambda}{(a+2b\lambda)^2} \qquad a = q^2,\ b = v\cdot(q+p)$$

$$(q^2 + 2\lambda v\cdot q + 2\lambda v\cdot p + i0)^2 = (t^2 - A)^2, \qquad t = q + \lambda v$$
$$A = \lambda(\lambda - 2v\cdot p - i0)$$

$$= (-C_F g^2)\, \frac{2i\,\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^\infty d\lambda\ \lambda^{d/2-2}(\lambda - 2v\cdot p)^{d/2-2}$$

$$= (-C_F g^2)\, \frac{2i\,\Gamma(\epsilon)}{16\pi^2}\, \frac{\Gamma(3/2 - d/2)\,\Gamma(d/2-1)}{2\sqrt{\pi}}\, (-v\cdot p - i0)^{d-3}$$

$$= \frac{-iC_F g^2}{8\pi^2}\, v\cdot p\, \frac{1}{\epsilon} + \cdots$$



$$= i(Z_h - 1)\, v\cdot p \quad \text{so} \quad Z_h = 1 + \frac{C_F g^2}{8\pi^2\epsilon}$$ (not $= Z_\psi$ in QCD)

Local Ops
$$O_\Gamma^{(0)} = \bar{q}^{(0)}\, \Gamma\, Q_v^{(0)}$$
$$O_\Gamma = \frac{1}{Z_O} O_\Gamma^{(0)} = \bar{q}\,\Gamma Q_v + \left(\frac{\sqrt{Z_q Z_h}}{Z_O} - 1\right)\bar{q}\,\Gamma Q_v$$
c.t.'s

+ w.fn Z's gives
$$Z_O = 1 + \frac{g^2}{8\pi^2\epsilon}, \qquad \gamma_O = \frac{-g^2}{4\pi^2}$$
indep of Γ

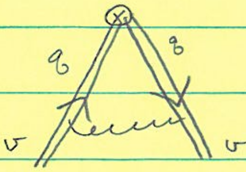
running below M_Q

More interesting case

<u>Heavy - Heavy</u> $T_\Gamma = \bar{Q}_{v'} \Gamma Q_v + \left(\frac{Z_h}{Z_T} - 1 \right) Q_{v'} \Gamma Q_v$

eg. $B \to D^* \ell \nu$ where v is for b, v' is for c

$(b\bar{u}) \to (c\bar{u}) \ell \nu$



$$= -i C_F g^2 v \cdot v' \int \frac{d^d q}{q^2 \, v \cdot q \, v' \cdot q}$$ UV & IR divergent

UV counterterm : $Z_T = 1 - \frac{g^2}{6\pi^2 \epsilon} \left[\omega r(\omega) - 1 \right]$ $\omega = v \cdot v'$

$$r(\omega) = \frac{\ln\left(\omega + \sqrt{\omega^2 - 1}\right)}{\sqrt{\omega^2 - 1}}$$

$$\gamma_T = \frac{g^2}{3\pi^2} \left[\omega r(\omega) - 1 \right]$$

Notes : ① depends on $\omega = v \cdot v'$ $J_{v,v'} = \bar{Q}_{v'} \Gamma Q_v$

↑ act like indices on currents

Wilson Coeff's also depend on ω:

$$C(\mu, \alpha_s(\mu), m_b v^\mu, m_c v'^\mu) = C(\mu, \alpha_s(\mu), m_b^2, m_c^2, v \cdot v')$$

in $B \to D^* \ell \nu$ we let $p_B^\mu = m_B v^\mu = m_{D^*} v'^\mu + q^\mu$

$$q^2 = m_B^2 + m_{D^*}^2 - 2 m_B m_{D^*} v \cdot v' \Rightarrow \omega = \frac{m_B^2 + m_{D^*}^2 - q^2}{2 m_B m_D}$$ related to mom. transfer

$1 \le \omega \le 1.5$

<u>external</u> <u>variable</u>
fixed by kinematics

② γ_T indep of spin structure Γ (HQS)

③ $\ln(m_Q / \Lambda_{QCD})$ in QCD becomes $\ln(\mu / \Lambda_{QCD})$ in HQET operators & $\ln(m_Q/\mu)$ in coeff.

Anomalous dimension sums these large logs

[in QCD vector current $\bar{q}_1 \gamma^\mu q_2$ is conserved for massless quarks so no anom. dim. Masses don't spoil this ($\mu \gg m$). In HQET the scales are $\mu \lesssim m$ and currents $\bar{q} \gamma^\mu Q_v$, $\bar{Q}_{v'} \gamma^\mu Q_v$ are <u>not</u> <u>conserved</u>]

④ LL solution match at $\mu = m_Q$, say $C(m_Q) = 1$

$$C_{LL}(\mu) = C(m_Q)\,U(m_Q,\mu) = \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{-\gamma/2\beta_0}$$

similar to Weak Hamiltonian.

Here γ^{LL} = constant for $\bar q\,\Gamma Q_v$, $\gamma^{LL} = \gamma^{LL}(\omega)$ for $\bar Q_{v'}\Gamma Q_v$

⑤ Corresponding μ dependence in HQET matrix elts

eg. Decay constant $\langle 0|(\bar q\,\gamma^\mu\gamma_5 Q_v)|P(v)\rangle = -i\,a(\mu)\,v^\mu$

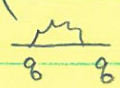
(want $\mu \gtrsim \Lambda_{QCD}$ so $a(\mu)$ has no large logs, ...etc)

Matching pert. corrections in $\alpha_s(m_Q)$, use $\overline{MS}$ everywhere

Heavy-Light

QCD: $\langle q(0,s')|\,\bar q\,\gamma^\mu Q\,|Q(p,s)\rangle = \left[R^{(Q)}R^{(q)}\right]^{1/2}\bar u(0,s')\left[\gamma^\mu + V_1^\mu\,\alpha_s(\mu)\right]u(p,s)$

UV finite residue corrections in $\overline{MS}$:



, vertex

HQET: $\langle q(0,s')|\,\bar q\,\Gamma Q_v\,|Q(v,s)\rangle = \left[R^{(h)}R^{(q)}\right]^{1/2}\bar u(0,s')\left[1 + V_1^{\text{eff}}\alpha_s(\mu)\right]\Gamma\,u(v,s)$

vector current has two operators $\quad C_1^{(v)}\,\bar q\,\gamma^\mu Q_v + C_2^{(v)}\,\bar q\,v^\mu Q_v$

($\bar q\,\sigma^{\mu\nu}v_\nu Q_v$ reducable)

$$C_1^{(v)} = 1 + \frac{\alpha_s(\mu)}{\pi}\left[\ln\frac{m_Q}{\mu} - \frac{4}{3}\right]$$

$$C_2^{(v)} = \frac{2}{3}\,\frac{\alpha_s(\mu)}{\pi}$$

<u>Nice trick</u>: pick IR regulator to make calc. very simple

Use dim reg for UV (in $\overline{MS}$) & IR

→ all HQET graphs with on-shell external lines are scaleless $\propto \left(1/\epsilon_{UV} - 1/\epsilon_{IR} \right)$, $1/\epsilon_{UV}$ removed by counterterms

→ UV renormalized QCD graphs have form

$$\underbrace{\#/\epsilon_{IR}}_{\text{match with HQET}} + \underbrace{\# \ln(\mu/m_Q) + \#}_{\text{hence } \underline{\text{this}} \ \underline{\text{is}} \text{ the matching result}}$$

Non-Perturbative Corrections & Reparameterization

Recall
$$\mathcal{L} = \bar{Q}_v \, i v\cdot D \, Q_v - \bar{B}_v \left(i v\cdot D + 2m_Q \right) B_v + \bar{Q}_v \, i \not{D}_T \, B_v + \bar{B}_v \, i \not{D}_T \, Q_v$$

$\not{v} Q_v = Q_v$

$\not{v} B_v = -B_v$

At tree-level B_v can be integrated out using e.o.m.

$$\frac{\delta \mathcal{L}}{\delta \bar{B}_v} = 0 \; : \qquad (i v\cdot D + 2m_Q) B_v = i \not{D}_T Q_v$$

$$B_v = \frac{1}{(2m_Q + i v\cdot D)} \, i \not{D}_T Q_v$$

$$\mathcal{L} = \bar{Q}_v \left(i v\cdot D + i \not{D}_T \frac{1}{2m_Q + i v\cdot D} i \not{D}_T \right) Q_v$$

$$= \bar{Q}_v \, i v\cdot D \, Q_v - \frac{1}{2m_Q} \bar{Q}_v \not{D}_T \not{D}_T Q_v + \cdots = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \cdots$$

$$[D^\mu, D^\nu] = i g G^{\mu\nu} \, , \qquad \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

$$\not{D}_T \not{D}_T = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} D_T^\mu D_T^\nu + \frac{1}{2} [\gamma^\mu, \gamma^\nu] D_T^\mu D_T^\nu$$
$$= D_T^2 + \frac{g}{2} \sigma_{\mu\nu} G^{\mu\nu}$$

$$\mathcal{L}^{(1)} = - \bar{Q}_v \frac{D_T^2}{2m_Q} Q_v - g \, \bar{Q}_v \frac{\sigma_{\mu\nu} G^{\mu\nu}}{4m_Q} Q_v$$

"Kinetic Energy"
breaks flavor symmetry

"magnetic moment" $\sim \sigma \cdot B$
breaks flavor & spin symmetries

More General Procedure

- write down all possible operators constrained by symmetries

① Power counting, powers of $1/m_Q$ tell us dimension of fields needed

② Gauge Symmetry D^μ

③ Discrete Symmetries C, P, T

④ Lorentz Invariance (?)

Rotations $M^{\mu\nu}_{TT}$ M^{12}, M^{13}, M^{23} ✓

Boosts $v_\mu M^{\mu\nu_T} \xrightarrow[v=v_r=(1,\vec{0})]{} M^{01}, M^{02}, M^{03}$ no

Introducing v^μ breaks part of Lorentz Inv. (gives preferred frame)

→ Restored by "Reparameterization Invariance" (RPI) on v^μ order by order in $1/m_Q$

$P_Q^\mu = m_Q v^\mu + k^\mu$ split is somewhat arbitrary

inv. under $v^\mu \to v^\mu + \epsilon^\mu/m_Q$

$k^\mu \to k^\mu - \epsilon^\mu$

ϵ^μ is infinitesimal & counting is $\epsilon^\mu/m_Q \sim 1/m_Q$

$v^2=1$ so $\epsilon \cdot v = 0$, 3 parameter family of trnsfms

Change to Field? $\not{v}\, Q_v(0) = Q_v(\overset{x=0}{0})$

$$\to (\not{v} + \frac{\not{\epsilon}}{m_Q})(Q_v + \delta Q_v) = (Q_v + \delta Q_v)$$

$$(1-\not{v})\,\delta Q_v = \frac{\not{\epsilon}}{m_Q} Q_v$$

$\delta Q_v = \dfrac{\not{\epsilon}}{2m_Q} Q_v$ is solution

Reparameterization inv. is :

$$\left.\begin{aligned} v &\to v + \epsilon/m_Q \\ Q_v &\to \underbrace{e^{i\epsilon\cdot x}}_{\substack{\text{encodes} \\ i\partial^\mu \to i\partial^\mu - \epsilon^\mu \\ \text{ie}\;\; k^\mu \to k^\mu - \epsilon^\mu}} \left(1 + \frac{\not{\epsilon}}{2m_Q}\right) Q_v \end{aligned}\right]$$

this restores invariance under "small boosts" (!) ($\epsilon \sim \Lambda_{QCD}$) which are all we care about

General $1/m_Q$ operators

$$\mathcal{L}^{(1)} = -C_K\, \bar{Q}_v \frac{D_T^2}{2m_Q} Q_v - C_F \frac{g\, \bar{Q}_v\, \sigma_{\mu\nu} G^{\mu\nu} Q_v}{4m_Q}$$

where C_K, C_F are wilson coeff's (here $C_F \neq 4/3$, sorry & standard notation)

RPI phase is only LO change , $v\cdot\epsilon = 0$

$$\mathcal{L}^{(0)} + \delta\mathcal{L}^{(0)} = \bar{Q}_v \left(1 + \frac{\not{\epsilon}}{2m_Q}\right) e^{-i\epsilon\cdot x}\, i\left(v + \frac{\epsilon}{m_Q}\right)\cdot D\, e^{i\epsilon\cdot x} \left(1 + \frac{\not{\epsilon}}{2m_Q}\right) Q_v$$

use $\frac{(1+\not{v})}{2} \not{\epsilon} \frac{(1+\not{v})}{2} = \epsilon\cdot v = 0$

$$\delta\mathcal{L}^{(0)} = \bar{Q}_v \frac{i\epsilon\cdot D}{m_Q} Q_v$$

$$\delta\mathcal{L}_K^{(1)} = -C_K\, \bar{Q}_v \frac{i\epsilon\cdot D_T}{m_Q} Q_v \qquad \text{from phase change}$$

$$\delta\mathcal{L}_F^{(1)} = 0 \qquad \text{at this order}$$

$\therefore$ $C_K = 1$ to all orders in α_s [schemes which don't break RPI]

$C_F(\mu)$ is true Wilson Coefficient (Matching, Running ...)

$\downarrow$ L10

LL $\quad C_F(\mu) = \left[\dfrac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right]^{C_A/\beta_0}$, $\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f$

$C_A = 3$

Note: $\quad \mathcal{V}^\mu = v^\mu + \dfrac{i D^\mu}{m_Q}$ is RPI invariant

RPI can also affect $1/m_Q$ suppressed currents, by relating their Wilson Coeff's to that of the leading current (analog of $C_K = 1$)

Masses $\quad m_H = m_Q + \overset{}{\underset{\uparrow\, O(1)}{\bar\Lambda}} + O(1/m_Q)$ $\qquad v = v_r = (1, \vec{0})$

$\mathcal{L}^{(0)} + \mathcal{L}_{QCD}^{light} \to \mathcal{H}^{(0)}$, $\quad \dfrac{\langle H^{(Q)} | \mathcal{H}^{(0)} | H^{(Q)} \rangle}{\langle H^{(Q)} | H^{(Q)} \rangle} \equiv \bar\Lambda$ $\quad$ indep of spin, flavor

$\bar\Lambda$ only depends on S_ℓ^π , same <u>within</u> each multiplet

<u>$O(1/m_Q)$</u> $\quad \mathcal{H}_1 = -\mathcal{L}_1 = \bar{Q}_v \dfrac{D_T^2}{2 m_Q} Q_v + C_F\, g\, \bar{Q}_v \dfrac{\sigma_{\alpha\beta} G^{\alpha\beta}}{4 m_Q} Q_v$

$$2 \lambda_1 \equiv - \langle H^{(Q)} | \bar{Q}_{vr} D_T^2 Q_{vr} | H^{(Q)} \rangle$$

$$16\, \vec{S}_Q \cdot \vec{S}_\ell \, \lambda_2(m_Q) = C_F(\mu) \langle H^{(Q)} | \underbrace{\bar{Q}_{vr}\, g\, \sigma_{\alpha\beta} G^{\alpha\beta}}_{} Q_{vr} | H^{(Q)} \rangle$$

$\bar{Q}_{vr}\, \sigma \cdot B\, Q_{vr}$, $\quad \bar{Q}_{vr} \sigma^i Q_{vr} \sim S_Q^i$

time reversal, $B^i \sim S_\ell^i$

$$\vec{S}_Q \cdot \vec{S}_\ell = \frac{J^2 - S_Q^2 - S_\ell^2}{2}$$

$\lambda_2(m_Q)$ has $\ln(m_Q)$ dependence (only) from $C_F(\mu, m_Q)$

<u>expect</u> $\quad \left.\begin{array}{l} \lambda_1 \sim \Lambda_{QCD}^2 \\ \lambda_2 \sim \Lambda_{QCD}^2 \end{array}\right\}$ non-perturbative hadronic parameters

$$M_B = m_b + \bar\Lambda - \frac{\lambda_1}{2m_b} - \frac{3}{2}\frac{\lambda_2(m_b)}{m_b}$$

$$M_{B^*} = m_b + \bar\Lambda - \frac{\lambda_1}{2m_b} + \frac{\lambda_2(m_b)}{2m_b}$$

$$M_D = m_c + \bar\Lambda - \frac{\lambda_1}{2m_c} - \frac{3}{2}\frac{\lambda_2(m_c)}{m_c}$$

$$M_{D^*} = m_c + \bar\Lambda - \frac{\lambda_1}{2m_c} + \frac{\lambda_2(m_c)}{2m_c}$$

[other states have different $\bar\Lambda, \lambda_1, \lambda_2$, eg. Λ_b, (Σ_b, Σ_b^*) (D_1, D_2^*)

$$M_{\Lambda b} = m_b + \bar\Lambda_\Lambda - \frac{\lambda_{1\Lambda}}{2m_b}$$

$$M_{\Lambda c} = m_c + \bar\Lambda_\Lambda - \frac{\lambda_{1\Lambda}}{2m_c}$$
]

$$\bar m_P \equiv \frac{3M_{P^*} + M_P}{4}, \quad P = B, D \quad \text{is indep of } \lambda_2$$

$$M_{B^*}^2 - M_B^2 \simeq .49\ \text{GeV}^2 \simeq 4\lambda_2(m_b) \simeq 4 * (.12\ \text{GeV}^2)$$

$$M_{D^*}^2 - M_D^2 \simeq .55\ \text{GeV}^2 \simeq 4\lambda_2(m_c)$$

$$1.12 \overset{\text{expt}}{\simeq} \frac{\lambda_2(m_c)}{\lambda_2(m_b)} \overset{\substack{\text{LL}\\ \text{RGE}}}{\approx} \left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{1/3} \simeq 1.17 \qquad n_f = 3$$

not bad!

[We'll discuss higher order $1/m_Q$ & α_s corrections later]

Phenomenology

A simple class of B-decays are semileptonics:

exclusive

$B \to D\ell\bar\nu$, $B \to D^*\ell\bar\nu$ form factors (H.Q.S.), $1/m_Q$ corr, $\alpha_s(m_Q)$

inclusive

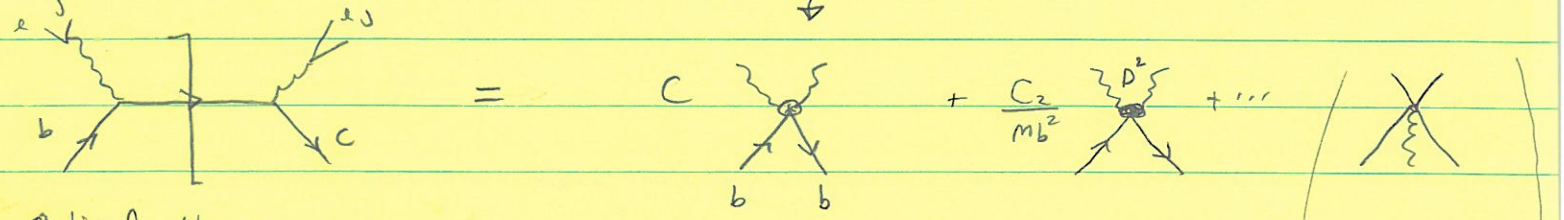
$B \to X_c\ell\bar\nu$ OPE also HQET constraints, & compute $1/m_Q$ power corr ($\overset{\text{same}}{\underset{\text{appear}}{\lambda_1, \lambda_2 !}}$), $\alpha_s(m_Q)$ corr.

here $X_c = \{D, D^*, D\pi, D\pi\pi\pi, \ldots\}$

Inclusive Spectrum $\dfrac{d\Gamma}{dq^2\, dE_e\, dM_X^2}$ $\quad q \equiv (p_e + p_\nu) = (p_B - p_X)$

OPE means "expand in Λ_{QCD}/m_Q for this process with $\sum_X$"
[the $\sum_X$ is important, and demonstrating that this makes the OPE valid requires more work; extra reading will be suggested]

LO $B \to X_c \ell \bar\nu$ is $b \to c \ell \bar\nu$ including $\alpha_s^k(m_Q)$ corrections



optical theorem
$\sim |A|^2$ $\qquad \mathcal{O}_0 = \bar b b$ $\qquad O_1 = \bar b D_T^2 b$ $\qquad O_1' = \bar b \sigma \cdot G b$

$\langle B | \bar b b | B \rangle = 1$
no non-pert paramter

power correction

$C = C(\alpha_s(m_Q), \text{kin. vars})$ is $b \to c \ell \bar\nu$ decay rate!

This OPE requires that kinematic variables are "hard" $\sim m_Q$
or integrated over a region of phase space $\sim m_Q$, $\int_0^{\sim m_B^2} dM_X^2$

[If we restrict phase space this introduces new scales in the problem, and hence need not be described by HQET with $m_Q \gg \Lambda_{QCD}$. More later.]

NLO $\mathcal{O}\left(\frac{\Lambda_{QCD}}{m_b}\right) = 0$ argument uses e.o.m $iv \cdot D h_v = 0$

<u>NNLO</u> $\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{m_B^2}\right)$ spectrum depends on λ_1, λ_2
<u>and</u> <u>that's</u> <u>it</u>

This OPE is phenomenologically very successful.
eg $|V_{cb}| = (41.6 \pm 0.6) \times 10^{-3}$ is measured comparing expt. rates ⇔ OPE

Rather than exploring phenomenology, we'll use B-decays to explore the topic of

<u>Renormalons</u>

<u>Idea</u>

- there is a freedom in defining perturbative series <u>&</u> <u>simultaneously</u> Lagrangian parameters (like masses) or matrix elements (like λ_1, λ_2). A poor choice leads to lousy convergence $(\sum \alpha_s^n)$ on the one hand, and irreducible uncertainty in the meaning of a parameter like λ_1 on the other.
- poor choices are plagued by "renormalons"
- What goes wrong is that short distance coefficients may have hidden (power law) sensitivity to IR
Matrix elements have sensitivity to UV.

eg. $b \to u e \bar{\nu}$ at LO $(B \to X_u e \bar{\nu})$

K_i = #'s

$\epsilon = 1$ tracks order

$$\Gamma(b \to u e \bar{\nu}) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} \underbrace{m_b^5} \left[1 + K_1 \frac{\alpha_s(m_b)}{\pi} \epsilon + K_2 \frac{\alpha_s^2(m_b)}{\pi^2} \epsilon^2 + \cdots \right]$$

pole scheme: $\quad " \quad (m_b^{pole})^5 \left[1 - 0.17\,\epsilon - 0.13\,\epsilon^2 + \cdots \right]$

$\overline{MS}$ scheme: $\quad " \quad (\overline{m}_b)^5 \left[1 + 0.30\,\epsilon + 0.19\,\epsilon^2 + \cdots \right]$

1S scheme: $\quad " \quad (m_b^{1S})^5 \left[1 - 0.115\,\epsilon - 0.035\,\epsilon^2 + \cdots \right]$

Mass schemes

$$m_b^{pole} = \overline{m}_b(m_b)\left(1 + \frac{4}{3}\frac{\alpha_s(m_b)}{\pi}\epsilon + 13\frac{\alpha_s^2}{\pi^2}\epsilon^2 + \dots\right)$$
$$= \overline{m}_b(m_b)\left(1 + 0.09\epsilon + 0.06\epsilon^2 + \cdots\right)$$
$$m_b^{pole} = m_b^{1S}\left(1 + 0.011\epsilon + 0.016\epsilon^2 + \dots\right)$$

Choice of mass scheme effects perturbative series for rate.

Physically

- m_b^{pole} is a poor choice. Due to confinement the notion of a "pole in a quark propogator" is only perturbatively meaningful & is ambiguous by $\Delta m_b^{pole} \sim \Lambda_{QCD}$
  Our setup for HQET used m_{pole} $\quad e^{-i m_b^{pole} v\cdot x}$
  because we expand about the mass shell.

Other mass choices can be implemented $\quad m_b^{pole} = \underset{\substack{\uparrow\\ \text{some}\\ \text{scheme}}}{m} + \underset{\substack{\uparrow\\ \text{some } \alpha_s\\ \text{series}}}{\delta m}$

but this requires $\mathcal{L}_{\delta m} = -\delta m\, \bar{Q}_v Q_v$ in HQET $\mathcal{L}$.

- $\overline{MS}$ has $\delta m = \overline{m}_b \alpha_s + \dots$, and parametrically/numerically δm is too big for HQET power counting (dynamics)

- 1S mass: $\quad m_b^{1S} = \frac{1}{2}$(mass $b\bar{b}$ bound state in pert theory) $= \frac{1}{2} M_\Upsilon^{pert}$
  $\delta m = m_b^{1S} \alpha_s^2$ $\quad$ small enough

- General: $\delta m = R\,\alpha_s + \dots$ $\quad$ R tuneable so we can satisfy power counting

Mathematically

- QFT perturbative series are usually not convergent but rather asymptotic series

$$f(\alpha) \overset{\text{asymptotic}}{\doteq} \sum_{n=-1}^{\infty} f_n \, \alpha^{n+1} \quad \text{iff} \quad \left| f(\alpha) - \sum_{n=-1}^{N} f_n \alpha^{n+1} \right| < K_{N+2} \, \alpha^{N+2}$$

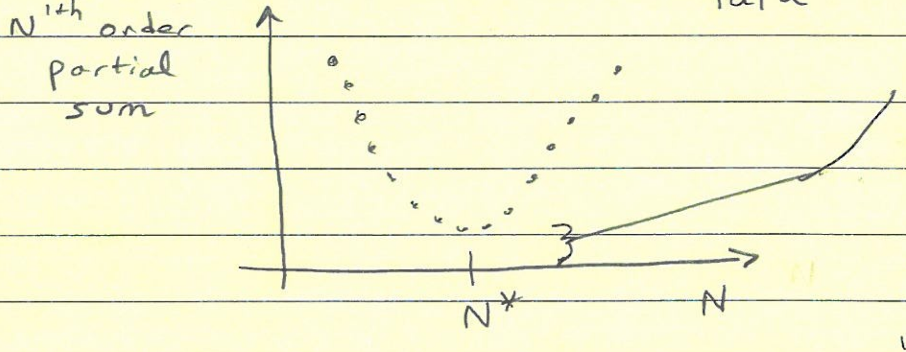
for some numbers K_{N+2}

In QFT its typical that $f_n \sim a^n \, n!$ as $n \to \infty$

for fixed $\alpha \ll 1$ truncation error can grow $K_N \sim N! \, a^N$

series has zero radius of convergence in α

Series will decrease until $N_* \sim \dfrac{1}{|a| \alpha}$



for $N^* \gg 1$

$K_{N^*} \alpha^{N^*}$

$\sim \exp\left(\dfrac{-1}{|a| \alpha} \right)$

for "best approx"

We can classify how poorly convergent a series is with a

Borel Transform $\quad f(\alpha) \Leftrightarrow F(b)$

Let $\quad F(b) \equiv f_{-1} \, \delta(b) + \displaystyle\sum_{n=0}^{\infty} \frac{f_n \, b^n}{n!} \quad \leftarrow$ improved convergence

$$f(\alpha) = \int_0^\infty db \; e^{-b/\alpha} \, F(b) \qquad \text{is inverse transform}$$

For a convergent series $\sum_n f_n \alpha^{n+1}$ we get back original $f(\alpha)$

For divergent series where $F(b)$ exist $_\wedge$ and inverse transform we can use it to define $f(\alpha)$

eg. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \alpha^{n+1}$ for $\alpha > 1$ $\qquad \left(\text{for } \alpha < 1 \text{ its } \dfrac{\alpha}{1+\alpha} \right)$

here $F(b) = \displaystyle\sum_{n=0}^{\infty} \frac{(-b)^n}{n!} = e^{-b}$

$$\int_0^\infty db \; e^{-b/\alpha} \, e^{-b} = \frac{\alpha}{1+\alpha}$$ $\leftarrow$ convergent integral for $\alpha > 1$

If inv. transform doesn't exist poles in $F(b)$ tell us about severity of divergence

eg. $f_n = \bar{a}^n (n+K)!$ then $B(b) = \frac{k!}{(1-b/a)^{K+1}}$ + less singular

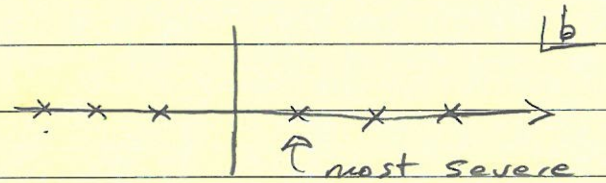
has a pole at $b=a$

Call this a "$b=a$ renormalon"

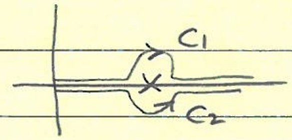
If $a<0$ then inv. transform exists

If $a>0$ pole is on integration contour $\int_0^\infty db$, no inv. transform

Smaller a's give poles closer to origin

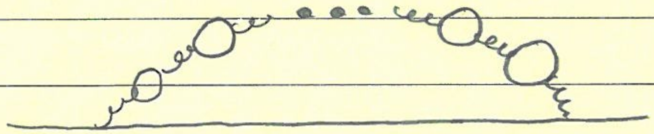


ambiguity is given by residue of pole: $C_1 - C_2$

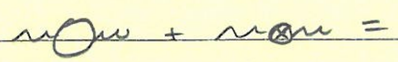


Consider pole versus $\overline{MS}$ mass

Obviously we can't compute all the f_n's for QCD. We'll have to be satisfied with a unique ∞-subset of terms.

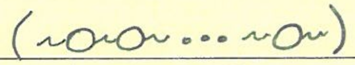


bubble sum



$$= \frac{-i}{p^2+i0}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta^{ab}\left(\frac{-n_f \alpha_s}{6\pi}\right)\left[\underbrace{\ln\left(\frac{-\mu^2}{p^2}\right) + \frac{5}{3}}_{\ln\left(\frac{-\mu^2 e^{\bar{C}}}{p^2}\right)} + \cancel{\frac{1}{\epsilon}} - \cancel{\frac{1}{\epsilon}}\right]$$

$\bar{C} = \frac{5}{3}$

$G^{\mu\nu}_{\text{bubble}}(p, \alpha_s) = \sum_n$ () + c.t

n-bubbles

(use Landau Gauge)

$$= \frac{-i}{p^2+i0}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta^{ab}\sum_{n=0}^{\infty}\left(\frac{\beta_0 \alpha_s}{4\pi}\right)^n \ln^n\left(\frac{-\mu^2 e^{\bar{C}}}{p^2}\right)$$

n_f^n determines $\beta_0^n = \left(-\frac{2}{3} n_f + \frac{11}{3} C_A\right)^n$

$n_f^n N_c^K$ vs $\beta_0^n N_c^K$ basis

Asymptotics

Big O $f(x) = O(g(x))$ as $x \to \infty$ iff $\exists M > 0$ so $|f(x)| \leq M|g(x)|$ $\forall x > x_0$

$f(x) = O(g(x))$ as $x \to a$ iff $\exists M, \delta > 0$ so $|f(x)| \leq M|g(x)$ for $|x-a|<\delta$

or for non zero $g(x)$'s iff $\limsup_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty$

for a sum of terms keep the one with largest growth rate

eg $6x^4 - 2x^3 + 5 \sim O(x^4)$

Little o $f(x) \in o(g(x))$ means $g(x)$ grows much faster than $f(x)$

$f(x) = o(g(x))$ as $x \to \infty$ iff $\forall \epsilon > 0$ $\exists N \ni$ $|f(x)| \leq \epsilon |g(x)|$ for all $x \geq N$

stronger statement than big O (implies big O)

and for non-zero $g(x)$ its iff $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

Asymptotic Expn

truncation after finite # of terms provides approx as argument tends toward valu

f has asymptotic expn of order N as $x \to L$ iff

$$f(x) - \sum_{n=0}^{N-1} a_n \varphi_n(x) = o(\varphi_{N-1}(x)) \quad \text{as } x \to L$$

or $$f(x) - \sum_{n=0}^{N-1} a_n \varphi_n(x) = O(\varphi_N(x)) \quad \text{as } x \to L$$

If this hold for all N we say $f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x)$ as $x \to L$

Convergent For any $\epsilon > 0$ $\exists N'$ such that for all $N \geq N'$

$$\left| \sum_{n=1}^{N} a_n \varphi_n(x) - f(x) \right| < \epsilon$$

variable "b" previously : $\alpha^{n+1} \to b^n/n!$

Borel transform in variable u : $\left(\frac{\beta_0 \alpha_s(\mu)}{4\pi}\right)^{n+1} \to \frac{u^n}{n!}$

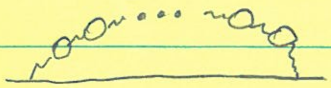
An extra $g^2 = 4\pi\alpha_s = \frac{16\pi^2}{\beta_0}\left(\frac{\alpha_s\beta_0}{4\pi}\right)$ comes from ends
so $n \geq 0$ here

$$[g^2 G^{\mu\nu}_{bubble}](p,u) = \frac{-i}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta^{ab}\frac{16\pi^2}{\beta_0}\sum_{n=0}^{\infty} \frac{u^n}{n!}\ln^n\left(\frac{-\mu^2 e^{\bar c}}{p^2}\right)$$

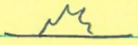
$$= \frac{-i}{p^2}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\delta^{ab}\frac{16\pi^2}{\beta_0}\exp\left[u\ln\left(\frac{-\mu^2e^{\bar c}}{p^2}\right)\right]$$

$$= \frac{-i}{(-p^2)^{2+u}}\left(p^2g^{\mu\nu}-p^\mu p^\nu\right)\delta^{ab}\frac{16\pi^2}{\beta_0}\left(\mu^2e^{\bar c}\right)^u$$

acts like a modified gluon propogator. Therefore to compute



is about as hard as



≡ Σ bubbles

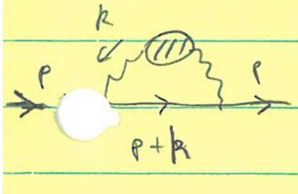
Calculate $\Sigma^{bubbles}$ in terms of $\bar m$ (the $\overline{MS}$ mass), cancel $1/\epsilon_{UV}$ poles. These come from $u=0$ (log div.) while values $u>0$ and $u<0$ probe power divergences through $1/(-p^2)^{2+u}$

Propogater $\frac{i}{\not p - \bar m - \Sigma(p,\bar m)}$, $\Sigma(p,\bar m) = \bar m\,\Sigma_1(p^2,\bar m,\alpha_s) + (\not p - \bar m)\,\Sigma_2(p^2,\bar m,\alpha_s)$

Pole mass : $\left[\not p - \bar m - \Sigma(p,\bar m)\right]\Big|_{p^2=m_{pole}^2} = 0$

$(1-\Sigma_2)\not p = (1-\Sigma_2+\Sigma_1)\bar m$

$m_{pole}^2 = \not p^2 = \bar m^2\left(\frac{1-\Sigma_2+\Sigma_1}{1-\Sigma_2}\right)^2 \nearrow \boxed{m_{pole} = \bar m\,(1+\Sigma_1+\cdots)}$



$$= i^3 C_F\int đ^d k\, \frac{\bar u(p)\,\gamma^\mu(\not p+\not k+\bar m)\gamma^\nu u(p)}{\left[(p+k)^2-\bar m^2\right]}\,[g^2G^{bubble}_{\mu\nu}](k,\alpha_s)$$

Borel $\to$ $[g^2G^{bubble}_{\mu\nu}](k,u)$ in place of $[g^2G^{bubble}_{\mu\nu}](k,\alpha_s)$

[suffices as long as $u\neq 0$ ie for probe]

Combine Denominators $\frac{1}{a^n b} = \frac{\Gamma(n+1)}{\Gamma(n)} \int_0^1 dx \frac{x^{n-1}}{[ax+b(1-x)]^{n+1}}$

... algebra ...

get $\Sigma_1 (p^2 = \bar{m}^2, \bar{m}, u)$ ← Borel Space

$$M_{pole} = \bar{m} \left[\delta(u) - \frac{C_F}{6\pi\beta_0} \left(\frac{\mu^2 e^{\bar{c}}}{\bar{m}^2}\right)^u 6(1-u) \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} \right.$$

$$\left. \begin{array}{l} + \text{ pole } \frac{1}{u} \text{ rendering it regular at } u=0 \\ + \text{ regular terms in } u \end{array} \right]$$

↑ not determined, but not needed

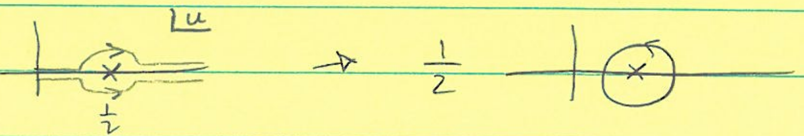
Strongest pole at $u = 1/2$ from $\Gamma(1-2u)$
"$u = 1/2$ renormalon"

$$M_{pole} = \bar{m} \left[\delta(u) - \underbrace{\frac{C_F}{6\pi\beta_0} \left(\frac{\mu}{\bar{m}} e^{\bar{c}/2}\right) \frac{(-2)}{(u - 1/2)}}_{B(u)} + \cdots \right]$$

Inverse Borel

$$B(\alpha_s) = \int_0^\infty du \, \exp\left(-u \frac{4\pi}{\beta_0 \alpha_s(\mu)}\right) B(u)$$

Has ambiguity



$$|2\pi i| \frac{1}{2} \oint du \, e^{-u\, 4\pi/\beta_0 \alpha_s} \frac{1}{(u-1/2)} \frac{\mu e^{\bar{c}/2}}{\cancel{\bar{m}}} \left(\frac{C_F}{3\pi\beta_0}\right) \cancel{\bar{m}}$$

$$= e^{\bar{c}/2} \frac{C_F}{3\beta_0} \underbrace{\left(\mu \exp\left(\frac{-2\pi}{\beta_0 \alpha_s}\right)\right)}_{\Lambda_{QCD}} = \underbrace{e^{\bar{c}/2} \frac{C_F}{3\beta_0} \Lambda_{QCD} \simeq \delta M_{pole}}_{\text{an } \mathcal{O}(\Lambda_{QCD}) \text{ ambiguity as expected.}}$$

Facts

① Ambiguity didn't depend on use of $\overline{m}$, it is strictly associated to m_{pole}

② Ambiguity is μ independent, $\delta m \propto \Lambda_{QCD}$.
Residue of pole is μ-dependent. When we express $\Gamma(b \to u e \bar{\nu}) = (m_b^{pole})^5 [1 + \alpha_s + \alpha_s^2 + \ldots]$ in terms of $\overline{m}_b$, the poles in the two series cancel $\frac{1}{(u-\frac{1}{2})} - \frac{1}{(u-\frac{1}{2})}$ as long as <u>both</u> are expanded in <u>same</u> <u>$\alpha_s(\mu)$</u>.

Recall Borel variable $u \Leftrightarrow \frac{4\pi}{\beta_0 \alpha_s(\mu)}$, must have same meaning for both series.

③ Poles always cancel for observables.

④ Removing the ambiguity involves introducing a new scale "R"

$$M_{pole} = m(R) + \underbrace{R \sum_{n=1}^{\infty} \sum_{k} a_{nk} \ln^{k}(\mu/R)}_{\delta m} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n}$$

is a general scheme change. $m(R) = M_{pole} - \delta m$ will be renormalon free if the series δm properly subtracts the pole mass renormalon,

large n $\quad a_{(n+1)0} \sim n!\, 2^n \beta_0^n \quad$ for n_f-bubbles

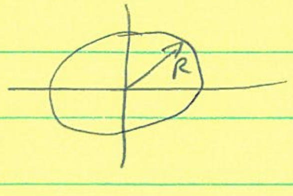
In $\overline{MS}$ $\quad R = \bar{m}(\mu = \bar{m})$.

In 1S $\quad R = M_{1S}\,\alpha_s(\mu)$ $\quad$ (inverse Bohr radius)

can make R a free parameter

Ambiguity is $\delta m \sim \Lambda_{QCD}$ & independent of R

R sets the scale for absorbing IR fluctuations together with Mpole (point particle mass) to yield well defined $m(R)$

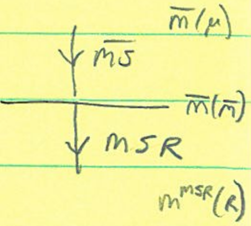


mass $m(R)$

eg <u>MSR mass</u> $\quad \mu = R$, use $\overline{MS}$ scheme to define a_{n0}'s but do not set $R = \bar{m}$

$$M_{pole} = M^{MSR}(R) + R \sum_{n=1}^{\infty} \bar{a}_{n0} \left(\frac{\alpha_s(R)}{4\pi}\right)^{n}$$

good for physics at scale $\sim R$ $\quad$ $\overline{MS}$ values



Note: $M^{MSR}(R = \bar{m}) = \bar{m}(\mu = \bar{m})$, matches onto $\overline{MS}$

$\boxed{\text{Aside}}$ Λ_{QCD} defined at higher orders (LL, NLL, ...)

$\overline{MS}$

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = \beta[\alpha_s] = -2\alpha_s(\mu) \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{n+1}$$

↑ #'s with color factors

$$\ln \frac{R_1}{R_0} = \int_{R_0}^{R_1} \frac{dR}{R} = \int_{\alpha_s(R_0)}^{\alpha_s(R_1)} \frac{d\alpha}{\beta[\alpha]}$$

let $t \equiv \frac{-2\pi}{\beta_0 \alpha_s(R)}$, $dt = \frac{2\pi}{\beta_0} \frac{d\alpha_s}{\alpha_s^2}$

$$t_0 = \frac{-2\pi}{\beta_0} \frac{1}{\alpha_s(R_0)}, \quad t_1 = \frac{-2\pi}{\beta_0} \frac{1}{\alpha_s(R_1)}$$

$$\ln \frac{R_1}{R_0} = \int_{t_1}^{t_0} dt \, \underbrace{\hat{b}(t)} = G(t_0) - G(t_1)$$

$$\hat{b}(t) = 1 + \frac{\hat{b}_1}{t} + \frac{\hat{b}_2}{t^2} + \frac{\hat{b}_3}{t^3} + \dots \quad \leftarrow \quad \frac{d\alpha}{\left(-\frac{\beta_0}{2\pi} \alpha_s^2\right)} = -dt \text{ etc}$$

where $\hat{b}_1 = \frac{\beta_1}{2\beta_0^2}$, $\hat{b}_2 = \frac{\beta_1^2 - \beta_0\beta_2}{4\beta_0^4}$, $\hat{b}_3 = \frac{\beta_1^3 - 2\beta_0\beta_1\beta_2 + \beta_0^2\beta_3}{8\beta_0^6}$

$$G(t) = t + \hat{b}_1 \ln(-t) - \frac{\hat{b}_2}{t} - \frac{\hat{b}_3}{2t^2} + \dots$$

$$G'(t) = \hat{b}(t)$$

$$G(t_1) = G(t_0) - \ln R_1/R_0 \quad \leftarrow$$

all orders relation between $\alpha_s(R_1)$ & $\alpha_s(R_0)$

And rearranging $R_1 e^{G(t_1)} = R_0 e^{G(t_0)} \equiv \Lambda_{QCD}$

$$\Lambda_{QCD} = \mu \exp\left[\underbrace{\frac{-2\pi}{\beta_0 \alpha_s(\mu)}}_{\text{LL result}} + \hat{b}_1 \underbrace{\ln\left(\frac{2\pi}{\beta_0 \alpha_s(\mu)} \right)}_{\text{NLL}} + \dots \right]$$

$\boxed{\text{End Aside}}$

Lets treat R like a variable which parameterizes a mass scheme. Vary scale R in MSR scheme much like we vary μ in $\overline{MS}$.

set $\mu = R$

$$\boxed{\text{R-RGE}} \qquad R\, d/_{dR}\, m_{pole} = 0 = R\, d/_{dR}\, m(R) + \underbrace{R\, d/_{dR}\, \delta m(R)}$$

$$\equiv R\, \gamma_R[\alpha_s(R)]$$

$$\gamma_R[\alpha_s(R)] = \sum_{n=0}^{\infty} \gamma_n^R \left[\frac{\alpha_s(R)}{4\pi}\right]^{n+1}$$

perturbative series
($\mu = R$ avoids $\ln(\mu/R)$
terms that could be
large if we vary R &
have $\mu \gg R$ in some region)

$$\boxed{R\, d/_{dR}\, m(R) = -R\, \gamma_R[\alpha_s(R)]}$$

Solution: $$m(R_1) - m(R_0) = -\int_{\ln R_0}^{\ln R_1} d\ln R\;\, R\, \gamma_R[\alpha_s]$$

$$R = \Lambda_{QCD}\, e^{-G(t)} \quad , \quad d\ln R = dt\left[-G'(t)\right]$$

$$R\, d\ln R = \Lambda_{QCD}\, dt\left[d/_{dt}\, e^{-G(t)}\right]$$

$$\boxed{m(R_1) - m(R_0) = \Lambda_{QCD} \int_{t_1}^{t_0} dt\; \underbrace{\gamma_R(t)}_{\text{well defined integral}}\; d/_{dt}\, e^{-G(t)}}$$

Problem Set uses ⊛ this

evolution $R_0 \to R_1$ yields new well defined $m(R_1)$ which absorbs different amount of IR fluctuations.

eg. LL solution $\quad \gamma_R[\alpha_s] = \gamma_0^R \frac{\alpha_s(R)}{4\pi}$, $\quad \gamma_R(t) = -\frac{\gamma_0^R}{2\beta_0}\frac{1}{t}$

$$G(t) = t$$

$$m(R_1) - m(R_0) = \frac{\Lambda_{QCD}^{(0)}\gamma_0^R}{2\beta_0} \int_{t_1}^{t_0} \frac{e^{-t}}{t}$$

$$= \frac{\Lambda_{QCD}^{(0)}\gamma_0^R}{2\beta_0} \left[\Gamma(0,t_1) - \Gamma(0,t_0)\right]$$

↑ incomplete Gamma Function

Last Time

$\downarrow$ L12

$$M_{pole} = \overset{\substack{\text{mass in} \\ \text{MSR scheme} \\ \swarrow}}{m(R)} + \underset{\substack{\uparrow \\ \text{power law} \\ \text{cutoff parameter}}}{R} \sum_{n=1}^{\infty} \underset{\substack{\uparrow \\ \overline{MS} \text{ values}}}{\bar{a}_{n0}} \left(\frac{\alpha_s(R)}{4\pi}\right)^n$$

R-RGE $\quad R \frac{d}{dR} m(R) = -R\, \gamma_R[\alpha_s(R)]$

$$R = \Lambda_{QCD}\, e^{-G(t)}$$

$$m(R_1) - m(R_0) = \Lambda_{QCD} \int_{t_1}^{t_0} dt\; \gamma_R(t) \frac{d}{dt} e^{-G(t)} \qquad \circledast$$

LL solution $\quad \gamma_R(t) = -\frac{\gamma_0^R}{2\beta_0} \frac{1}{t}, \quad G(t) = t$

$$m(R_1) - m(R_0) = \frac{\Lambda_{QCD}^{(0)}\, \gamma_0^R}{2\beta_0} \left[\Gamma(0,t_1) - \Gamma(0,t_0)\right]$$

Expanding about $\alpha_s = 0$, $t = +\infty$, we have asymptotic expansion

$$\Lambda_{QCD}^{(0)}\, \Gamma[0,t] = -2R \sum_{n=0}^{\infty} \underset{\uparrow}{2^n}\, n! \left[\frac{\beta_0\, \alpha_s(R)}{4\pi}\right]^{n+1}$$

$u = \frac{1}{2}$ type renormalon

But expanding in $\alpha_s(R_1)$

$$m(R_1) - m(R_0) = -\frac{\gamma_0^R}{2\beta_0} R_1 \sum_{n=0}^{\infty} \left[\frac{\beta_0\, \alpha_s(R_1)}{2\pi}\right]^{n+1} n! \left[1 - \frac{R_0}{R_1} \sum_{k=0}^{n} \frac{1}{k!} \ln^k \frac{R_1}{R_0}\right]$$

$$= \frac{-\gamma_0^R R_0}{2\beta_0} \sum_{n=0}^{\infty} \left[\frac{\beta_0\, \alpha_s(R_1)}{2\pi}\right]^{n+1} \sum_{k=n+1}^{\infty} \frac{n!}{k!} \ln^k \frac{R_1}{R_0}$$

renormalon free & sums $\ln R_1/R_0$ logs.

We can connect physics at scales $R_1 \gg R_0$ in renormalon free fashion [which is not possible in general with $\overline{MS}$ & μ-RGE]

All Order Generalization

Write ⊛'s integrand: $e^{-t-\hat{b}_1 \ln(-t)} \left[\underbrace{1 + \frac{\#}{t} + \frac{\#}{t^2}}_{\text{from } \gamma_R(t),\ \text{higher } \hat{b}_i\text{'s}} + \cdots\right]$

$$-\gamma_R(t)\, G'(t)\, e^{-G(t)} = \frac{e^{-t}\,(-t)^{-\hat{b}_1}}{t} \sum_{j=0}^{\infty} S_j \frac{1}{(-t)^j}$$

Integrate

$$\left[m(R_1) - m(R_0)\right]^{N^kLL} = \Lambda_{QCD}^{(k)} \sum_{j=0}^{k} S_j\, (-1)^j\, e^{i\pi \hat{b}_1} \left[\Gamma(-\hat{b}_1 - j, t_1) - \Gamma(-\hat{b}_1 - j, t_0)\right]$$

where $S_0 = \tilde{\gamma}_0^R$

$S_1 = \tilde{\gamma}_1^R - (\hat{b}_1 + \hat{b}_2)\, \tilde{\gamma}_0^R \qquad \tilde{\gamma}_k^R \equiv \dfrac{\gamma_k^R}{(2\beta_0)^{k+1}}$

$S_2 = \ldots$

Note: $\gamma_R(\alpha_s(R)) = {}^{d}/_{dR}\, \delta m(R)$ is free of $\delta m \sim \Lambda_{QCD}$ renormalon

Find $\gamma_0^R = a_1$ $\qquad a_n \equiv a_{n0}$

$\gamma_1^R = a_2 - 2\beta_0 a_1$

$\gamma_2^R = a_3 - 4\beta_0 a_2 - 2\beta_1 a_1$ ← contains higher order (non-bubble) info too

$\vdots$

$\gamma_n^R = \underbrace{a_{n+1}} - \underbrace{(2n\beta_0)\, a_n} + \cdots$

n_f-bubbles: $n!\,(2\beta_0)^n \qquad (2n\beta_0)\,(n-1)!\,(2\beta_0)^{n-1}$ ambiguities cancel

We'll return to practical applications of the R-RGE in a bit.

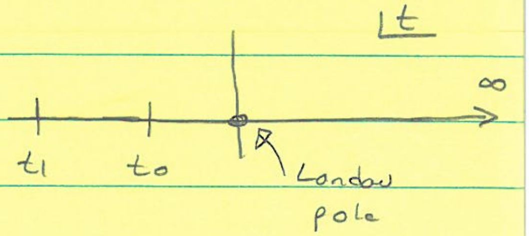
recall $t = \frac{-2\pi}{\beta_0 \alpha_s(R)}$

Renormalon Probe

$$m(R_1) - m(R_0) = \Lambda_{QCD} \int_{t_1}^{t_0} dt\, \gamma_R(t)\, {}^{d}/_{dt}\, e^{-G(t)}$$

consider $R_0 \to 0$ then $m(R_0) \to m_{pole}$

$t_0 = -\ln\left(\frac{R_0}{\Lambda_{QCD}}\right) \to +\infty$



must continue past Landau pole at $t = 0$
which introduces an ambiguity

$$m(R_1) - m_{pole} = \Lambda_{QCD} \int_{t_1}^{\infty} dt\, \gamma_R(t)\, {}^{d}/_{dt}\, e^{-G(t)}$$

$$\overset{LL}{=} \frac{\Lambda^{(0)}_{QCD}\, \gamma_0^R}{2\beta_0} \int_{t_1}^{\infty} dt\, \frac{e^{-t}}{t}$$

... Manipulations ... (P.Set)

$$= \int_0^\infty du\, F(u)\, e^{-u \frac{4\pi}{\beta_0 \alpha_s(R)}}$$

where $F(u) \sim \frac{1}{u - 1/2}$

Landau Pole ⟺ Borel Pole

Sum Rule for renormalon

formally taking the Borel transform to all orders yields

$$P_{1/2} = \sum_{k=0}^{\infty} \frac{\overset{\text{RGE coefficients}}{\overset{\swarrow}{S_k}}}{\Gamma(1+\hat{b}_1+k)} \qquad \text{for } \underline{\text{residue}}\ P_{1/2} \text{ of } u=\tfrac{1}{2} \text{ pole}$$

$P_{1/2} \neq 0$ means $u=\frac{1}{2}$ renormalon is present. This allows us to detect / probe for renormalons without bubble chains.

$\boxed{\text{Renormalons in Operator Product Expansions}}$ (OPE)

Consider OPE in $\overline{MS}$

$$\underset{\substack{\uparrow\\ \text{dimensionless}\\ \text{observable}}}{\sigma} = \underset{\substack{\uparrow\\ \text{dimless}\\ \overline{MS}\text{ Wilson}\\ \text{coefficient}}}{\bar{C}_0(Q,\mu)}\ \underset{\substack{\uparrow\\ \overline{MS}\text{ matrix}\\ \text{element}\\ (\text{dim}=0)}}{\bar{\Theta}_0(\mu)} + \underset{\substack{\uparrow\\ \text{dim}=0}}{\bar{C}_1(Q,\mu)}\ \frac{\overset{\text{dim}=1}{\overset{\swarrow}{\bar{\Theta}_1(\mu)}}}{Q} + \cdots$$

$$\text{here}\quad \bar{C}_0(Q,\mu) = 1 + \sum_{n=1}^{\infty} \underbrace{b_n(\mu/Q)}_{=\sum_k b_{nk} \ln^k(\mu/Q)} \left[\frac{\alpha_s(\mu)}{4\pi}\right]^n$$

- only logs, no (μ/Q) power law terms
- Lorentz, gauge inv,
- simple for multiloop calculations

but • has renormalons generically $\Delta\bar{C}_0 \sim \frac{\Lambda_{QCD}}{Q}$, $\Delta\bar{\Theta}_1 \sim \Lambda_{QCD}$ which is a $u=1$ renormalon

Take a toy integral to see what's going on

$$\sigma \sim \int_0^\infty d^{d-3}k \;\frac{f(k^2,\Lambda_{QCD}^2)}{(k^2+Q^2)^{1/2}}\;\mu^{2\epsilon}$$

$\overline{MS}$: $\sigma \sim \mu^{2\epsilon}\int_0^\infty d^{d-3}k\,\frac{[f(k^2,0)+\cdots]}{(k^2+Q^2)^{1/2}} + \mu^{2\epsilon}\int_0^\infty d^{d-3}k\; f(k^2,\Lambda_{QCD}^2)\left[\frac{1}{Q}+\cdots\right]$

is

$$\sigma = \overset{\downarrow}{\bar{C}_0(Q,\mu)}\;\overset{\swarrow 1}{\bar{O}_0} + \overset{\downarrow 1}{\bar{C}_1}\;\overset{\downarrow}{\frac{\bar{O}_1(\mu)}{Q}}$$

$\overline{MS}$ properly separates short & long distance scales for logs, but for powers relies on scaless integrals ⇔ 0. This treatment leaves residual sensitivity to power divergences from including the wrong regions of momentum space in integrals ⇔ renormalons.

Wilsonian: $\sigma \sim \int_{\Lambda_f}^\infty dk\,\frac{[f(k^2,0)+\cdots]}{(k^2+Q^2)^{1/2}} + \int_0^{\Lambda_f} dk\; f(k^2,\Lambda_{QCD}^2)\left[\frac{1}{Q}+\cdots\right]$

$$\sigma = \overset{\downarrow}{C_0^W(Q,\Lambda_f)}\;\underset{\uparrow 1}{O_0^W(\Lambda_f)} + \underset{\uparrow 1}{C_1^W(Q,\Lambda_f)}\;\overset{\downarrow}{\frac{O_1(\Lambda_f)}{Q}}$$

it strictly separates scales, but causes difficulties for symmetry & calculations

R-scheme OPE : write $\bar{O}_1(\mu) = O_1(R,\mu) - R\sum_{n=1}^\infty d_n(\mu/R)\left[\frac{\alpha_s(\mu)}{4\pi}\right]^n \bar{O}_0$

↑

$\bar{C}_1 = 1$ for simplicity

$$\bar{C}_0(Q,\mu) = C_0(Q,R,\mu) + \frac{R}{Q}\sum_{n=1}^\infty d_n(\mu/R)\left[\frac{\alpha_s(\mu)}{4\pi}\right]^n$$

then

$$\sigma = C_0(Q,R,\mu)\,\bar{O}_0 + \frac{O_1(R,\mu)}{Q}$$

pick d_n coefficients to remove $u=1$ $\overline{MS}$ renormalon

- still gauge & Lorentz inv.
- has power law dependence on R that removes sensitivity of $\overline{C}_0$ to small momenta
  - → perturbs $\overline{MS}$ towards Wilsonian in gauge inv. way

eg. MSR scheme for OPE

reuse $\overline{MS}$ coefficients $b_n(\mu/Q)$ at a different scale R

$$d_n(\mu/R) \equiv b_n(\mu/R)$$

$$C_0(Q,R,\mu) = \sum_{n=1}^{\infty} \underbrace{\left\{ b_n(\mu/Q) - \frac{R}{Q}\, b_n\left(\frac{\mu}{R}\right) \right\}}_{\text{renormalon is } Q \,\&\, R \text{ independent, cancels in difference}} \left[\frac{\alpha_s(\mu)}{4\pi}\right]^n$$

$$= \overline{C}_0(Q,\mu) - \underbrace{\frac{R}{Q}\,\overline{C}_0(R,\mu)}_{\text{acts as IR cutoff to ensure } C_0 \text{ is short dist.}}$$

Set $\mu = R$, pretend $\overline{C}_0$ has no anom. dim. then, similar to mass:

$$R\frac{d}{dR}\, C_0(Q,R,R) = -\frac{R}{Q}\, \gamma[\alpha_s(R)]$$

solution

$$C_0(Q,R_1,R_1) = C_0(Q,R_0,R_0) + \frac{\Lambda_{QCD}}{Q} \sum_j S_j\, (-1)^j\, e^{i\pi \hat{b}_1} \left[\Gamma(-\hat{b}_1 - j, t_0) - \Gamma(-\hat{b}_1 - j, t_1)\right]$$

$$= C_0(Q,R_0,R_0)\, \mathcal{U}(Q,R_0,R_1)$$

eg. consider $r \equiv \dfrac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2}$

$\overline{MS}$ OPE in HQET:

$$r = \underbrace{\frac{\overline{C}_F(m_b,\mu)}{\overline{C}_F(m_c,\mu)}}_{\text{lowest order purely perturbative}} + \underbrace{\frac{\overline{\Sigma}_p(\mu)}{\mu_G^2(\mu)}}_{\sim \frac{\Lambda_{QCD}^3}{\Lambda_{QCD}^2} = \Lambda_{QCD}} \left(\frac{1}{m_b} - \frac{1}{m_c}\right) + \cdots$$

$\dfrac{\Lambda_{QCD}}{m_Q}$ term

$$r^{\text{perturbative}} = 1 - 0.1113\big|_{\alpha_s} - 0.078\big|_{\alpha_s^2} - 0.0755\big|_{\alpha_s^3}$$

$$\overset{\text{or}}{=} \underbrace{(0.8517)_{LL}}_{\text{near expt.}} + (-0.0696)_{ANLL} + (-0.0908)_{ANNLL} \qquad \text{bad!}$$

$\overline{C}_F$ has $u=1$ renormalon

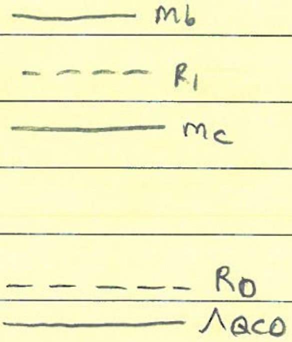
define $C_F(m_Q, R, R)$ as above in MSR scheme

$$r = \frac{C_F(m_b, R_0, R_0)}{C_F(m_c, R_0, R_0)} + \frac{\Sigma_e(R_0,R_0)}{\mu_G^2(R_0)}\left(\frac{1}{m_b} - \frac{1}{m_c}\right)$$

$\mu = R_0 \gtrsim \Lambda_{QCD}$

here choice $R_0 \gtrsim \Lambda_{QCD}$ preserves power counting for new $\Sigma_e(R_0,R_0) = \overline{\Sigma}_e - \underbrace{R\,\mu_G^2}_{\sim\Lambda_{QCD}^3} \sum \alpha_s$

Use R-RGE to sum logs between $R_0 \to m_Q$



$$r = \underbrace{\frac{C_F(m_b,R_1,R_1)\,U(m_b,R_1,R_0)}{C_F(m_c,R_1,R_1)\,U(m_c,R_1,R_0)}}_{} + \underbrace{\frac{\Sigma(R_0,R_0)}{\mu_G^2(R_0)}\left(\frac{1}{m_b} - \frac{1}{m_c}\right)}_{}$$

converges @ $\mathcal{O}(\alpha_s^3)$

$1 \to .88 \to .862 \to .860$

smaller than in $\overline{MS}$

$$r^{NNLL} = 0.860 \pm (0.065)_{\Sigma_e} \pm (0.008)_{pert}$$

↑ R_0 variation

↑ perturbative R_1 variation (like μ variation)

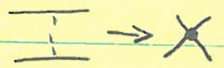
Note! since R_0 dependence cancels between leading power term & $1/Q$ term, it gives us a method for estimating the size of the power correction (much like μ does for perturbative corrections in $\overline{MS}$)

EFT with a fine tuning

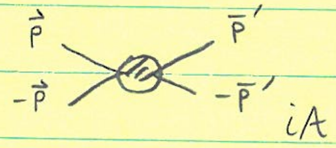
Investigate an EFT where a naively irrelevant operator must be promoted to being relevant.

Two Nucleon Non-relativistic EFT

$p << m_\pi$ integrate out all exchanged particles including pions



eg. elastic scattering



Quantum mechanics [single p.wave]

$|\vec{p}| = |\vec{p}\,'|$

$$S = e^{2i\delta} = 1 + \frac{ipM}{2\pi} A$$

$$A = \frac{4\pi}{mp} \frac{e^{2i\delta} - 1}{2i} = \frac{4\pi}{mp} \frac{1}{\cot\delta - i} = \frac{4\pi}{m} \frac{1}{p\cot\delta - ip}$$

I claim for <u>any</u> short range potential

$$p^{2l+1} \cot\delta = -\frac{1}{a} + \frac{r_0 p^2}{2} + \dots$$

Effective Range Expansion

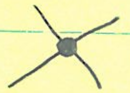
Lets prove it with EFT

$$\mathcal{L} = N^\dagger \left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \dots \right) N - \sum_s \sum_{m=0}^{\infty} C_{2m}^{(s)} O_{2m}^{(s)} + \dots$$

nucleon field
spin-½
isospin-$\frac{1}{2}$

$^{2S+1}L_J$

$(N^\dagger N)^2$ with $2m$ ∂'s



$\sim p^{2m}$

Fermions so

$I = 1$, $(-1)^{S+L}$ = even

$I = 0$, $(-1)^{S+L}$ = odd

[mixing $\overset{in}{L} \to \overset{out}{L'}$ $\quad S=0 \quad L=L'$; $\quad S=1 \quad , |L-L'| = 0, 2$]
conserve J

$$\sum_{s,m} C_{2m}^{(s)} O_{2m}^{(s)} = C_0^{(s)} \left(N^T P_i^{(s)} N\right)^\dagger \left(N^T P_i^{(s)} N\right) - \frac{C_2^{(s)}}{8} \left[\left(N^T P_i^{(s)} N\right)^\dagger \left(N^T P_i^{(s)} \overleftrightarrow{\nabla}^2 N\right) + h.c. \right] + \cdots$$

$\uparrow \; \overleftarrow{\nabla}^2 - 2 \overleftarrow{\nabla} \cdot \overrightarrow{\nabla} + \overrightarrow{\nabla}^2$

$$P_i^{({}^1S_0)} = \frac{1}{\sqrt{8}} (i\sigma^2)(i\tau_2 \tau_i) \quad , \quad P_i^{({}^3S_1)} = \frac{1}{\sqrt{8}} (i\sigma_2 \sigma_i)(i\tau_2)$$

Feyn. Rules

$= -iC_0$

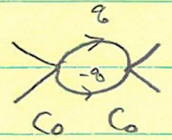
$= -iC_2 \, p^2$ etc


$\equiv \sum_m$

$= -i \sum_m C_{2m} \, p^{2m}$ complete tree level amplitude

Loops

$E=0$



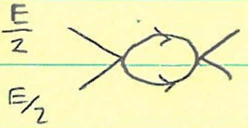
$$= (-iC_0)^2 \int đ^d q \, \frac{i}{(q^0 + i\epsilon)} \, \frac{i}{(-q^0 + i\epsilon)}$$

pinch singularity

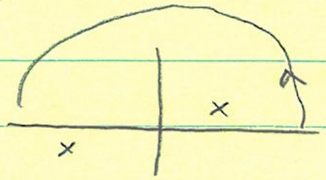
missing something

* Kinetic energy relevant in Q.M. , $\underbrace{E \sim \frac{p^2}{2m}}_{\text{both leading order}}$

2 Heavy particles differs from HQET



$$= (-iC_0)^2 \int đ^d q \, \frac{i \qquad\qquad i}{\left(\frac{E}{2} + q^0 - \frac{\vec{q}^{\,2}}{2m} + i\epsilon\right)\left(\frac{E}{2} - q^0 - \frac{\vec{q}^{\,2}}{2m} + i\epsilon\right)}$$



$$\frac{E}{2} - \frac{\vec{q}^{\,2}}{2m} + i\epsilon = q^0$$

$$= i C_0^2 \int đ^n q \, \frac{M}{(\vec{q}^{\,2} - ME - i\epsilon)} \overset{\text{dim. reg.}}{\downarrow}{=} -i C_0^2 \, \frac{M \sqrt{-ME - i\epsilon}}{4\pi}$$

$$= -i C_0^2 \left(\frac{-iMp}{4\pi} \right)$$

M in numerator ?

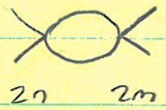
Count m's holding p fixed. $k^0 \sim 1/m$, $t \sim m$

$$\underset{m}{\int d^4x}\; \underset{m^0}{N^\dagger}\,(i\partial_t \underset{\frac{1}{m}}{-\frac{\nabla^2}{2m}})\,\underset{m^0}{N} \Rightarrow N \sim m^0$$

$$\underset{m}{\int d^4x}\; \underset{\underset{1/m}{\uparrow}}{C^{2m}}\; \underset{\underset{m^0}{\uparrow}}{O^{2m}} \quad , \ \underline{\text{any}}\ C_{2m} \sim \frac{1}{m}$$

dimension $[C_{2m}] = -2-2m$, expect $C_{2m} \sim \frac{1}{m\Lambda^{2m+1}}$

$p \ll \Lambda$



$$\int \bar{d}^n q \;\; \vec{q}^{\,2n} \; \frac{m}{(\vec{q}^{\,2} - mE - i\epsilon)} \; \vec{q}^{\,2m}$$

$$= \int \bar{d}^n q \; \frac{(\vec{q}^{\,2} - mE + mE)^{n+m}}{(\vec{q}^{\,2} - mE + i\epsilon)} = (mE)^{n+m} \int \frac{\bar{d}^n q}{\vec{q}^{\,2} - mE}$$

$iA=$

$$= -i \left(\sum_m C_{2m} p^{2m}\right)^K \left(-\frac{imp}{4\pi}\right)^{K-1}$$

sum them , geometric series

$$A = \frac{-\left(\sum_m C_{2m} p^{2m}\right)}{1 + \frac{imp}{4\pi}\left(\sum_m C_{2m} p^{2m}\right)} = \frac{4\pi}{m} \frac{1}{\underbrace{\left(\frac{-4\pi}{m}\frac{1}{\sum_m C^{2m} p^{2m}}\right)}_{p\cot\delta} - ip}$$

$$p\cot\delta = \frac{-1}{\sum_n \hat{C}_{2m} p^{2m}} \qquad \hat{C}_{2m} \equiv \underbrace{\frac{m C_{2m}}{4\pi}}_{\text{no m's here}}$$

S-wave , $l=0$

$$p\cot\delta = \frac{-1}{\hat{C}_0 + \hat{C}_2 p^2 + \cdots} = \frac{-1}{\hat{C}_0} + \frac{\hat{C}_2}{\hat{C}_0^2} p^2 + \cdots$$

$$= \frac{-1}{a} + \frac{r_0}{2} p^2 + \cdots$$

P-wave , $l = 1$ no $\hat{C}_0$

$$p^3 \cot\delta = \frac{-p^2}{\hat{C}_2 p^2 + \hat{C}_4 p^4 + \cdots} = \frac{-1}{\hat{C}_2} + \frac{\hat{C}_4}{\hat{C}_2^2} p^2 + \cdots$$

We've just proven a thm that's quite difficult to prove in N.R. Q.Mechanics

<u>"Matching"</u>

$$\hat{C}_0 = a \qquad C_0 = \frac{4\pi}{m} a$$

$$\hat{C}_2 = \frac{r_0}{2} a^2 \qquad C_2 = \frac{4\pi}{m} \frac{a^2 r_0}{2}$$

Pretend expt tells us $a, r_0, \ldots$

etc

if $a, r_0, \ldots \sim \frac{1}{\Lambda}$ reproduce our $p \ll \Lambda$ power counting $(E \sim p^2)$

But in nature a is large , <u>C_0 is large</u>

$$a^{(^1S_0)} = -23.714 \pm 0.013 \text{ fm} , \quad a^{(^3S_1)} = 5.425 \pm .001 \text{fm}$$

$$1/a = -8.3 \text{ MeV} \ll m_\pi \qquad 1/a = 36 \text{ MeV}$$

$$\boxed{?}$$

a has a <u>Fine Tuning</u> from EFT pt. of view while $r_0 \sim 1/m_\pi , \ldots$ are okay

Want power counting $a p \sim 1$ actually $ap >> 1$

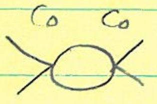
$r_0 p << 1$ etc

ie C_0 relevant

$p >> \frac{1}{a}$ think of $a \to \infty$ $C_0 \to \infty$?

Problem with $\overline{MS}$ scheme (what we used so far)

Try offshell mom. subtraction  $\Big|_{p=i\mu_R} \equiv -i \sum_m C_{2m}^{(\mu_R)} p^{2m}$



$+$


$$= \frac{iM}{4\pi} C_0(\mu_R)^2 (ip + \mu_R)$$

↑ power divergence

$$C_0^{bare} = C_0(\mu_R) + \delta C_0(\mu_R)$$

large anom. dim.

$$\mu_R \frac{\partial}{\partial \mu_R} C_0(\mu_R) = -\mu \frac{\partial}{\partial \mu} \delta C_0 = \frac{M}{4\pi} \mu_R C_0(\mu_R)^2$$

$$C_0(0) = \frac{4\pi a}{m} = C_0^{\overline{MS}}$$

Solution: $C_0(\mu_R) = \frac{-4\pi}{m} \frac{1}{\mu_R - 1/a}$

If $\mu \sim p >> 1/a$ $C_0 \sim \frac{1}{m\mu}$ we swapped $\frac{1}{\Lambda} \to \frac{1}{\mu}$!

↑ relevant coupling now

if $\mu \sim p << 1/a$ $C_0 \sim \frac{a}{m} \sim \frac{1}{m\Lambda}$ for $a \sim \frac{1}{\Lambda}$

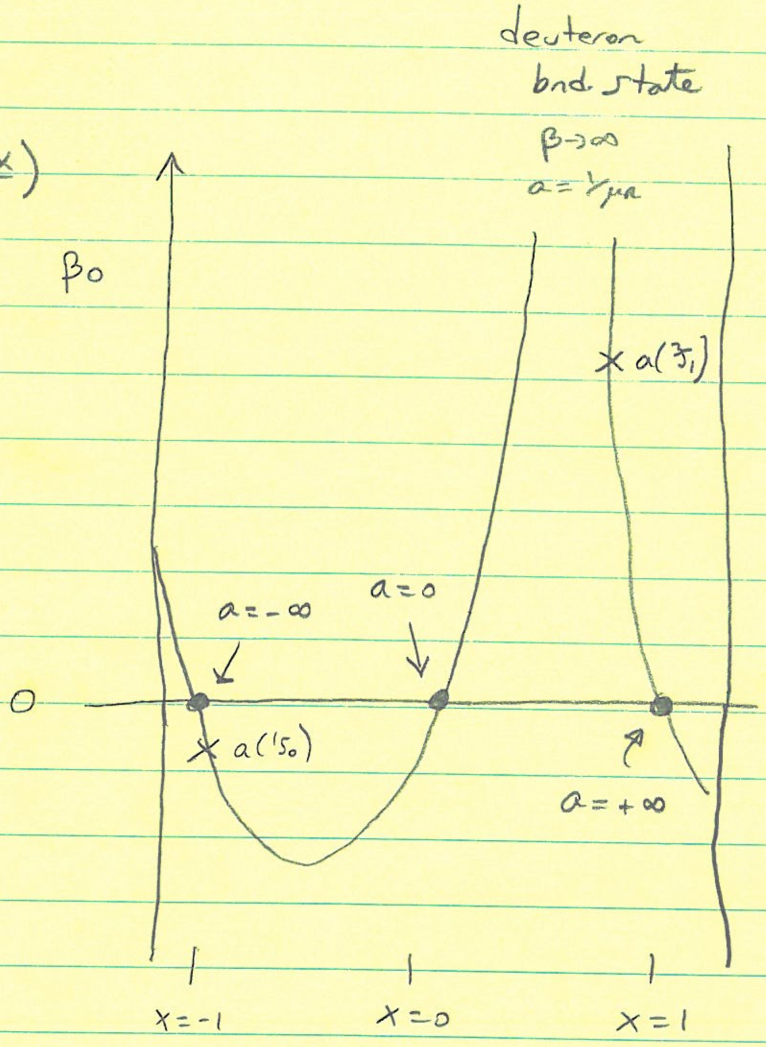
$$\beta_0 \propto \frac{a\,\mu_R}{(1-a\,\mu_R)^2}$$

map to compact interval: $a\,\mu_R = \tan\left(\frac{\pi x}{2}\right)$

plot $\beta_0(x)$ vs x

<u>or</u> $\beta_0(a\mu_R)$ vs a

holding μ_R fixed



Comments

- 3 fixed points

$a = 0$ non-interacting

$a = \pm\infty$ interacting!

Classically "a" measures interaction size

— either so big or so small that its the same on all scales

- $\beta \to \infty$ at $a = 1/\mu_R$, <u>deuteron</u> bnd state.

From $a = +\infty$ side this is true pole in our amplitude

deuteron exists <u>in</u> theory

From $a = 0$ side can never see deuteron in p. theory

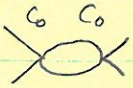
- $a = \pm\infty$ are <u>Conformal</u> fixed pts

with an enhanced <u>SU(4)</u> symmetry

Another Scheme: Power Divergence Subtraction (PDS)

Subtract poles in $d=4$ $\ln(\Lambda)$

$d=3$ Λ



$$= iC_0^2\left(\frac{\mu}{2}\right)^{4-d}\int\frac{d^nq_0\;M}{(\vec{q}_0^{\,2}-ME-i\epsilon)} = \frac{iC_0^2M\,\Gamma\!\left(\frac{3-d}{2}\right)}{(4\pi)^{(d-1)/2}}\left(-ME-i\epsilon\right)^{(d-3)/2}\left(\frac{\mu}{2}\right)^{4-d}$$

$n=d-1$; $\Gamma(1-n/2)$

↑ 2 part of scheme

$$\overset{d=4}{=}\ \frac{iC_0^2\,M\,(-2\sqrt{\pi})}{(4\pi)^{3/2}}(-ip) = +i\left(\frac{+iMp}{4\pi}\right)$$

$$\overset{d=3}{=}\ iC_0^2\,\frac{M}{4\pi}\,\frac{\cancel{2}}{3-d}\,\frac{\mu}{\cancel{2}}$$ ← add c.t. to cancel this

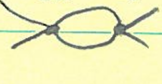
$$\times = \frac{-iM}{4\pi}\,\frac{\mu}{3-d}\,C_0^2$$

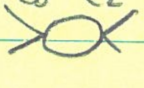
$$+\times\Big|_{d=4} = i\,\frac{M}{4\pi}\,(ip+\mu)\,C_0(\mu)^2$$

$$\mu\,d/d\mu\; C_0(\mu) = \frac{M}{4\pi}\,C_0(\mu)^2$$ same as before

Easier in general than $p=i\mu$ (OS-scheme)

eg $C_2(\mu)$





have p^2

$$\mu\,d/d\mu\; C_2(\mu) = \frac{M\mu}{4\pi}\,2\,C_0(\mu)\,C_2(\mu)$$

$$C_2(0) = \frac{4\pi}{M}\,a^2 r_0$$

Solution: $$C_2(\mu) = +\frac{4\pi}{M}\left(\frac{1}{\mu-1/a}\right)^2\frac{r_0}{2}$$

RGE

Determines enhancement due to "fine tuning" $a\to\infty$

$C_{2k}(\mu)$ $\quad \mu\, d/d\mu\ C_{2k}(\mu) = \frac{m\mu}{4\pi} \sum_{i=0}^{k} \underbrace{C_{2i}\, C_{2(k-i)}}$

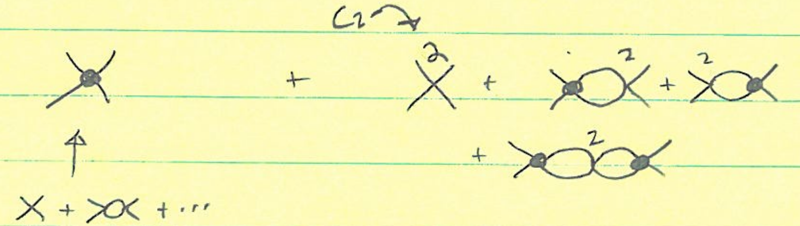
p^{2k} derivatives

Naive $p << 1/a$	Improved $pa >> 1$
$\hat{C}_0 \sim 1/\Lambda$	$\hat{C}_0 \sim 1/\mu$
$\hat{C}_2 \sim 1/\Lambda^3$	$\hat{C}_2 \sim \frac{1}{\mu^2 \Lambda}$ irrelevant still
$\hat{C}_4 \sim 1/\Lambda^5$	$\hat{C}_4 \sim \underset{\uparrow\ \text{no new constant}}{\frac{1}{\mu^3\Lambda^2}} + \underset{\uparrow\ \text{new constant}}{\frac{1}{\mu^2\Lambda^3}}$
	$\hat{C}_{2n} \sim \frac{1}{\mu^{n+1}\Lambda^n} + \cdots$

$$\frac{mA}{4\pi} = \frac{-\sum_n \hat{C}_{2n}\, p^{2n}}{1 + (\mu + ip) \sum_n \hat{C}_{2n}\, p^{2n}}$$

$$= \frac{-\hat{C}_0(\mu)}{1 + \hat{C}_0(\mu)(\mu + ip)} - \frac{\hat{C}_2(\mu)\ p^2}{\left[1 + \hat{C}_0(\mu)(\mu+ip)\right]^2} + \left[\frac{(\hat{C}_2 p^2)^2 (\mu + ip)}{[\cdots]^3} - \frac{\hat{C}_4\, p^4}{[\cdots]^2}\right] + \cdots$$

=



+ ...

organizes geometric series

$$= \frac{-1}{(1/a + ip)} - \frac{\frac{r_0}{2}\, p^2}{(1/a + ip)^2} - \frac{(r_0/2)^2\, p^4}{(1/a + ip)^3} - \frac{(r_1/2\Lambda^2)\, p^4}{(1/a + ip)^2}$$

Conformal Invariance for Non-Rel. Field Theory

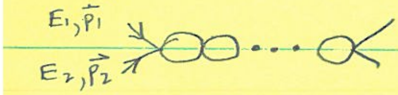
translation	4 gens	
rotation	3 gens	
Galilean Boosts	3 gens	$\vec{x}' = \vec{x} + \vec{c}t \ , \ t' = t$
Scale	1 gen	$\vec{x}' = e^s \vec{x} \ , \ t' = e^{2s} t$
Conformal	1 gen	$\vec{x}' = \dfrac{\vec{x}}{1+ct} \ , \ \dfrac{1}{t'} = \dfrac{1}{t} + c$

Schrödinger Group

$a \to \infty \qquad C_0(\mu) = -\dfrac{4\pi}{m\mu}$, scaling $\mu \to e^s \mu$ the $\mathcal{L}$ is scale inv.

Green's functions are invariant under conformal too

e.g.



$$A^{LO} = \frac{8\pi}{M} \frac{1}{\sqrt{-4M(E_1+E_2) + (\vec{p}_1+\vec{p}_2)^2 - i0}}$$

is scale & conformal invariant

$$\sigma \sim \frac{4\pi}{p^2}$$

SU(4) Spin-Isospin Symmetry = Wigner's SU(4)

$$\delta N = i \, \alpha_{\mu\nu} \, \sigma^\mu \tau^\nu N \qquad \sigma^\mu = (1, \vec{\sigma}) \qquad \tau^\nu = (1, \vec{\tau})$$

remove $U(1)$ = baryon #

$$\mathcal{L} = -\frac{1}{2} \underset{\substack{\uparrow \\ SU(4) \text{ symmetric}}}{C_0^S} (N^\dagger N)^2 - \frac{1}{2} C_0^T (N^\dagger \vec{\sigma} N)^2$$

different basis than before

$$= -\frac{1}{4}\left[C_0^{(^1S_0)} + 3C_0^{(^3S_1)}\right](N^\dagger N)^2 - \frac{1}{8}\left[C_0^{(^3S_1)} - C_0^{(^1S_0)}\right](N^\dagger \vec{\sigma} N)^2$$

so $C_0^S = \frac{-4\pi}{M\mu}$ for $a^{(^1S_0)} \to \infty$, $a^{(^3S_1)} \to \infty$

$$-\frac{M}{4\pi} C_0^T(\mu) = \frac{1}{\mu - 1/a^{(^3S_1)}} - \frac{1}{\mu - 1/a^{(^1S_0)}} = \frac{1/a^{(^3S_1)} - 1/a^{(^1S_0)}}{\left(\mu - 1/a^{(^3S_1)}\right)\left(\mu - 1/a^{(^1S_0)}\right)}$$

so $C_0^T \to 0$ for $a^i \to \infty$

(also for $a^{(^1S_0)} \simeq a^{(^3S_1)}$ but nature is very far from this limit)

Theory has SU(4) symmetry because both a's are large (not because they are nearly equal)

Deuteron $d = np$ bound state , $I = 0$ $^3S_1$

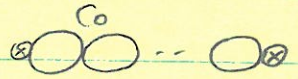
ie $S = 1$

interpolating field $d_i \equiv N^T P_i^{(^3S_1)} N$

Is d in our theory? Look for pole

$$G(\bar{E})\,\delta_{ij} = \int d^4x \; e^{-ip\cdot x} \langle 0|\, T\, d_i^{+}(x)\, d_j(0)\,|0\rangle \overset{?}{\sim} \frac{\delta_{ij}\, iZ(\bar{E})}{\bar{E} + B_d + i0}$$

2 nucleon cm energy $\bar{E} = E - \frac{\bar{p}^2}{4m} + \cdots$



$$G = \frac{\Sigma}{1 + iC_0\,\Sigma}$$

$\Sigma = $ 2PI^{C_0} graphs

$\Sigma^{(1)} =$  $= \frac{-iM}{4\pi}\left(\mu - \sqrt{-M\bar{E} - i0}\right)$

$\bar{E} = -E_B$, $\sqrt{ME_B} \equiv \gamma_B > 0$ so $\sqrt{-M\bar{E} - i0} = -ip = \gamma_B$

$$G \propto \frac{1}{1/a + ip} = \frac{1}{1/a - \gamma_B}$$ pole $\gamma_B = 1/a^{(^3S_1)}$!

and $\frac{1}{a^{(^3S_1)}} = 36$ MeV (>0)

$E_B = \frac{\gamma_B^2}{M} = 1.4$ MeV

$(E_d = 2.2$ MeV$)$

$\frac{1}{a^{(^1S_0)}} < 0$ no bnd state (pole near threshold)

Deuteron E.M. form factor

A nice example of LSZ for bound states

$\langle p', j\,|\, J^\mu_{em}\,|\, p, i\rangle \propto$ $F_C(q^2)$, $F_M(q^2)$, $F_Q(q^2)$

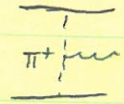
electric magnetic quadrupole

$F_C(0) = 1$ $\frac{e\,F_M(0)}{2M_d} = \mu_M$

$$D_\mu N = \left[\partial_\mu + i e \overset{Q_{em}}{\begin{pmatrix}1&0\\0&0\end{pmatrix}} e A_\mu\right] N \qquad \text{etc.}$$

Also $\mathcal{L} = e L_2(\mu) \left[N^T P_i N\right]^\dagger \left(N^T P_i \vec{\sigma}\cdot\vec{B} N\right) + \text{h.c.}$

comes from eg



$$L_2(\mu) \sim \frac{1}{M\mu^2\Lambda}$$

LSZ

$$\langle p', j | J^\mu_{em} | p, i\rangle = Z \left[G^{-1}(\bar{E})\, G^{-1}(\bar{E}')\, G^\mu_{ij}(\bar{E}, \bar{E}'; q)\right]\Big|_{\bar{E},\bar{E}' \to -B}$$

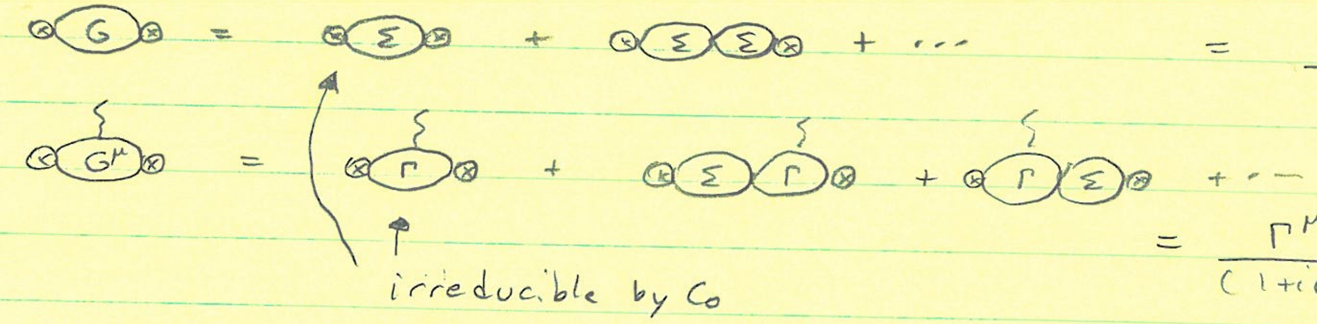
↑ bound-state Z-factor

↑ ↑ truncation by 2-pt functions

↑ 3-pt function

↑ on shell

$$G^\mu_{ij} = \int d^4x\, d^4y\, e^{-ip\cdot x} e^{ip'\cdot y} \langle 0 | T\, d_i^\dagger(x)\, J^\mu_{em}(0)\, d_j(y) | 0\rangle$$



$$= \frac{\Sigma}{1 + iC_0\Sigma}$$

$$= \frac{\Gamma^\mu}{(1 + iC_0\Sigma)(1 + iC_0\Sigma)}$$

$$G = \frac{i Z(\bar{E})}{\bar{E} + B + i\epsilon}, \qquad Z = Z(-B) = -i \left[\frac{d G^{-1}(\bar{E})}{d\bar{E}}\right]^{-1}\Bigg|_{\bar{E} = -B}$$

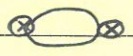
$$= \ldots = \frac{i\,\Sigma^2}{d\Sigma/d\bar{E}}\Bigg|_{\bar{E} = -B}$$

All together:

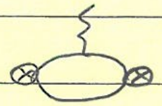
$$\langle p', j | J^\mu_{em} | p, i\rangle = i \frac{\Gamma^\mu_{ij}(\bar{E}, \bar{E}'; q)}{d\Sigma/d\bar{E}}\Bigg|_{\bar{E},\bar{E}' = -B}$$

LO J^0_{em} for electric

$$\left.\frac{d\Sigma^{(1)}}{dE}\right|_{E=-B} = \frac{-i\,M^2}{8\pi\gamma_B}$$

from 

$$\Gamma^{(-1)}_{ij} = -e\,\frac{M^2}{2\pi q}\tan^{-1}\left(\frac{q}{4\gamma_B}\right)\delta_{ij}$$



NLO need $\Sigma^{(2)}$, $\Gamma^{(0)}$

↑ if we left out $\Sigma^{(2)}$ we would mess up $F_C(0)=1$, charge of deuteron

Other Processes : $np \to d\gamma$ BBN N^4LO 1%

$(\gamma d \to np)$

$\nu d \to ppe^-$ CC process at SNO

$\nu d \to pn\nu$ NC

eg. $NN \to NN$ + axion

$E_{axion} \sim E_{nucleon}$,

implements multipole expansion, $\vec{k}_{axion} \ll \vec{P}_{nucleon}$

$$\mathcal{L}_{int} = g_0\,(\nabla^j \chi^0)\Big|_{\vec{x}=0}\underbrace{N^\dagger\sigma^j N}_{\to Q_{j0}} + g_1\,(\nabla^j \chi^0)\Big|_{\vec{x}=0}\underbrace{N^\dagger\sigma^j\tau^3 N}_{\to Q_{j3}}$$

$Q_{\mu\nu} = \int d^3x\, N^\dagger \sigma_\mu \tau_\nu N$ charges of SU(4) symmetry

∴ time independent, axion has 0 energy, ∴ no scattering

$NN(^1S_0) \to NN(^1S_0)\,X^0$ vanishes by ang. mom. (X^0 in p-wave)

$NN(^3S_1) \to NN(^3S_1)\,X^0$ vanishes for any a's (Q_{j0} is conserved spin)

$NN(^1S_0) \to NN(^3S_1)\,X^0$ vanishes for $a^i \to \infty$

$$A \propto \vec{k}\cdot\vec{E}(^3S_1)\left(\frac{1}{a^{(^1S_0)}} - \frac{1}{a^{(^3S_1)}}\right)\frac{1}{1/a^{(^1S_0)}+ip}\,\frac{1}{1/a^{^3S_1}+ip}$$

-skip- Homework

4 body $\mathcal{L}_4 = E_0 (N^+N)^4$ only 4-states so one coupling

3-body $\mathcal{L}_3 = -\frac{D_0}{6} (N^+N)^3$ only 1 operator so also symmetric

Count operators with SU(4) for 3-body:

$$\left.\begin{array}{ll} NNN & \sim \bar{4} \\ N^+N^+N^+ & \sim \underset{\uparrow}{4} \end{array}\right\} \quad 1 \oplus \underset{\uparrow}{15}$$

antisymmetric

no spin or isospin singlet under SU(2) subgroup

2-body

$$\left.\begin{array}{ll} NN & \sim 6 \\ N^+N^+ & \sim \bar{6} \end{array}\right\} \quad \underset{\uparrow}{1} \oplus 15 \oplus \underset{\uparrow}{20}$$

only these have singlets

Soft-Collinear Effective Theory (SCET)

Lecture Notes
Iain Stewart

Formalism & Applications
[more Refs online as we go along]

Partial Topic List

(i) Intro, Degrees of Freedom, Scales,
expansion of spinors, propagators,
power counting see ②,③
(ii) Construction of $\mathcal{L}_{SCET}$, Currents
Multipole Expn, Labels,
Zero-bin, I.R. divergences see ②,③,⑩
(iii) SCET$_I$, Gauge Symmety ③,④,⑥
Reparameterization Invariance
(iv) Ultrasoft-Collin. Factorization
Hard-Collinear Factorization
Matchg & Runnig for Hard Fn's ④,①,② ③
(v) DIS, how SCET p.c. includes
twist expansion as special case
renormalization with convolutions
(vi) SCET$_{II}$ Soft-Collinear Interactions
use of auxillary Lagrangians
Power Counting formula, Rapidity
Divergences ④,⑦,⑩,⑤ ⑫
(vii) Power Corrections,
including SCET$_{II}$ from SCET$_I$

Refs I used

① hep-ph/0005275 (d.o.f.)
② hep-ph/0011336 (d.o.f, $\mathcal{L}$, W,...)
③ hep-ph/0107001 (hard-collin fact.)
④ hep-ph/0109045 (Gauge Inv. soft-collin)
⑤ hep-ph/0205289 (power counting)
⑥ hep-ph/0204229 (RPI)
⑦ hep-ph/0303156 (Gauge Inv. $O(\lambda^2)$ $\mathcal{L}$)
⑧ hep-ph/0202088 (Hard Scattering)
⑨ hep-ph/0107002 ($B \to D\pi$)
⑩ hep-ph/0605001 (0-bin)
⑪ hep-ph/0211069 (SCET$_I$ $\to$ SCET$_{II}$)
⑫ arXiv: 1202.0814 (rapidity RGE)

Processes: $e^+e^- \to$ jets, $B \to D\pi$, $e^-p \to e^-X$, $pp \to$ Higgs + jets
$B \to \pi \ell \nu$, $\gamma^* \gamma \to \pi^0$, ...

$\boxed{\text{Section 1}}$ Intro, Degrees of Freedom, Coordinates

SCET: an EFT for energetic hadrons $E_H \simeq Q \gg \Lambda_{QCD} \sim M_H$

an EFT for energetic jets $E_J \simeq Q \gg M_J = \sqrt{p_J^2}$

an EFT for massless hard ⇔ collinear ⇔ soft interactions

Why? • "Factorization" Our main probe of short distance physics is hard collisions ($e^+e^- \to$ stuff, $pp \to$ stuff). Disentangling the physics of QCD & other interactions requires a separation of scales → EFT → SCET

• jets, energetic hadrons are very common

eg. Hard Scattering $e^- p \to e^- X$ (DIS), $p\bar{p} \to X \overset{\text{Drell Yan}}{\downarrow} \ell^+\ell^-$, $pp \to HX$

$\gamma^* \gamma \to \pi^0$, $e^+e^- \to$ jets, $e^+e^- \to J/\psi\, X$, ...

jet substructure

eg B-decays $B \to X_s \gamma$, $B \to X_u e \bar{\nu}$, $B \to D\pi$, $B \to \pi \ell \nu$

$B \to \pi\pi$, ...

$M_B = 5.279$ GeV $\gg \Lambda_{QCD}$

• Need to separate perturbative $\alpha_s(Q) \ll 1$ & non-perturbative effects in QCD (eg. hard scattering vs. parton distn's)

• Sum large Sudakov double logs $\sim (\alpha_s \ln^2)^k$

• New EFT tools

Prelude (What Makes SCET different from other EFT's)

- We will have multiple fields for the same particle
  ξ_n = collinear quark field
  q_s = soft " "
- We will integrate out offshell modes but not entire d.o.f. (like HQET)
- SCET has convolutions $\sum_i C_i O_i \rightarrow \int d\omega\, C(\omega)\, O(\omega)$
- power counting parameter $\lambda \ll 1$ is not related to mass dimension of fields
- Wilson Lines $P \exp\left(ig \int ds\, n\cdot A(ns)\right)$ appear everywhere, subtle & interesting gauge symmetry structure
- $1/\epsilon^2$ divergences at 1-loop that require UV counterterm

Degrees of freedom for SCET:

eg 1 $B \to D\pi$ hadrons

in B rest frame $p_\pi^\mu = (2.310\text{ GeV}, 0, 0, -2.306\text{ GeV})$

$\simeq Q\, n^\mu$ to good approx.

where $n^\mu = (1,0,0,-1)$, $n^2 = 0$ light-like vector

$\uparrow$ 0,1,2,3 basis

$Q \gg \Lambda_{QCD}$

Light-Cone Coordinates

Basis vectors $n^\mu, \bar{n}^\mu$

$n^2 = 0, \quad \bar{n}^2 = 0, \quad n\cdot\bar{n} = 2$

vectors $p^\mu = \frac{n^\mu}{2}\, \bar{n}\cdot p + \frac{\bar{n}^\mu}{2}\, n\cdot p + p_\perp^\mu$

Notation

$p^+ \equiv n\cdot p \,, \quad p^- \equiv \bar{n}\cdot p$

$$p^2 = n\cdot p\,\bar{n}\cdot p + p_\perp^2 = p^+p^- + p_\perp^2 = p^+p^- - \vec{p}_\perp^{\,2}$$

metric $g^{\mu\nu} = \frac{n^\mu \bar{n}^\nu}{2} + \frac{\bar{n}^\mu n^\nu}{2} + g_\perp^{\mu\nu}$

epsilon $\epsilon_\perp^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\, \frac{\bar{n}_\alpha n_\beta}{2}$

- $n^2 = 0$ requires complementory vector $\bar{n}^\mu$ for decomposition (dual vector for orthogonality)
- choice $n^\mu = (1,0,0,-1)$, $\bar{n}^\mu = (1,\underbrace{0,0}_{\perp},1)$ works

but other choices do too [eg $n = (1,0,0,-1)$, $\bar{n} = (3,2,2,1)$] (more later)

Constituent Quark & Gluons:

In $B \to D\pi$ the B, D are soft $E_H \sim M_H$, use HQET for their constituents. quark & gluons with $p^\mu \sim \Lambda$

Pion is "collinear" $E_\pi \gg M_\pi$, is highly boosted

- In rest frame $\textcircled{\pi}$ has quark & gluon constituents $p^\mu \sim (\overset{+}{\Lambda}, \overset{-}{\Lambda}, \overset{\perp}{\Lambda})$
- boost along $\hat{z}$, $\kappa \gg 1$ $\quad \leftarrow \textcircled{\pi}$ has constituents $p^\mu \sim \underbrace{\left(\frac{\Lambda^2}{Q}, Q, \Lambda\right)}_{\text{relative scaling}}$

  $p^- \to \kappa p^-$, $p^+ \to \frac{p^+}{\kappa}$

  $p_\perp \to p_\perp$

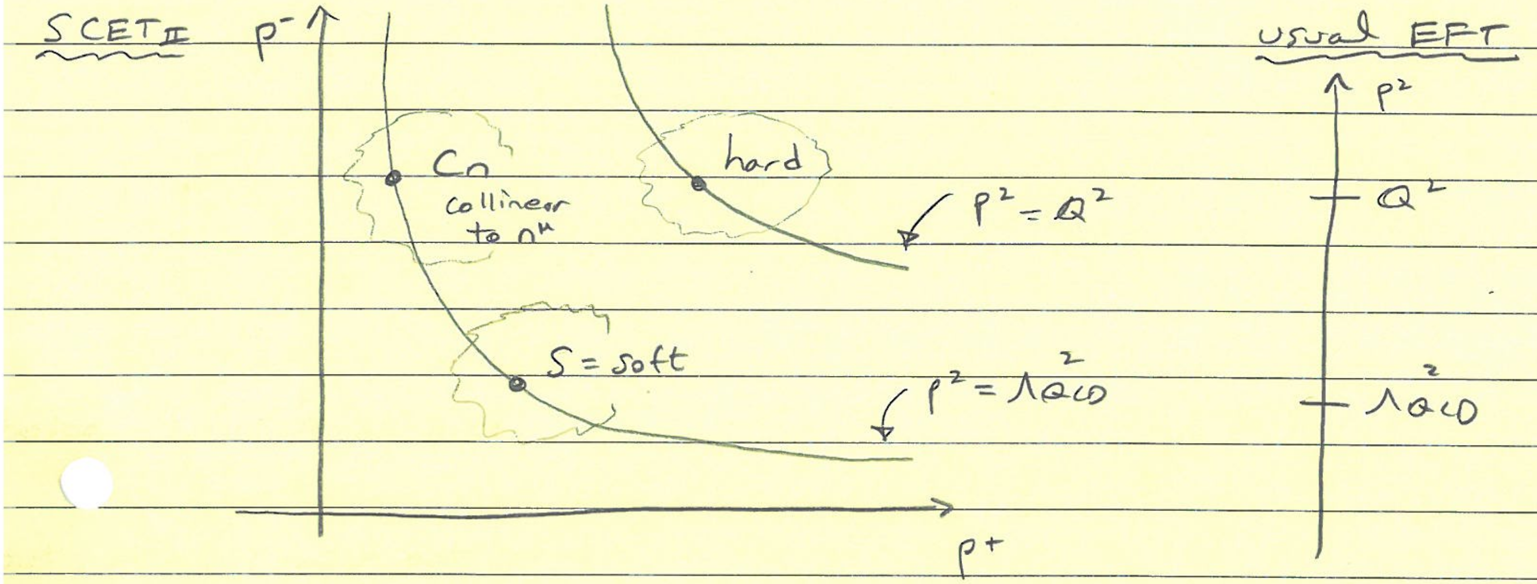
$p^- \gg p_\perp \gg p^+$ defines collinear

fluctuations about $(0,Q,0) = p_\pi^\mu$

Generically $(p^+, p^-, p^\perp) \sim Q(\lambda^2, 1, \lambda)$ is collinear

where $\lambda \ll 1$ is small parameter (our eg. has $\lambda = \frac{\Lambda}{Q}$)

Degrees of freedom occupy momentum regions in SCET

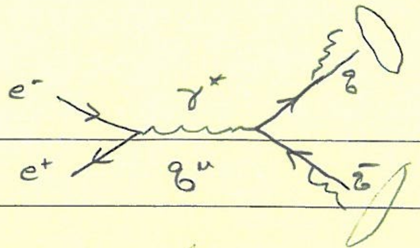


Comments

- $p^2 = p^+p^- - \vec{p}_\perp^{\,2}$, enough to characterize d.o.f in p^+-p^- plane since $\vec{p}_\perp^{\,2} \sim p^+p^-$ for modes that can go on-shell

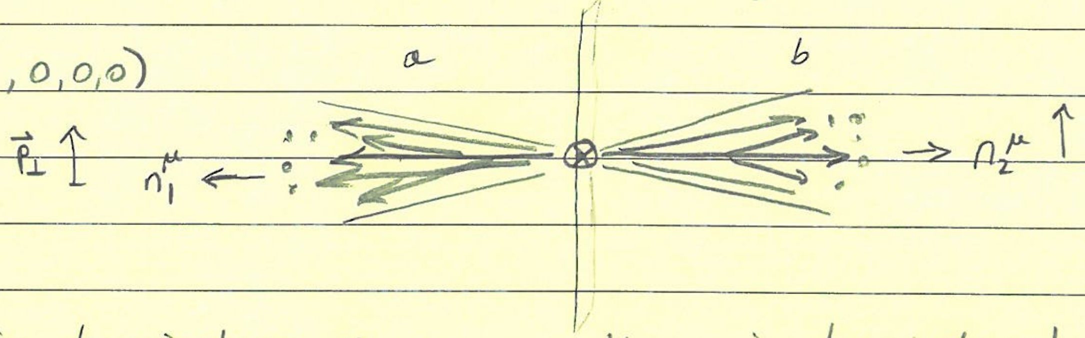
- boundary of regions would be a cutoff in Wilsonian EFT, but we'll use dim. reg to preserve symmetries. Still the correct picture but region overlaps a bit more tricky

- the theory with c_n & s d.o.f. is known as SCET$_{II}$ & it applies for energetic hadron production eg 2. $e^+e^- \to$ dijets



CM frame $q^\mu = (Q, 0,0,0)$

back-to-back jets



jet of hadrons in hemisphere a , another in hemisphere b

n₁-collinear jet

jet constituents have $p_\perp \sim \Delta \ll p_- \sim Q$

$$(p^+, p^-, p_\perp) \sim \left(\underset{\uparrow}{\frac{\Delta^2}{Q}}, Q, \Delta\right) \sim Q(\lambda^2, 1, \lambda)$$

collinear

fixed by $p^+p^- \sim p_\perp^2$

Jet Mass $\quad m_J^2 = \left(\sum_{i\varepsilon a} p_i^\mu\right)^2 \sim p^-p^+ \sim \Delta^2 \ll Q^2$

(another way to characterize that its a jet)

here $\lambda = \frac{\Delta}{Q} \ll 1$

If $\Delta \sim Q$ we don't have dijets (inclusive sum over many (hadrons in all directions) jets, local OPE region)

$\Delta \sim \Lambda_{QCD}$ we have energetic hadrons , jets are so narrow that all constituents bind into a hadron

n₂-collinear jet $\quad$ take $n_1 = n$

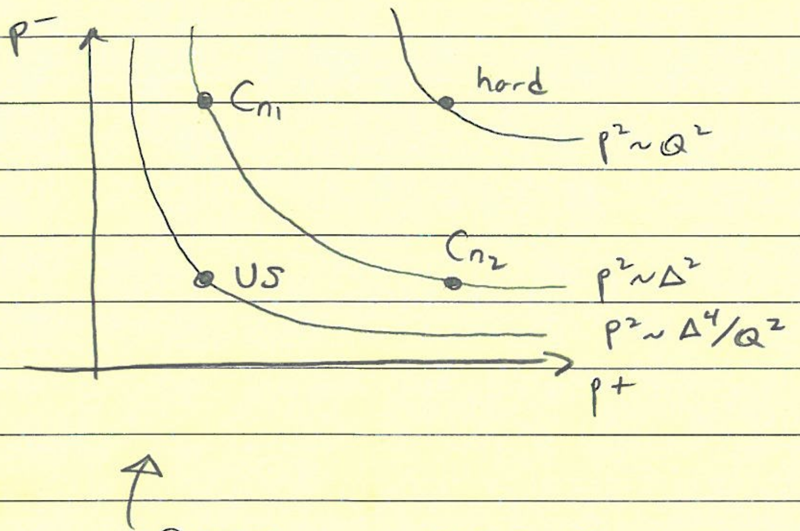
$n_2 = \bar{n}$ for simplicity

just mirror of above with $+ \leftrightarrow -$

Another important d.o.f. are ultrasoft modes "US" that can communicate between jets

$$p^\mu \sim \left(\underset{+}{\frac{\Delta^2}{Q}}, \underset{-}{\frac{\Delta^2}{Q}}, \underset{\perp}{\frac{\Delta^2}{Q}} \right)$$

"communicate" means sharing momenta of a common size



	$(+, -, \perp)$
n-collin	$(\lambda^2, 1, \lambda)\, Q$
$\bar{n}$-collin	$(1, \lambda^2, \lambda)\, Q$
usoft	$(\lambda^2, \lambda^2, \lambda^2)\, Q$

IR degrees of freedom with $p^2 \lesssim Q^2 \lambda^2$

SCET_{I}, EFT for energetic jets

[soft $(\lambda, \lambda, \lambda)\, Q$ in this notation]

Note (Discuss)

(i) multiple modes for IR <-> needed for p.c. <-> multiple fields

(ii) we integrate out modes above a given hyperbola in invariant mass (offshell modes)

(iii) important thing is relative scaling of momenta btwn modes (absolute scaling frame dependent, but relative scaling is frame independent) eg 3 1-jet only?

$b \to s \gamma$ $B \to X_s \gamma$

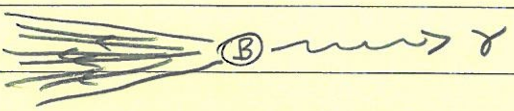
$\uparrow$ ≥ 1 hadron, summed over

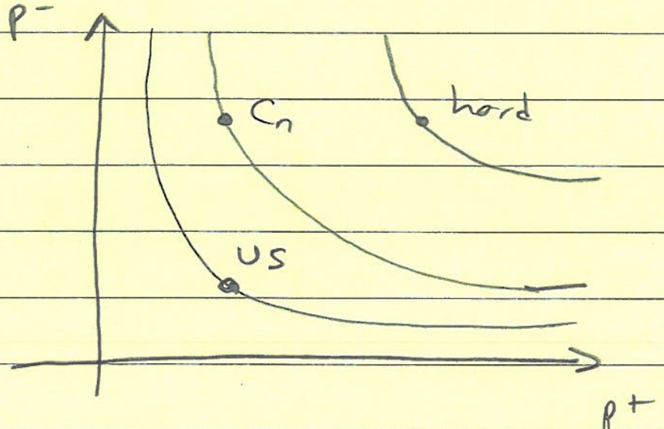
two-body kinematics $E_\gamma = \frac{m_B^2 - m_X^2}{2 m_B} \in \left[0, \frac{m_B^2 - m_{K^*}^2}{2 m_B} \right]$

for $m_X \in [m_B , m_K^*]$

$\Lambda_{QCD}^2 \ll m_X^2 \ll m_B^2 = Q^2$ gives

Jet





natural case

$p_{us}^2 \sim \Lambda_{QCD}^2 \sim \Delta^4 / Q^2$

$\Delta \sim \sqrt{\Lambda_{QCD} Q}$

ultrasoft modes are constituents of B-meson

Collinear Spinors

u_n labelled by direction n
(analog of HQET spinor u_v)

massless QCD spinors
(Dirac Rep.)

$$u(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0}\, u \end{pmatrix} , \quad v(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{p^0}\, v \\ v \end{pmatrix}$$

let $n^\mu = (1,0,0,1)$
$\bar{n}^\mu = (1,0,0,-1)$

expand $\bar{n}\cdot p = p^0 + p^3 \gg p_\perp \gg n\cdot p = p^0 - p^3$

$$\frac{\vec{\sigma}\cdot\vec{p}}{p^0} = \sigma^3$$

$$u_n = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ \sigma^3 u \end{pmatrix} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\} \quad \text{particles}$$

$$v_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma^3 v \\ v \end{pmatrix} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{antiparticles}$$

$$\not{n} = \begin{pmatrix} \mathbb{1} & -\sigma^3 \\ \sigma^3 & -\mathbb{1} \end{pmatrix} \quad \text{so} \quad \boxed{\not{n}\, u_n = \not{n}\, v_n = 0}$$

$$\frac{\not{n}\not{\bar{n}}}{4} = \frac{1}{2} \begin{pmatrix} \mathbb{1} & \sigma^3 \\ \sigma^3 & \mathbb{1} \end{pmatrix} \quad \text{so} \quad \boxed{\frac{\not{n}\not{\bar{n}}}{4} u_n = u_n , \quad \frac{\not{n}\not{\bar{n}}}{4} v_n = v_n}$$

$\uparrow$

Projection Operator

Decompose $\mathbb{1} = \frac{\not{n}\not{\bar{n}}}{4} + \frac{\not{\bar{n}}\not{n}}{4}$

$$\mathbb{1}\, \psi^{QCD} = \xi_n + \varphi_{\bar{n}}$$

[← slightly different from above spinors, more later]

At high energy we produce/annihilate the components ξ_n, not the "small" components $\varphi_{\bar{n}}$

Collinear Propagators $\quad p^2+i0 = \bar{n}\cdot p \, n\cdot p + p_\perp^2 + i0$

$\sim \lambda^0 * \lambda^2 + \lambda * \lambda$ same size

λ suppressed

Fermions $\quad \dfrac{i\not{p}}{p^2+i0} = \dfrac{i\not{n}}{2}\,\dfrac{\bar{n}\cdot p}{p^2+i0} + \cdots$

$$\underset{p}{\longrightarrow} \qquad = \frac{i\not{n}}{2}\,\frac{1}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p} + i0\,\mathrm{sign}(\bar{n}\cdot p)} + \cdots$$

both particles $\bar{n}\cdot p>0$

& antiparticle $\bar{n}\cdot p<0$

from $T\{\xi_n(x), \bar{\xi}_n(0)\}$

Power counting of fields from free kinetic term

$$\mathcal{L} = \int d^4x \quad \bar{\xi}_n \frac{\bar{\not{n}}}{2}\left[\,i n\cdot\partial + \cdots\right]\xi_n$$

$$\lambda^{-4} \quad \lambda^a \quad \left[\lambda^2 + \cdots\right] \quad \lambda^a \quad = \lambda^{2a-2}$$

set $\mathcal{L} \sim \lambda^0$, normalize kinetic term so no λ's

then $\boxed{\xi_n \sim \lambda}$

Note: mass dimension $[\xi_n] = 3/2$

λ dimension $[\xi_n]^\lambda = 1$

$\boxed{\text{Collinear Gluons}}$

consider general covariant gauge

$$\int d^4x \; e^{ik\cdot x} \langle 0 | T A_n^\mu(x) A_n^\nu(0) | 0 \rangle = \frac{-i}{k^2}\left(g^{\mu\nu} - \overset{\text{gauge param.}}{\tau} \frac{k^\mu k^\nu}{k^2} \right)$$

as above $k^2 = k^+ k^- + k_\perp^2 \sim \lambda^2$, no expansion

Also $g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$ has two terms of same size

eg. $g_\perp^{\mu\nu} \sim \lambda^0 \sim \frac{k_\perp^\mu k_\perp^\nu}{k^2} \sim \frac{\lambda^2}{\lambda^2}$, $g^{+-} \sim \lambda^0 \sim \frac{k^+ k^-}{k^2} \sim \frac{\lambda^2 \lambda^0}{\lambda^2}$

dot $n_\mu n_\nu$: $g^{++} = 0$, $\frac{(n\cdot k)^2}{k^2} \sim \frac{\lambda^4}{\lambda^2} = \lambda^2$

$d^4x \sim \lambda^{-4} \sim \frac{1}{(k^2)^2}$ so $\underline{A_n^\mu \sim k^\mu} \sim \overset{+,-,\perp}{(\lambda^2, 1, \lambda)}$

$A_n^\mu = (A_n^+, A_n^-, A_n^\perp) \sim (\lambda^2, 1, \lambda)$

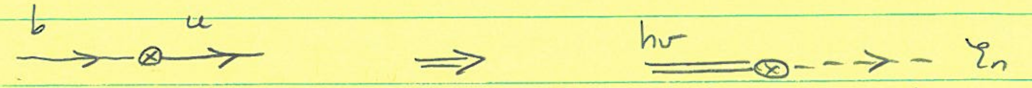
ie $k^\mu + gA^\mu = iD^\mu$ homogeneous covariant derivative

Note: $A_n^- \sim \lambda^0$ no suppression to add A_n^- fields

To see how this has an impact, consider an external weak $\boxed{\text{current}}$

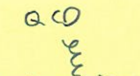
eg. $b \to u e \bar{\nu}$ QCD $J = \bar{u} \Gamma b$ $\Gamma = \gamma^\mu (1-\gamma_5)$

consider heavy b (HQET), energetic u (SCET)

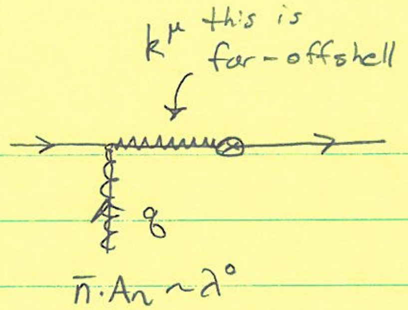


dashed collinear quark

$J_{eff} = \bar{\xi}_n \Gamma h_v$



$= igT^A\gamma^\mu$ sign convention

Consider 

$$k^\mu = m_b v^\mu + \frac{n^\mu}{2}\,\bar{n}\cdot q + \cdots$$

$$k^2 = m_b^2 + n\cdot v\, m_b\, \bar{n}\cdot q + \cdots$$

$$k^2 - m_b^2 \sim m_b^2 \quad \text{for} \quad \bar{n}\cdot q \sim \lambda^0 \sim m_b$$

no power suppression for these gluons

Find

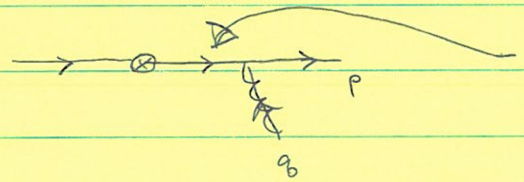
$$A_{n\mu}\,\bar{\xi}_n \Gamma \frac{i(\slashed{k}+m_b)}{k^2-m_b^2}\, igT^A\gamma^\mu h_v = -g A_n^{\mu A}\bar{\xi}_n\Gamma \frac{\left[\cancel{m_b}(1+\slashed{v}) + \cancelto{0=\slashed{n}^2}{\frac{\slashed{n}}{2}\bar{n}\cdot q}\right]}{n\cdot v\,\cancel{m_b}\,\bar{n}\cdot q}\frac{\slashed{n}}{2}\bar{n}_\mu T^A h_v$$

$$+ \cdots$$

$$= \frac{-g\,\bar{n}\cdot A^A}{\bar{n}\cdot q}\;\bar{\xi}_n \Gamma T^A\left[\frac{+\cancelto{0}{\frac{\slashed{n}}{2}(1-\slashed{v})} + \cancel{n\cdot v}}{\cancel{n\cdot v}}\right] h_v \qquad \slashed{v}h_v = h_v$$

$$= \frac{-g}{\bar{n}\cdot q}\;\bar{\xi}_n\Gamma\,\bar{n}\cdot A\, h_v =$$ 

same order in λ.

Consider 

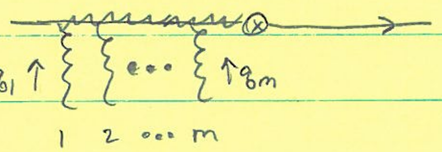
$p-q$ = collinear for

p & q both collinear,

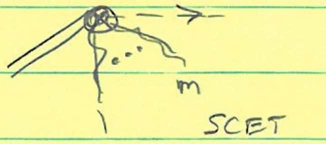
so <u>not offshell</u>

⇔ Lagrangian interaction

QCD graph

Consider 

+ crossed gluon graphs →



SCET graph

$$= (-g)^m \;\Gamma \sum_{\substack{\text{perms}\\ \{1,\ldots,m\}}} \frac{\bar{n}^{\mu_m}T^{A_m}\cdots\bar{n}^{\mu_1}T^{A_1}}{[\bar{n}\cdot q_1]\,[\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot\sum_{i=1}^{m} q_i]}$$

when we write fields for external lines we must be a bit careful

Since SCET vertex is localized with m identical fields

$$\rightarrow \frac{(\bar{n}\cdot A)^m}{m!}$$

Complete tree level matching is

$$\bar{u}\,\Gamma\, b \;\rightarrow\; \bar{\xi}_n W\, \Gamma\, h_v$$

where $$W = \sum_k \sum_{\text{perms}} \frac{(-g)^k}{k!} \left(\frac{\bar{n}\cdot A_{q_1} \cdots \bar{n}\cdot A_{q_k}}{[\bar{n}\cdot q_1][\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot \sum_{i=1}^{k} q_i]} \right)$$

is momentum space Wilson Line

position space Wilson line is

$$W(0,-\infty) = \underset{\uparrow}{P} \exp\left(ig \int_{-\infty}^{0} ds\; \bar{n}\cdot A_n(\bar{n}s) \right)$$

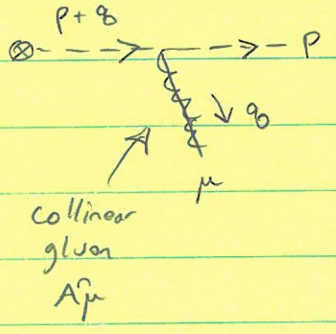
path ordering puts fields with larger arguement to the left $\bar{n}\cdot A_n(\bar{n}s)\, \bar{n}\cdot A_n(\bar{n}s')$

for $s > s'$

Effectively: $\bar{n}\cdot A$ field gets traded for $W[\bar{n}\cdot A]$

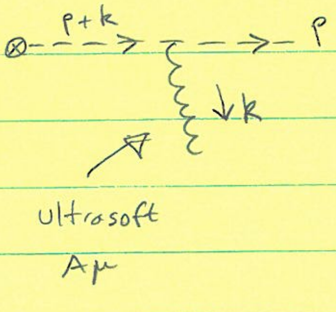
Consider SCET_I, collinear & usoft

$(\lambda^2, 1, \lambda)$ $(\lambda^2, \lambda^2, \lambda^2)$



$$\text{propagator} = \frac{\bar{n}\cdot(q+p)}{n\cdot(q+p)\,\bar{n}\cdot(q+p) + (q_\perp + p_\perp)^2 + i0}$$

$q^\mu \sim p^\mu$ so nothing dropped in denominator



here $k^\mu \sim \lambda^2$

$\bar{n}\cdot k \ll \bar{n}\cdot p \sim \lambda^0$

$k_\perp^\mu \ll p_\perp^\mu \sim \lambda$

$n\cdot k \sim n\cdot p$

$$\text{propogator} = \frac{\bar{n}\cdot p}{n\cdot(k+p)\,\bar{n}\cdot p + p_\perp^2 + i0} + \cdots$$

higher order terms

SCET Collinear Quark Lagrangian

Should:
- yield propagator & have interactions with both collinear gluons and usoft gluons
- have both quarks and antiquarks
- must yield LO propagator for different situations (without requiring an additional expansion)
- should be setup so we do not have to revist LO result when formulating power corrections

[we'll meet & resolve some technical hurtles along the way]

Step 1: Start with $\mathcal{L}_{QCD} = \bar\psi \, i \not{D} \, \psi$

Write $\psi = \xi_n + \varphi_{\bar n}$ where $\xi_n = \frac{\not{n}\not{\bar n}}{4}\psi$, $\varphi_{\bar n} = \frac{\not{\bar n}\not{n}}{4}\psi$ $\qquad \frac{\not{n}\not{\bar n}}{4}\xi_n = \xi_n$, $\frac{\not{\bar n}\not{n}}{4}\varphi_{\bar n} = \varphi_{\bar n}$

$$\mathcal{L} = (\bar\varphi_{\bar n} + \bar\xi_n)\left(i\frac{\not{\bar n}}{2} n\cdot D + i\frac{\not{n}}{2}\bar n\cdot D + i\not{D}_\perp\right)(\xi_n + \varphi_{\bar n})$$

$$= \bar\xi_n \frac{\not{\bar n}}{2} i n\cdot D \, \xi_n + \bar\varphi_{\bar n} \, i\not{D}_\perp \xi_n + \bar\xi_n \, i\not{D}_\perp \varphi_{\bar n} + \bar\varphi_{\bar n}\frac{\not{n}}{2} i\bar n\cdot D \, \varphi_{\bar n} \qquad \circledast$$

other terms are zero eg. $\bar\xi_n \, i\not{D}_\perp \xi_n = \bar\xi_n \, i\not{D}_\perp \frac{\not{n}\not{\bar n}}{4}\xi_n = \underbrace{\bar\xi_n \frac{\not{n}\not{\bar n}}{4}}_{0} i\not{D}_\perp \xi_n$

So far this $\mathcal{L}$ is just QCD written in terms of ξ_n, $\varphi_{\bar n}$ vars.

- $\varphi_{\bar n}$ corresponds to subleading spinor components. We will not consider a source for $\varphi_{\bar n}$ in the path integral
  $\therefore$ we can do path integral over $\varphi_{\bar n}$

e.o.m. $\frac{\delta}{\delta \bar{\varphi}_{\bar{n}}}$: $\frac{\not{n}}{2}\, i\bar{n}\cdot D\, \varphi_{\bar{n}} + i\not{D}_\perp \xi_n = 0$

$$i\bar{n}\cdot D\, \varphi_{\bar{n}} + \frac{\not{\bar{n}}}{2}\, i\not{D}_\perp \xi_n = 0$$

$$\varphi_{\bar{n}} = \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \frac{\not{\bar{n}}}{2}\xi_n \quad , \quad \psi = \left(1 + \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \frac{\not{\bar{n}}}{2}\right)\xi_n$$

Plug back into ⊛: already used/satisfied 2nd & 4th terms, 1st & 3rd give

$$\mathcal{L} = \bar{\xi}_n \left(in\cdot D + i\not{D}_\perp \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp \right) \frac{\not{\bar{n}}}{2}\, \xi_n \qquad \boxed{**}$$

← insert (107.5) Aside

We're not yet done. We still need to:

② separate collinear & usoft gauge fields

③ " " " " momenta

④ expand and put pieces together

Step ②: $A_n^\mu \sim (\lambda^2, 1, \lambda) \sim p_n^\mu$, $A_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2) \sim k_{us}^\mu$

write $A^\mu = A_n^\mu + A_{us}^\mu + \cdots$

like a classical background field to ξ_n, A_n^μ

$p_{us}^2 \sim Q^2\lambda^4 \ll p_c^2 \sim Q^2\lambda^2$

← long wavelength

there are some more terms that will matter for power corrections (& are fixed by gauge invariance). Ignore them for now.

Power counting

$$\left.\begin{array}{l} \bar{n}\cdot A_n \sim \lambda^0 \gg \bar{n}\cdot A_{us} \\ A_{\perp n}^\mu \sim \lambda \gg A_{us}^\perp \\ n\cdot A_n \sim \lambda^2 \sim n\cdot A_{us} \end{array}\right\}$$

so $A_{us}^\perp$ & $\bar{n}\cdot A_{us}$ can be dropped at leading order

107.5

What does $\frac{1}{i\bar{n}\cdot\partial}$ mean ?

Its the analog of how you define $\frac{1}{\hat{r}}$ in quantum mechanics, you use the eigenbasis:

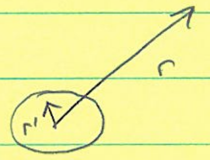
$$\frac{1}{i\bar{n}\cdot\partial}\,\phi(x) = \frac{1}{i\bar{n}\cdot\partial}\int d^4p\, e^{-ip\cdot x}\,\phi(p) = \int d^4p\, e^{-ip\cdot x}\,\frac{1}{\bar{n}\cdot p}\,\phi(p)$$

Step ③ We had a λ-expansion for a propagator carrying collinear & soft momenta

$$\frac{1}{(p_n+k_{us})^2} = \underset{\sim\lambda^{-2}}{\frac{1}{p_n^-(p_n^+ + k_{us}^+) + p_{n\perp}^2}} - \underset{\sim\lambda^{-1}}{\frac{2\,k_{us}^\perp\cdot p_n^\perp}{\left[p_n^-(p_n^+ + k_{us}^+) + p_{n\perp}^2\right]^2}} + \dots$$

There must be Feyn. Rules in SCET to reproduce 2nd term too, so when we expand $k_{us}^\perp \ll p_n^\perp$, $k_{us}^- \ll p_n^-$ we can't just ignore $k_{us}^{\perp,-}$. We need a systematic (gauge invariant) <u>multipole</u> <u>expansion</u>.

Recall E&M



$r' \ll r$

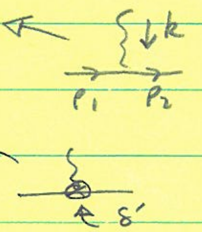
$$V(r) = \frac{1}{r}\int d^3r'\,\rho + \frac{1}{r^2}\int r'\cos\theta\,\rho\,d^3r' + \dots$$

<u>Position Space</u> (1-dim), consider

- $\int dx\ \bar\psi(x)\,A(0)\,\psi(x) = \int dx \int dp_1\,dp_2\,dk\ e^{ip_1\cdot x}\,e^{-ik\cdot 0}\,e^{-ip_2\cdot x}\,\bar\psi(p_1)A(k)\psi(p_2)$

  $= \int dp_1\,dp_2\,dk\ \delta(p_1-p_2)\ \bar\psi(p_1)\,A(k)\,\psi(p_2)$
- $\int dx\ \bar\psi(x)\ x\cdot i\partial\,A(0)\,\psi(x)$

  $= \int dp_1\,dp_2\,dk\ \delta'(p_1-p_2)\ k\ \bar\psi(p_1)A(k)\,\psi(p_2)$



k gets dropped [momentum not conserved]

must int. by parts...

We will carry out the multipole expansion in <u>momentum</u> <u>space</u>

- more directly get mom. space Feyn. Rules
- simplifies formulation of gauge transformations
- mom. expansion sits in propagators rather than vertices

eg. $\dfrac{k_{us}^\perp\cdot p_n^\perp}{[\cdots]^2}$ ~

propagator insertion

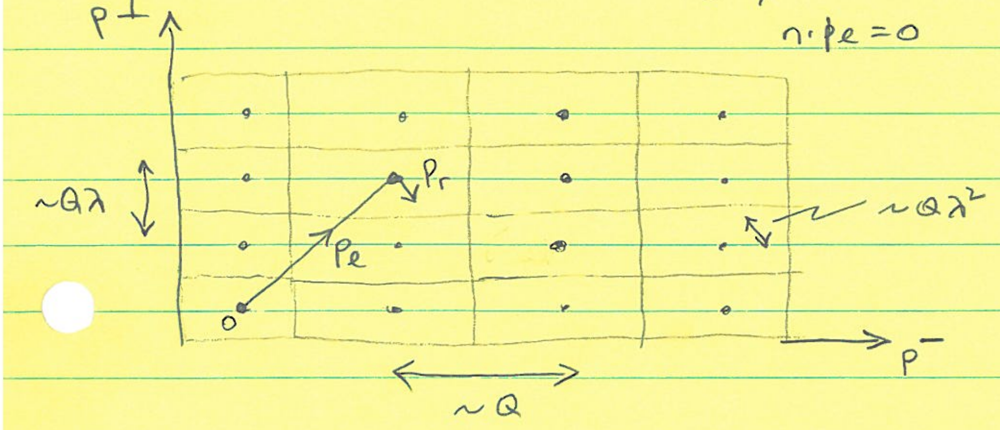
Call $\xi_n(x)$ field from

Eq. (**) pg. (107) $\to \hat{\xi}_n(x)$. [Consider only quark part, $a^{s}_{\vec{p}}$, to start.]

Let $\tilde{\xi}_n(p) = \int d^4x \; e^{ip\cdot x} \; \hat{\xi}_n(x)$

$$\begin{aligned} &\text{Analogy} \quad \text{HQET:} \quad p^\mu = \overset{\text{label}}{m v^\mu} + \overset{\text{residual}}{k^\mu} \\ &\qquad\qquad\;\; \text{SCET:} \quad p^\mu = \underset{\substack{\uparrow \\ (p_\ell^-,\, p_\ell^\perp) \sim (1,\lambda) \\ \text{only}}}{p_\ell^\mu} + \underset{\substack{\uparrow \\ p_r^\mu \sim (\lambda^2,\lambda^2,\lambda^2)}}{p_r^\mu} \end{aligned}$$

$n\cdot p_\ell = 0$



p_ℓ^μ discrete grid points

p_r^μ continuous

$p^\mu = p_\ell^\mu + p_r^\mu$ unique for given grid

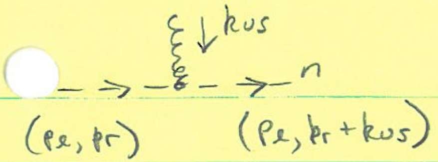
$\int d^4p = \sum_{p_\ell \neq 0} \int d^4 p_r$ for collinear p
[$p_\ell = 0$ is not collinear]

$\int d^4 p = \int d^4 p_r$ for usoft p [usoft has $p_\ell = 0$]

Write: $\tilde{\xi}_n(p) \to \tilde{\xi}_{n,p_\ell}(p_r)$

Note: We have separate conservation of label & residual momenta

$$\int d^4x \; e^{i(p_\ell - q_\ell)\cdot x} \; e^{i(p_r - q_r)\cdot x} = \delta_{p_\ell, q_\ell} \; \delta^4(p_r - q_r) \; (2\pi)^4$$



"non-conservation" of momenta is replaced by two separate conservations where some fields don't carry label momenta.

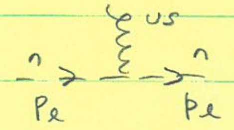
Final Step

since all fields carry residual momenta the conservation law just corresponds to locality with respect to Fourier transform pr → x

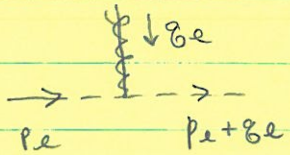
$$\xi_{n,p_\ell}(x) = \int \frac{d^4 p_r}{(2\pi)^4}\, e^{-i p_r \cdot x}\, \tilde{\xi}_{n,p_\ell}(p_r)$$

↑ build action from these fields

- usoft gluons leave labels conserved



- collinear gluons change labels



- label n for collinear direction always preserved by usoft & collinear gluons only a hard interaction can couple fields with different n's eg

back-to-back production by hard γ*

All together

$$\hat{\xi}_n(x) = \int d^4p\, e^{-ip\cdot x}\, \tilde{\xi}_n(p) = \sum_{p_\ell \neq 0} \int d^4 p_r\, e^{-ip_\ell \cdot x}\, e^{-ip_r\cdot x}\, \tilde{\xi}_{n,p_\ell}(p_r)$$

$$= \sum_{p_\ell \neq 0} e^{-ip_\ell \cdot x}\, \xi_{n,p_\ell}(x)$$

Define <u>two</u> derivative operators:

$$i\partial^x_\mu \xi_{n,p_\ell}(x) \sim \lambda^2 \xi_{n,p_\ell}(x) \qquad \text{residual}$$

$$\mathcal{P}_\mu \xi_{n,p_\ell}(x) \equiv p_\ell^\mu \xi_{n,p_\ell}(x) \sim (\overset{+}{0},\overset{-}{1},\overset{\perp}{\lambda})\, \xi_{n,p_\ell}(x)$$

$$\Rightarrow \quad i\bar{n}\cdot\partial \ll \bar{\mathcal{P}} = \bar{n}\cdot\mathcal{P}\,, \qquad i\partial_\perp^\mu \ll \mathcal{P}_\perp^\mu$$

implements multipole expansion

similar structure to expansion for gauge fields → gauge symmetry easier

Notation is friendly:

$$\hat{\xi}_n(x) = \sum_{p_\ell\neq 0} e^{-ip_\ell\cdot x}\xi_{n,p_\ell}(x) = e^{-i\mathcal{P}\cdot x}\sum_{p_\ell\neq 0}\xi_{n,p_\ell}(x)$$

$$\equiv e^{-i\mathcal{P}\cdot x}\underbrace{\xi_n(x)}_{\sum_{p_\ell\neq 0}\xi_{n,p_\ell}(x)}$$

← suppress labels if we don't need them explicitly

Field products

$$\hat{\xi}_n(x)\,\hat{\xi}_n(x) = e^{-i\mathcal{P}\cdot x}\,\xi_n(x)\,\xi_n(x)$$

↑ acts on both fields & just gives <u>label</u> conservation

Last Step is to consider anti-quarks & gluons

Mode Exp^n

$$\psi(x) = \int d^4p \;\; \delta(p^2)\,\theta(p^0)\,\left[\,u(p)\,a(p)\,e^{-ip\cdot x} + v(p)\,b^\dagger(p)\,e^{ip\cdot x}\right]$$

$$= \psi^+ + \psi^-$$

QCD

Write
$$\left.\begin{aligned}\psi^+(x) &= \sum_{p_\ell\neq 0} e^{-ip_\ell\cdot x}\,\xi^+_{n,p_\ell}(x)\\ \psi^-(x) &= \sum_{p_\ell\neq 0} e^{ip_\ell\cdot x}\,\xi^-_{n,p_\ell}(x)\end{aligned}\right\}$$
both have $\theta(p^0) = \theta(\bar{n}\cdot p)$

$\not{n}\,\xi^\pm_{n,p_\ell} = 0$

Define $\xi_{n,p_\ell}(x) \equiv \xi^+_{n,p_\ell}(x) + \xi^-_{n,-p_\ell}(x)$ any p_ℓ signs

$\bar{n}\cdot p_\ell > 0$ particles destroy $\bar{\xi}_{n,p_\ell}$ $\bar{n}\cdot p_\ell > 0$ part. create

$\bar{n}\cdot p_\ell < 0$ antiparticles create $\bar{n}\cdot p_\ell < 0$ anti, destroy

p_ℓ carries same sign as mom. flow along fermion # $\xrightarrow[p_\ell]{}$

then $\hat{\xi}_n(x) = e^{-i\mathcal{P}\cdot x}\,\xi_{n,p_\ell}(x)$ as before

Collinear Gluons $A^\mu_{n,q_\ell}(x)$, $\left[A^\mu_{n,q_\ell}(x)\right]^* = A^\mu_{n,-q_\ell}(x)$

$q_\ell > 0$ destroy

$q_\ell < 0$ create $\hat{A}_n(x) = e^{-i\mathcal{P}\cdot x}\,A_n(x)$

$\uparrow \sum_{q_\ell} A_{n,q_\ell}(x)$

$\boxed{\text{General Results}}$

sign on † fields

$$\mathcal{P}^\mu\left(\phi^\dagger_{q_1}\,\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots\right) = \left(p_1^\mu + p_2^\mu + \cdots - q_1^\mu - q_2^\mu - \cdots\right)\left(\phi^\dagger_{q_1}\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots\right)$$

eigenvalue eqtn

$$i\partial^\mu \sum_p e^{-ip\cdot x}\,\phi_{n,p}(x) = \sum_p e^{-ip\cdot x}\left(\mathcal{P}^\mu + i\partial^\mu\right)\phi_{n,p}(x)$$

$$= \underbrace{e^{-i\mathcal{P}\cdot x}}_{}\left(\mathcal{P}^\mu + i\partial^\mu\right)\phi_n(x)$$

later we'll suppress this & recall that labels are conserved

Step ④ Expand $\mathcal{L} = \bar{\hat{\xi}}_n(x)\left[in\cdot D + i\slashed{D}_\perp \frac{1}{i\bar{n}\cdot D}\, i\slashed{D}_\perp \right]\frac{\bar{\slashed{n}}}{2}\,\hat{\xi}_n(x)$

$$iD^\mu = \mathcal{P}^\mu + gA_n^\mu + i\partial^\mu + gA_{us}^\mu + \cdots$$

$$in\cdot D = in\cdot\partial + g\,n\cdot A_n + g\,n\cdot A_{us} \qquad (\text{exact, all} \sim \lambda^2)$$

$$iD_\perp = \underbrace{(\mathcal{P}_\perp + gA_n^\perp)}_{\lambda} + \underbrace{(i\partial_\perp + gA_\perp^{us})}_{\lambda^2} + \cdots$$

$$i\bar{n}\cdot D = \underbrace{(\bar{\mathcal{P}} + g\bar{n}\cdot A_n)}_{\lambda^0} + \underbrace{(i\bar{n}\cdot\partial + g\bar{n}\cdot A_{us})}_{\lambda^2} + \cdots$$

From before $\hat{\xi}_n(x) \sim \lambda \overset{\text{so}}{\sim} \xi_n(x)$

$$d^4x\, \underbrace{e^{-ix\cdot\mathcal{P}}}_{} \sim \lambda^{-4}$$

$\mathcal{O}(1)$ phases implies $x^- \sim 1/p^+$, $x^+ \sim \frac{1}{p^-}$

$x^\perp \sim 1/p_\perp$

<u>Leading Order $\mathcal{L}$ is $\mathcal{O}(\lambda^4)$</u>

$$\mathcal{L}^{(0)}_{\xi\xi} = e^{-ix\cdot\mathcal{P}}\, \bar{\xi}_n \left[in\cdot D + i\slashed{D}_\perp^n \frac{1}{i\bar{n}\cdot D_n}\, i\slashed{D}_\perp^n \right] \frac{\bar{\slashed{n}}}{2}\, \xi_n$$

where $\left.\begin{array}{l} iD_\perp^{n\,\mu} = \mathcal{P}_\perp^\mu + gA_n^{\perp\mu} \\ i\bar{n}\cdot D_n = \bar{\mathcal{P}} + g\bar{n}\cdot A_n \end{array}\right\}$ collinear cov. derivatives

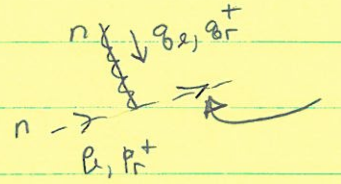
Note:

- both terms $\sim \lambda\cdot\lambda^2\cdot\lambda \sim \lambda^4$
- all fields at x, derivatives $i\partial \sim \lambda^2$, action is explicitly <u>local</u> at $Q\lambda^2$ scale

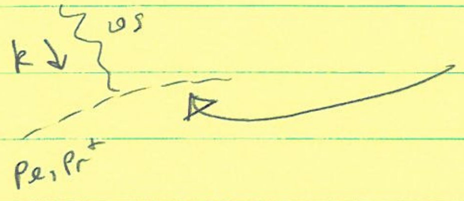
  also local at $Q\lambda$ too ($D_\perp^n$ in numerator, momentum space version of locality)

  <u>only</u> nonlocal at $\sim Q$ from $\frac{1}{\bar{n}\cdot p}$ factors

- Collinear propogators



$$\frac{\bar n\cdot(q_\ell+p_\ell)}{\bar n\cdot(q_\ell+p_\ell)\, n\cdot(q_r+p_r)+(q_\ell^\perp+p_\ell^\perp)^2+i0}$$

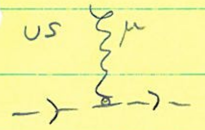


$$\frac{\bar n\cdot p_\ell}{\bar n\cdot p_\ell\; n\cdot(p_r+k)+(p_\ell^\perp)^2+i0}$$

because <u>no</u> $i\bar n\cdot\partial$ or $i\partial^\perp$ in $\mathcal{L}_{\xi\xi}^{(0)}$

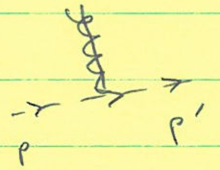
$\mathcal{L}_{\xi\xi}^{(0)}$ knows how to give LO propogator in both situations <u>without</u> further expansions

<u>Feyn. Rules</u>



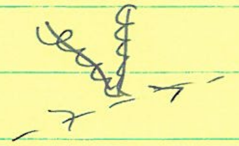
$$= ig\frac{\not{\bar n}}{2}n^\mu T^A$$

only $n\cdot A_{us}$ gluons



$$= igT^A\frac{\not{\bar n}}{2}\left[n^\mu+\frac{\gamma_\perp^\mu\not p_\perp}{\bar n\cdot p}+\frac{\not p_\perp'\gamma_\perp^\mu}{\bar n\cdot p'}-\frac{\not p_\perp'\not p_\perp}{\bar n\cdot p'\,\bar n\cdot p}\bar n^\mu\right]$$

all 4 components couple



$= \cdots$

terms with ≥ 2 gluons also exist but have at most 2 $\perp$ gluons & rest $\bar n\cdot A$

trade $\bar{n}\cdot A_n \Leftrightarrow W$

Wilson Line Eqtns

$$i\bar{n}\cdot D_x\, W(x,-\infty) = 0$$

equivalent def'n to position space W-line is momentum space W_n

$$i\bar{n}\cdot D_n\, W_n = 0$$

$$(\bar{\mathcal{P}} + g\bar{n}\cdot A_n)\, W_n = 0$$

$$i\bar{n}\cdot D_n\, W_n\, \mathcal{O} = W_n\, \bar{\mathcal{P}}\, \mathcal{O}$$

$\uparrow$ some operator

so $\boxed{i\bar{n}\cdot D_n\, W_n = W_n\, \bar{\mathcal{P}}}$ as operator equation

and since $(W(x,-\infty))^{\dagger}\, W(x,-\infty) = 1$

$$W_n^{\dagger}\, W_n = 1$$

we have $\boxed{i\bar{n}\cdot D_n = W_n\, \bar{\mathcal{P}}\, W_n^{\dagger}}$ $\qquad \bar{\mathcal{P}} = W_n^{\dagger}\, i\bar{n}\cdot D_n\, W_n$

$$\frac{1}{\bar{\mathcal{P}}} = W_n^{\dagger}\, \frac{1}{i\bar{n}\cdot D}\, W_n \quad , \quad \frac{1}{i\bar{n}\cdot D_n} = W_n\, \frac{1}{\bar{\mathcal{P}}}\, W_n^{\dagger}$$

(easy to check that these are inverses)

$$\mathcal{L}_{\xi\xi}^{(0)} = e^{-ix\cdot\mathcal{P}}\, \bar{\xi}_n\, \frac{\bar{\not{n}}}{2}\left[\, in\cdot D + i\not{D}_{n\perp}\, W_n\, \frac{1}{\bar{\mathcal{P}}}\, W_n^{\dagger}\, i\not{D}_{n\perp} \right] \xi_n$$

Collinear Gluon Lagrangian

QCD
$$\mathcal{L} = \underbrace{-\frac{1}{2}\,\mathrm{tr}\{G^{\mu\nu}G_{\mu\nu}\}}_{\text{standard } -\frac{1}{4}G^{\mu\nu}_A G^{\mu\nu A}} + \tau\,\underbrace{\mathrm{tr}\{(i\partial_\mu A^\mu)^2\}}_{\text{gen. cov. gauge fixing}} + 2\,\underbrace{\mathrm{tr}\{\bar c\, i\partial_\mu\, iD^\mu c\}}_{\text{gen. cov. ghost}\left\{\begin{array}{l}\text{adjoint}\\ \text{scalar}\\ \text{fermi statistics}\end{array}\right.}$$

$$G^{\mu\nu} = G^{\mu\nu}_A T^A = \frac{i}{g}[D^\mu, D^\nu]$$

SCET: same steps as for quark action

Let $i\mathcal{D}^\mu = \frac{n^\mu}{2}(\bar{\mathcal{P}} + g\,\bar n\cdot A_n) + (\mathcal{P}_\perp^\mu + g A_{n\perp}^\mu) + \frac{\bar n^\mu}{2}(in\cdot\partial + g\,n\cdot A_n + g\,n\cdot A_{us})$

$\boxed{iD^\mu \to i\mathcal{D}^\mu}$ at LO

$$i\mathcal{D}_{us}^\mu = \frac{n^\mu}{2}\bar{\mathcal{P}} + \mathcal{P}_\perp^\mu + \frac{\bar n^\mu}{2}(in\cdot\partial + g\,n\cdot A_{us})$$

Recall A^μ_{us} behaves like background to A^μ_n. Maintaining gauge inv. for the background even in the A^μ_n gauge fixing terms requires

$\boxed{i\partial^\mu \to i\mathcal{D}^\mu_{us}}$ at LO In SCET this needed so collinear gauge fixing term does not break the usoft gauge inv.

$$\mathcal{L}_{cg}^{(0)} = \frac{1}{2g^2}\,\mathrm{tr}\{([i\mathcal{D}^\mu, i\mathcal{D}^\nu])^2\} + \tau\,\mathrm{tr}\{([i\mathcal{D}^\mu_{us}, A_{n\mu}])^2\} + 2\,\mathrm{tr}\{\bar c_n [i\mathcal{D}_\mu^{us}, [i\mathcal{D}^\mu, c_n]]\}$$

$$\boxed{\mathcal{L}^{(0)}_{SCET_1} = \mathcal{L}^{(0)}_{\xi\xi} + \mathcal{L}^{(0)}_{cg} + \underbrace{\mathcal{L}^{(0)}_{q}}_{} + \mathcal{L}^{(0)}_{A}}$$

full QCD actions for usoft quark q_{us} and for US gluon A^μ_{us}. These have no collinear fields

Analysis so far was tree level. To go further we need symmetries & power counting

① Gauge Symmetry

② Reparameterization Invariance

$\Big]$ very useful

③ Spin Symmetry ?

Lets first consider ③ :

revisit spinors $\quad \psi(x) = e^{-ix\cdot\tilde{p}} \left(1 + \frac{1}{i\bar{n}\cdot D_n} i\not{D}_{\perp n} \frac{\bar{\not{n}}}{2} \right) \xi_n(x)$

so $\quad u(p) = \left(1 + \frac{1}{\bar{n}\cdot p} \not{p}_\perp \frac{\bar{\not{n}}}{2} \right) u_n \quad , \quad u_n = \frac{\not{n}\bar{\not{n}}}{4} u$

$\left[\not{n}\, u_n = 0, \ \frac{\not{n}\bar{\not{n}}}{4} u_n = u_n \right]$

- consider $\sum_s u_n^s \bar{u}_n^s = \frac{\not{n}\bar{\not{n}}}{4} \sum_s \bar{u}^s u^s \frac{\bar{\not{n}}\not{n}}{4} = \frac{\not{n}\bar{\not{n}}}{4} \not{p} \frac{\bar{\not{n}}\not{n}}{4} = \frac{\not{n}}{2} \bar{n}\cdot p$

  → quantized ξ_n field does give collinear propogator, including numerator.

- u_n is not equal to expanded spinor $\sqrt{p^-/2} \begin{pmatrix} u \\ \sigma^3 u \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  even though it obeys the same relations

Instead

$$u_n = \frac{1}{2} \begin{pmatrix} \mathbb{1} & \sigma^3 \\ \sigma^3 & \mathbb{1} \end{pmatrix} \sqrt{p^0} \begin{pmatrix} u \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0} u \end{pmatrix} = \frac{\sqrt{p^0}}{2} \begin{pmatrix} \left(1 + p_3/p_0 - \frac{(i\vec{\sigma}\times\vec{P}_\perp)_3}{p^0} \right) u \\ \sigma_3 \left(1 + \frac{p_3}{p^0} - \frac{(i\vec{\sigma}\times\vec{P}_\perp)_3}{p_0} \right) u \end{pmatrix}$$

$$= \sqrt{\frac{p^-}{2}} \begin{pmatrix} \tilde{u} \\ \sigma^3 \tilde{u} \end{pmatrix}$$

Here $\quad \tilde{u} \equiv \sqrt{\frac{p^0}{2p^-}} \left(1 + \frac{p_3}{p^0} - \frac{(i\vec{\sigma}\times\vec{P}_\perp)_3}{p^0} \right) u \quad$ is two-componet spinor

$$\sum_s \tilde{u}^s \tilde{u}^{\dagger s} = \mathbb{1}_{2\times2} .$$

The extra terms in $\tilde{u}$ compared to u ensure proper structure under ② RPI. (In particular projetors $P_n' = \frac{\not{n}\bar{\not{n}}}{4} + \frac{\not{n}}{2}$, $P_{\bar{n}}' = \frac{\bar{\not{n}}\not{n}}{4} - \frac{\not{n}}{2}$ would give $\sqrt{\frac{p^-}{2}} \begin{pmatrix} u \\ \sigma 3 u \end{pmatrix}$ but are not RPI-III invariant.)

Spin Symmetry easiest to analyze in two-component form

$$\xi_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_n \\ \sigma^3 \varphi_n \end{pmatrix} \qquad \text{where } \dim \xi_n = \dim \varphi_n$$

$$\mathcal{L}^{(0)} = \varphi_n^{+} \left\{ in\cdot D + iD_{n\perp}^{\mu} \frac{1}{i\bar{n}\cdot D} iD_{n\perp}^{\nu} \left(g_{\mu\nu}^{\perp} + i\epsilon_{\mu\nu}^{\perp} \sigma_3 \right) \right\} \varphi_n$$

not $SU(2)$

just $U(1)$ helicity $h = \frac{i\epsilon_\perp^{\mu\nu}}{4} [\gamma_\mu, \gamma_\nu] \sim \sigma_3$ generator,

spin along the direction of collinear motion n

- broken by masses
- broken by non-perturbative effects
- useful in perturbation theory
- related to chiral rotation $\gamma_5 \xi_n = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_n \\ \sigma^3\varphi_n \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma^3\varphi_n \\ \varphi_n \end{pmatrix}$

  ie $\varphi_n \to \sigma_3 \varphi_n$

---

① Gauge Symmetry $\quad U(x) = \exp\left[i\, \alpha^A(x)\, T^A \right]$

Need to consider U's which leave us within EFT

eg. $i\partial^\mu \alpha^A \sim Q\, \alpha^A$ then $\xi_n' = U(x)\, \xi_n$ would no longer have $p^2 \lesssim Q^2\lambda^2$.

global $U = e^{i\alpha^A T^A}$

collinear $U_c(x) \qquad i\partial^\mu U_c(x) \sim Q(\lambda^2, 1, \lambda)\, U_c(x) \quad \leftrightarrow \quad A_n^\mu$

usoft $U_u(x) \qquad i\partial^\mu U_u(x) \sim Q(\lambda^2, \lambda^2, \lambda^2)\, U_u(x) \quad \leftrightarrow \quad A_{us}^\mu$

- two classes of gauge trnsfm for two gauge fields

- in label momentum space we have $\xi_{n,p_\ell}^{(x)} \to \sum_{q} (U_c)_{p_\ell - q_\ell}^{(x)}\, \xi_{n,q_\ell}^{(x)}$

  (analog of $\psi(x) \to U(x)\,\psi(x)$

  $\tilde{\psi}(p) \to \int dq\, \tilde{U}(p-q)\, \tilde{\psi}(q)$)

Let $(U_c)_{p_\ell - q_\ell} = \overset{\text{matrix}}{(U_c)_{p_\ell , q_\ell}}$ ie $\{p_\ell , q_\ell\}$'th entry is number $(U_c)_{p_\ell - q_\ell}$

For A_n^μ we let its U_c transformation be that of quantum gauge trnsfm of a quantum field in a A_{us}^μ background (in manner homogeneous in p.c.)

$U_c(x)$

- $\xi_n(x) \to U_c(x)\,\xi_n(x)$ using a matrix notation
- $A_n^\mu \to U_c \big(A_n^\mu + \frac{i}{g} \underbrace{D_{us}^\mu}_{\text{adjoint}} \big) U_c^\dagger$
- Also $q_{us} \xrightarrow{U_c} q_{us}$ since otherwise we give large momentum to usoft field
  $A_{us}^\mu \xrightarrow{U_c} A_{us}^\mu$

For $U_{us}(x)$ the fields ξ_n, A_n^μ transform like quantum fields under background gauge trnsfm. That is, they transform like matter fields of appropriate rep.

$U_{us}(x)$

- $\xi_n(x) \to U_{us}(x)\,\xi_n(x)$, $A_n^\mu \to U_{us} A_n^\mu U_{us}^\dagger$

  ↑ one number for all $\xi_{n,p}$ "vector" components

- $q_{us} \to U_{us}\, q_{us}$, $A_{us}^\mu \to U_{us} \big(A_{us}^\mu + \frac{i \partial^\mu}{g} \big) U_{us}^\dagger$

  ↑ ↗ usual gauge transformations

These transformations are fundamental, they are not corrected by power corrections.

$\mathcal{U}_c, \mathcal{U}_{us}$

Gauge transformations are homogeneous in λ

no mixing of term of different orders

eg. recall our heavy-to-light current

$\bar{\xi}_n \Gamma h_v^{us} \xrightarrow{\mathcal{U}_c} \bar{\xi}_n \mathcal{U}_c^\dagger \Gamma h_v^{us}$ is <u>not</u> gauge inv!

BUT recall offshell propagators generated Wilson line

$\bar{W}(x,-\infty)$

In general $\bar{W}(x,y) \to \mathcal{U}(x)\,\bar{W}(x,y)\,\mathcal{U}^\dagger(y)$. To avoid double counting with $\mathcal{U}_{global}$, we will take $\mathcal{U}_c^\dagger(-\infty) = 1$

$\bar{W}(x,-\infty) \to \mathcal{U}_c(x)\,\bar{W}(x,-\infty)$

Momentum Space $W = \sum_{m=0}^{\infty} \sum_{perms} \sum_{q_i} \frac{(-g)^m}{m!} \frac{\bar{n}\cdot A_{n,q_1}^{a_1}(x) \cdots \bar{n}\cdot A_{n,q_m}^{a_m}(x)\, T^{a_m}\cdots T^{a_1}}{\bar{n}\cdot q_1\ \bar{n}\cdot(q_1+q_2) \cdots \bar{n}\cdot(\Sigma q_i)}$

$$W(x) = \left[\sum_{perms} \exp\left(\frac{-g}{\bar{\mathcal{P}}}\,\bar{n}\cdot A_n(x)\right)\right]$$

the dependence on x encodes residual momenta in Wilson line. For $x=0$ the Fourier transform w.r.t p_r^- gives the line $\bar{W}(\cancel{x},-\infty)$ where $\cancel{x}$ is conjugate p_r^-.

* $W(x) \xrightarrow{\mathcal{U}_c} \mathcal{U}_c(x)\,W(x)$ in label matrix space.

* $W(x) \xrightarrow{\mathcal{U}_{u}} \mathcal{U}_{us}(x)\,W(x)\,\mathcal{U}_{us}^\dagger(x)$ from transformation of A_n directly.

$\bar{\xi}_n W \Gamma h_v \xrightarrow{\mathcal{U}_c} \bar{\xi}_n \cancel{\mathcal{U}_c^\dagger}\,\cancel{\mathcal{U}_c} W\Gamma h_v = \bar{\xi}_n W\Gamma h_v$ <u>invariant</u>

$\bar{\xi}_n W \Gamma h_v \xrightarrow{\mathcal{U}_{us}} \bar{\xi}_n \cancel{\mathcal{U}_{us}^\dagger}\,\cancel{\mathcal{U}_{us}} W \cancel{\mathcal{U}_{us}^\dagger}\,\Gamma\,\cancel{\mathcal{U}_{us}} h_v = \bar{\xi}_n W\Gamma h_v$ "

- the Wilson line carries n-collinear gluons, which in full QCD combine with attachments to ξ_n $--\!\!\rightarrow\!\!--$ to give gauge invariant answers.
- usoft can be taken to include global, and connects all fields.

Gauge Symmetry ties together

$$in\cdot D = in\cdot\partial + g n\cdot A_n + g n\cdot A_{us}$$

$$iD_{n\perp}^{\mu} = \mathcal{P}_\perp^{\mu} + g A_{n\perp}^{\mu}$$

$$i\bar{n}\cdot D_n = \bar{\mathcal{P}} + g\bar{n}\cdot A_n$$

$$iD_{us}^{\mu} = i\partial^{\mu} + g A_{us}^{\mu}$$ acting on usoft fields

Is Power Counting & Gauge Invariance enough to fix $\mathcal{L}_{\xi\xi}^{(0)}$?

$in\cdot D \sim \lambda^2$, $\frac{1}{\bar{\mathcal{P}}}(iD_\perp)^2 \sim \lambda^2$ ← no other $\mathcal{O}(\lambda^2)$ operators with correct mass dimension

but so far nothing rules out $\bar{\xi}_n\, iD_{n\perp}^{\mu}\, \frac{1}{i\bar{n}\cdot D_n}\, iD_{n\perp\mu}\, \frac{\bar{n}\!\!\!/}{2}\, \xi_n$.

② Reparameterization Invariance (RPI)

$n, \bar{n}$ break Lorentz Invariance (c.f. v^μ in HQET)

generators $\underbrace{n^{\mu} M_{\mu\nu}, \ \bar{n}^{\mu} M_{\mu\nu}}_{5 \text{ total}}$ ($M_{\mu\nu}$ usual 6 antisymm SO(3,1) generators)

only $\epsilon_\perp^{\mu\nu} M_{\mu\nu}$, rotations about $\vec{n}$ axis are preserved

3 types of RPI that keep $n^2=0$, $\bar{n}^2=0$, $n\cdot\bar{n}=2$

inf $\Delta_\perp$	inf $\epsilon_\perp$	finite α (simpler)
I. $n \to n+\Delta_\perp$	II. $n\to n$	III. $n\to e^{\alpha} n$
$\bar{n}\to\bar{n}$	$\bar{n}\to\bar{n}+\epsilon_\perp$	$\bar{n}\to e^{-\alpha}\bar{n}$

Power counting : $\Delta_\perp \sim \lambda$ eg: $n\cdot p \to n\cdot p + \Delta_\perp\cdot p_\perp \sim \lambda^2$

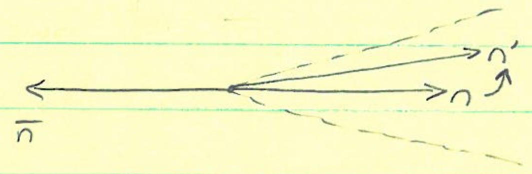
$$\left.\begin{aligned} \epsilon_\perp &\sim \lambda^0 \\ \alpha &\sim \lambda^0 \end{aligned}\right\} \text{unconstrained}$$

$\boxed{\text{type III}}$ simple, just implies for any operator with $\bar{n}^\mu$ in numerator there must be another n^μ in numerator, or $\bar{n}$ in denominator

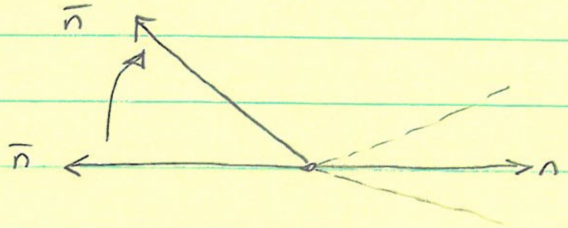
eg in $\mathcal{L}_{qq}^{(0)}$: had $\bar{\not{n}} \frac{1}{i\bar{n}\cdot D}$, $\bar{\not{n}}\, n\cdot D$ ✓

[no $\not{n}\; \bar{n}\cdot D$]

$\boxed{\text{Type I \& II}}$



type-I



type-II

We only can care about restoring Lorentz Inv, for the set of fluctuations described by SCET

Vector $p^\mu = \frac{n^\mu}{2}\bar{n}\cdot p + \frac{\bar{n}^\mu}{2} n\cdot p + p_\perp^\mu$ is invariant to choice for decomposition

→ implies transformations for $p_\perp^\mu$ to compensate $n, \bar{n}$'s.

<u>Find</u>

type-I	type-II
$n \to n + \Delta_\perp$	$\bar{n} \to \bar{n} + \epsilon_\perp$
$n\cdot D \to n\cdot D + \Delta_\perp \cdot D_\perp$	$n\cdot D \to n\cdot D$
$D_\perp^\mu \to D_\perp^\mu - \frac{\Delta_\perp^\mu}{2}\bar{n}\cdot D - \frac{\bar{n}^\mu}{2}\Delta^\perp\cdot D^\perp$	$D_\perp^\mu \to D_\perp^\mu - \frac{\epsilon_\perp^\mu}{2} n\cdot D - \frac{n^\mu}{2}\epsilon_\perp\cdot D^\perp$
$\bar{n}\cdot D \to \bar{n}\cdot D$	$\bar{n}\cdot D \to \bar{n}\cdot D + \epsilon_\perp\cdot D_\perp$
$\xi_n \to \left[1 + \frac{\not{\Delta}^\perp \bar{\not{n}}}{4}\right]\xi_n$	$\xi_n \to \left[1 + \frac{\not{\epsilon}^\perp}{2}\frac{1}{i\bar{n}\cdot D} i\not{D}_\perp\right]\xi_n$
$W \to W$	$W \to \left[\left(1 - \frac{1}{i\bar{n}\cdot D} i\epsilon^\perp\cdot D^\perp\right) W\right]$

[I write D^μ everywhere, but your free to think of it as $\mathcal{P}^\mu$ or $i\partial^\mu$ with appropriate gauging from symmetry ①] eg. $\delta^{(I)}\left(\bar{\xi}_n\, i\not{D}_{n\perp}\frac{1}{i\bar{n}\cdot D_n}\, i\not{D}_{n\perp}\frac{\bar{\not{n}}}{2}\xi_n\right) = -\,\bar{\xi}_n\, i\Delta^{\perp}\cdot D^{\perp}\frac{\bar{\not{n}}}{2}\xi_n$

$\delta^{(I)}\left(\bar{\xi}_n\, in\cdot D\,\frac{\bar{\not{n}}}{2}\xi_n\right) = \underbrace{+\,\bar{\xi}_n\, i\Delta^{\perp}\cdot D^{\perp}\frac{\bar{\not{n}}}{2}\xi_n}$

sum = 0, so connected by RPI, no non-trivial Wilson coefficient btwn them

type-II rules out the $\bar{\xi}_n\, iD^{\mu}_{n\perp}\frac{1}{i\bar{n}\cdot D_n}\, iD_{n\perp\mu}\frac{\bar{\not{n}}}{2}\xi_n$ operator in $\mathcal{L}^{(0)}_{\xi\xi}$.

So

$$\boxed{\mathcal{L}^{(0)}_{\xi\xi} = \bar{\xi}_n\left[in\cdot D + i\not{D}_{\perp n}\frac{1}{i\bar{n}\cdot D_n}\, i\not{D}_{n\perp}\right]\frac{\bar{\not{n}}}{2}\xi_n}$$

is unique LO $\mathcal{L}$ for ξ_n by p.c., gauge inv, RPI

MORE RPI: Freedom in the label + residual decomposition

$\bar{n}\cdot(p_\ell + p_r)$, $p^{\mu}_{\ell\perp} + p^{\mu}_{r\perp}$

$\mathcal{P}_\mu \to \mathcal{P}_\mu + \beta_\mu$, $i\partial_\mu \to i\partial_\mu - \beta_\mu$ with $n\cdot\beta = 0$

$\xi_{n,p}(x) \to e^{i\beta\cdot x}\,\xi_{n,p+\beta}(x)$

Connects: $\mathcal{P}^\mu + i\partial^\mu$ ie leading & subleading Wilson coefficients in $\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \cdots$ and in operators $C^{(0)}O^{(0)} + C^{(1)}O^{(1)} + \cdots$

Gauge This $n\cdot\mathcal{P} = 0$, so just $in\cdot\partial \to in\cdot D$

$in\cdot D \to U_c\, in\cdot D\, U_c^{\dagger}$, $U_u\, in\cdot D\, U_u^{\dagger}$ (with our gauge trnsfms)

Also

$iD^{\mu}_{n\perp} \to U_c\, iD^{\mu}_{n\perp}\, U_c^{\dagger}$ or $U_u\, iD^{\mu}_{n\perp}\, U_u^{\dagger}$

$i\bar{n}\cdot D_n \to U_c\, i\bar{n}\cdot D_n\, U_c^{\dagger}$ or $U_u\, i\bar{n}\cdot D_n\, U_u^{\dagger}$

$iD^{\mu}_{us} \to iD^{\mu}_{us}$ or $U_u\, iD^{\mu}_{us}\, U_c^{\dagger}$

$\therefore$ simplest idea $\left.\begin{array}{l} iD^{\mu}_{n\perp} + iD^{\mu}_{us\perp} \\ i\bar{n}\cdot D_n + i\bar{n}\cdot D_{us} \end{array}\right\}$ doesn't work due to lack of trnsfm of iD^{μ}_{us} under U_c

The object that can compensate is $W \to U_c W$.

The unique result that gauges $\mathcal{P}^\mu + i\partial^\mu$ (with our strictly LO, homogeneous transfms) is

$$\left.\begin{aligned} iD_{n\perp}^{\mu} + W\, iD_{\perp}^{us\,\mu}\, W^{\dagger} &\equiv iD_{\perp}^{\mu} \\ i\bar{n}\cdot D_n + W\, i\bar{n}\cdot D_{us}\, W^{\dagger} &\equiv i\bar{n}\cdot D \end{aligned}\right\} \text{combined result of RPI \& gauge inv.}$$

↑ the extra terms from $W, W^\dagger$ induce the $+\cdots$ in our earlier $A^\mu = A_n^\mu + A_{us}^\mu + \cdots$ expression

eg

from the $\bar{\xi}_n\, i\slashed{D}_{\perp n} \frac{1}{i\bar{n}\cdot D_n}\, i\slashed{D}_{\perp n} \frac{\bar{\slashed{n}}}{2}\, \xi_n$ term in $\mathcal{L}^{(0)}_{\xi\xi}$ we

get $\mathcal{L}^{(1)}_{\xi\xi} = (\bar{\xi}_n W)\, i\slashed{D}_{\perp}^{us} \frac{1}{\bar{\mathcal{P}}} (W^\dagger i\slashed{D}_{\perp n}\xi_n) + (\bar{\xi}_n\, i\slashed{D}_{\perp n} W) \frac{1}{\bar{\mathcal{P}}}\, i\slashed{D}_{\perp}^{us} (W^\dagger \xi_n)$

which is U_c & U_{us} gauge invariant & has no Wilson Coeff.

Like HQET, RPI also connects Wilson Coeff of leading & λ-suppressed external currents

Extension to more collinear fields — for >1 energetic hadron or >1 energetic jet

$$\sum_n \mathcal{L}_n^{(0)} = \sum_n \left[\mathcal{L}^{(0)}_{\xi_n \xi_n} + \mathcal{L}^{(0)}_{A_n} \right]$$

the sum is over inequivalent RPI equivalence classes

For $n_1, n_2, n_3, \dots$ the collinear modes are distinct only if $n_i \cdot n_j \gg \lambda^2$ for $i \neq j$

eg. $p_2 = Q n_2$ $\qquad n_1 \cdot p_2 = Q\, n_1 \cdot n_2 \sim \lambda^2$ if $n_1 \cdot n_2 \sim \lambda^2$

but then P_2 is n_1-collinear. So n_2 is within RPI equivalence class defined by $[n_1]$

All the things we derived with 1-collinear direction get repeated when we have more than one

- Collinear Gauge trnsfms: U_{n_1}, U_{n_2}, ...
- RPI: separate invariance for $\{n_1, \bar{n}_1\}$

  $\{n_2, \bar{n}_2\}$ etc

  here there is no simple connection to overall Lorentz Trnsfm (more like a type of Lorentz inv in each $[n_i]$ sector)

- Matching calculations generate Wilson lines

eg. $e^+ e^- \to \gamma^* \to$ two-jets

$$J^\mu = \bar{\psi} \gamma^\mu \psi \quad \to \quad J^\mu_{SCET} = \underbrace{\underbrace{(\bar{\xi}_{n_1} W_{n_1})}_{n_1 \text{ gauge inv}} \gamma^\mu \underbrace{(W^\dagger_{n_2} \xi_{n_2})}_{n_2 \text{ gauge inv}}}_{\text{usoft gauge inv}}$$

here $W_{n_1} = W_{n_1}[\bar{n}_1 \cdot A_{n_1}]$

$W_{n_2} = W_{n_2}[\bar{n}_2 \cdot A_{n_2}]$

} generated by integrating out offshell $p^2 \sim Q^2$ lines

Final Comment on Discrete Symmetries: $\qquad n = (1,0,0,1)$, $\bar{n} = (1,0,0,-1)$

$C^{-1} \xi_{n,p} C = - [\bar{\xi}_{n,-p} \mathcal{C}]^T$ $\qquad p = (p^+, p^-, p^\perp)$

$P^{-1} \xi_{n,p} P^{-1} = \gamma_0 \xi_{\bar{n},\tilde{p}}(x_P)$ $\qquad \tilde{p} = (p^-, p^+, -p^\perp)$

$\uparrow$ swaps role $n \leftrightarrow \bar{n}$

$T^{-1} \xi_{n,p} T = T \xi_{\bar{n},\tilde{p}}(x_T)$ $\qquad x_P = (x^-, x^+, -x^\perp)$

$x_T = (-x^-, -x^+, x^\perp)$

Study $\mathcal{L}_{\xi\xi}^{(0)}$

① Propagator

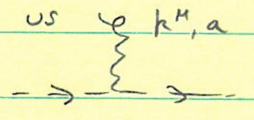
$$\frac{i\bar{n}\!\!\!/}{2}\,\frac{\theta(\bar{n}\cdot p)}{n\cdot p+\frac{p_\perp^2}{\bar{n}\cdot p}+i\epsilon}+\frac{i\bar{n}\!\!\!/}{2}\,\frac{\theta(-\bar{n}\cdot p)}{+n\cdot p+\frac{p_\perp^2}{\bar{n}\cdot p}-i\epsilon}=\frac{i\bar{n}\!\!\!/}{2}\,\frac{\bar{n}\cdot p}{n\cdot p\,\bar{n}\cdot p+p_\perp^2+i\epsilon}$$

particles $\bar{n}\cdot p>0$ anti $\bar{n}\cdot p<0$

✓ expn. of QCD

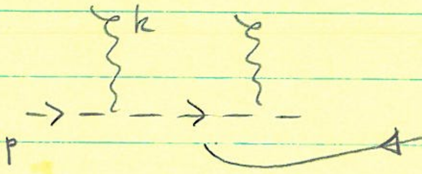
② Interactions

- for usoft gluons, only $n\cdot A_{us}$ at LO



$$= i g T^a n^\mu \frac{\bar{n}\!\!\!/}{2}$$

& only sees $n\cdot k$ usoft momentum (multipole expn.)



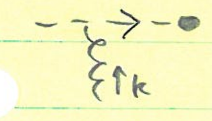
$$\frac{\bar{n}\cdot p}{\bar{n}\cdot p\; n\cdot(p+k)+p_\perp^2+i\epsilon}=\frac{\bar{n}\cdot p}{\bar{n}\cdot p\; n\cdot k+p^2+i\epsilon}$$

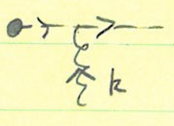
$$\underset{\text{on-shell}}{=}\frac{\bar{n}\cdot p}{\bar{n}\cdot p\; n\cdot k+i\epsilon}$$

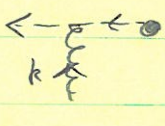
(Compare Collinear Gluon  $\frac{\bar{n}\cdot(p+q)}{(p+q)^2+i\epsilon}$)

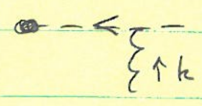
Propagator reduces to eikonal approx when appropriate

$\bar{n}\cdot p>0$ $\bar{n}\cdot p<0$





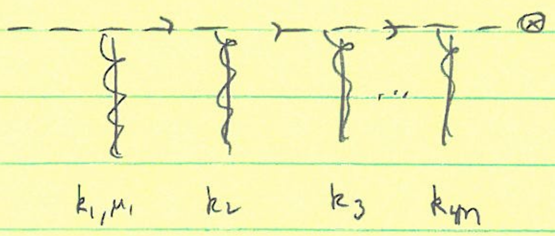




$\frac{n^\mu}{n\cdot k+i\epsilon}$ $\frac{n^\mu}{-n\cdot k+i\epsilon}$ $\frac{n^\mu}{-n\cdot k-i\epsilon}$ $\frac{n^\mu}{n\cdot k-i\epsilon}$

Usoft - Collinear Factorization

Consider



$$= \Gamma \sum_{m} \sum_{\text{perms}} \frac{(-g)^m \, n\cdot A^{a_1} \cdots n\cdot A^{a_m} T^{a_m} \cdots T^{a_n}}{n\cdot k_1 \; n\cdot(k_1+k_2) \cdots n\cdot(\Sigma k_i)}$$

$$* \, u_n$$

on-shell so $\frac{1}{n\cdot k + \frac{p^2}{\bar{n}\cdot p}} \to \frac{1}{n\cdot k}$

Motivates us to consider a field redefinition

$$\xi_{n,p}(x) = Y(x)\, \xi_{n,p}^{(0)}(x) \qquad A_{n,p} = Y A_{n,p}^{(0)} Y^{+}$$

$\uparrow$ adjoint version

$$Y(x) = P \exp\left(ig \int_{-\infty}^{0} ds \; n\cdot A_{us}^{a}(x+ns)\, T^a \right)$$

$$n\cdot D \, Y = 0 \quad , \quad Y^{+} Y = 1 \qquad \text{find} \quad W = Y W^{(0)} Y^{+}$$

$$\mathcal{L}_{\xi\xi}^{(0)} = \bar{\xi}_{n,p'} \frac{\bar{n}\!\!\!/}{2} \left[in\cdot D + \cdots \right] \xi_{n,p}$$

$$= \bar{\xi}_{n,p'}^{(0)} \frac{\bar{n}\!\!\!/}{2} \left[Y^{+} in\cdot D_{us} Y + Y^{+}(Y g \bar{n}\cdot A_n Y^{+}) Y + \cdots \right] \xi_{n,p}$$

$$= \bar{\xi}_{n,p'}^{(0)} \frac{\bar{n}\!\!\!/}{2} \Big[\underbrace{in\cdot\partial + g\bar{n}\cdot A_n}_{in\cdot D_c} + \cdots \Big] \xi_{n,p}$$

$\uparrow$ all $n\cdot A_{us}$'s disappear!

True for gluon action too

$$\mathcal{L}(\xi_{n,p}, A_{n,q}^{\mu}, n\cdot A_{us}) = \mathcal{L}(\xi_{n,p}^{(0)}, A_{n,q}^{(0)\mu}, 0)$$

Interactions don't disappear, but are moved out of L.O. $\mathcal{L}$ and into currents eg 1 $J_1 = \bar{\xi}_n W \Gamma h_v = \bar{\xi}_n^{(0)} Y^{\dagger} Y W^{(0)} Y^{\dagger} \Gamma h_v$

$= (\bar{\xi}_n^{(0)} W^{(0)}) \Gamma (Y^{\dagger} h_v)$

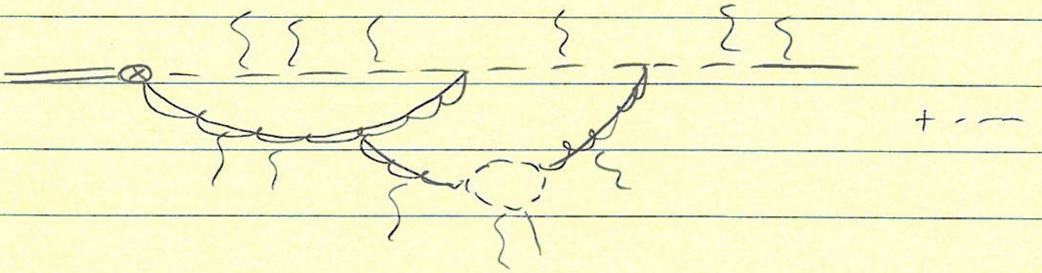
eg 2 $J_2 = (\bar{\xi}_{n_1} W_{n_1}) \Gamma (W_{n_2}^{\dagger} \xi_{n_2}) = (\bar{\xi}_{n_1}^{(0)} W_{n_1}^{(0)}) (Y_{n_1}^{\dagger} Y_{n_2}) \Gamma (W_{n_2}^{(0)\dagger} \xi_{n_2}^{(0)})$

eg 3 collinear parts are global color singlet set $n_1 = n_2$ above

$J_3 = (\bar{\xi}_n W_n) \Gamma (W_n^{\dagger} \xi_n) = (\bar{\xi}_n^{(0)} W_n^{(0)}) (\cancel{Y}^{\dagger} \cancel{Y}) \Gamma (W_n^{(0)\dagger} \xi_n^{(0)})$

Quite powerful, this "BPS field redefinition" sums an ∞ class of diagrams

in eg 1



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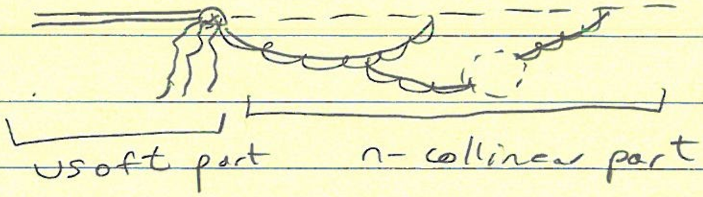
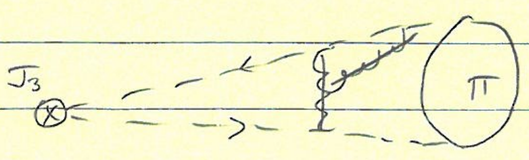


in eg 2 usoft gluons decouple at LO from any graph

This "color transparency"



- usoft gluons decouple from energetic partons in a color singlet state
- they just "see" overall color singlet due to the multipole expansion

U_c & U_{us} transformations post field redefinition

Our Logic with the $\xi_n \to Y \xi_n$ and $A_n \to Y A_n Y^\dagger$ field redefinitions is that they allow us to express in a simple way some of the consequences of dynamics <u>within</u> the EFT

Nevertheless, after the field redefinition we see that all operators become products of collinear & usoft blocks of fields, so its natural to ask about gauge symmetries that act seperately within those blocks. They can be derived in the following way:

<u>Original</u> U_c	$\xi_n \to U_c \xi_n$	U_{us}	$\xi_n \to U_{us} \xi_n$
	$A_n^\mu \to U_c \left(A_n^\mu + \frac{i}{g} \mathcal{D}_{us}^\mu \right) U_c$		$A_n \to U_{us} A_n U_{us}^\dagger$
	$q_{us} \to q_{us}$		$q_{us} \to U_{us}\, q_{us}$
	$A_{us}^\mu \to A_{us}^\mu$		$A_{us}^\mu \to U_{us} \left(A_{us}^\mu + \frac{i}{g} \partial^\mu \right) U_{us}^\dagger$

Consider $U_c^{(0)}$ defined by $U_c^{(0)}(x) = Y^\dagger(x)\, U_c(x)\, Y(x)$

$$U_c = Y(x)\, U_c^{(0)}(x)\, Y^\dagger(x)$$

- $\xi_n(x) = Y(x)\, \xi_n^{(0)}(x) \xrightarrow{U_c} U_c \xi_n = U_c Y(x) \xi_n^{(0)} = Y(x)\, U_c^{(0)} \xi_n^{(0)}$

$$\xrightarrow{U_{us}} U_{us}\, \xi_n = U_{us}\, Y(x)\, \xi_n^{(0)}$$

taking $Y(x) \to U_{us} Y(x)$ (so $U_{us}(-\infty) = 1$, distinguished from global)

we find $\xi_n^{(0)} \xrightarrow{U_c^{(0)}} U_c^{(0)} \xi_n^{(0)}$

$$\xi_n^{(0)} \xrightarrow{U_{us}} \xi_n^{(0)}.$$

- similarly

acts in adjoint rep: $\mathcal{D}_{us}^\mu \mathcal{Y}_{adj} = 0$

$$U_c \left(A_n^\mu + \frac{i}{g} \mathcal{D}_{us}^\mu \right) U_c^\dagger = Y(x)\, U_c^{(0)}\, Y^\dagger(x) \left[Y(x) A_n^{(0)} Y^\dagger(x) + \frac{i}{g} \mathcal{D}_{us}^\mu \right] Y(x)\, U_c^{(0)}\, Y^\dagger(x)$$

$$= Y(x) \left[U_c^{(0)} \left[A_n^{(0)\mu} + \frac{i}{g} \partial^\mu \right] U_c^{(0)\dagger} \right] Y^\dagger(x)$$

so $A_n^{(0)\mu} \xrightarrow{U_c^{(0)}} U_c^{(0)} \left[A_n^{(0)\mu} + \frac{i}{g} \partial^\mu \right] U_c^{(0)\dagger}$

$$A_n^{(0)\mu} \xrightarrow{U_{us}} A_n^{(0)\mu}$$

But <u>note</u> that this teaches us nothing new

What about Wilson Coefficients?

have $C(\bar{P},\mu)$ ie depend on large momenta picked out by label operator $\bar{P} \sim \lambda^0$

eg. $C(-\bar{P},\mu)\,(\bar{\xi}_n W)\,\Gamma\, h_v = (\bar{\xi}_n W)\,\Gamma\, h_v\, C(\bar{P}^\dagger)$

must act on product $(\bar{\xi} W)$ since only momentum of this combination is gauge invariant

Write $$(\bar{\xi} W)\,\Gamma\, h_v\, C(\bar{P}^\dagger) = \int d\omega\, C(\omega,\mu)\,\left[(\bar{\xi} W)\,\delta(\omega-\bar{P}^\dagger)\,\Gamma\, h_v\right]$$

$$= \int d\omega\; \underset{\uparrow}{C(\omega,\mu)}\;\; \underset{\uparrow}{O(\omega,\mu)}$$

convolution (as promised)

Hard-Collinear Factorization of "C" and collinear "O"

Recall defn of W, $\quad i\bar{n}\cdot D_c\, W = 0$, $\quad W^\dagger W = 1$

as operator
$$i\bar{n}\cdot D_c\, W = W\,\bar{P}$$
$$i\bar{n}\cdot D_c = W\,\bar{P}\,W^\dagger$$
$$(i\bar{n}\cdot D_c)^k = W\,\bar{P}^k\,W^\dagger$$

$$f(i\bar{n}\cdot D_c) = \underset{\uparrow}{W}\;\underset{\uparrow}{f(\bar{P})}\;\underset{\uparrow}{W^\dagger}$$

trades $\bar{n}\cdot A \to W$

Part of collin Op. $p^2 \sim \lambda^2 Q^2$

hard coefficient

$$= \int d\omega\;\; f(\omega)\;\; W\,\delta(\omega-\bar{P})\,W^\dagger$$

In general we can define a convenient set of (collinear gauge invariant) <u>building blocks</u> for operators:

- $\chi_n \equiv (W_n^\dagger \xi_n)$ "quark jet-field"

  $\chi_{n,\omega} \equiv \delta(\omega - \bar{\mathcal{P}})(W_n^\dagger \xi_n)$

  operators $\int d\omega_1\, d\omega_2\, C(\omega_1,\omega_2)\, \bar{\chi}_{n,\omega_1} \Gamma \chi_{n,\omega_2}$ etc.

- $g\,\mathcal{B}_{n\perp}^{\mu} = \left[\frac{1}{\bar{\mathcal{P}}} W_n^\dagger [i\bar{n}\cdot D_n, iD_{n\perp}^{\mu}] W_n\right] = g A_{n\perp}^{\mu} + \cdots$

  $\nwarrow$ derivatives act only inside $[\cdots]$

  "gluon jet-field" for two physical gluon-pol.

  $\mathcal{B}_{n\perp,\omega}^{\mu} = [\,\mathcal{B}_{n\perp}^{\mu}\, \delta(\omega - \bar{\mathcal{P}}^\dagger)\,]$

  $\uparrow$ convention/choice, acts left inside $[\cdots]$

<u>Building Blocks</u>

All operators can be constructed solely from

$\{\chi_n, \mathcal{B}_{n\perp}^{\mu}, \mathcal{P}_\perp^{\nu}\}$ + usoft fields & D_{us}^{μ}.

① Let $i\mathcal{D}_n^{\mu} = W_n^\dagger\, iD_n^{\mu}\, W_n$ where iD_n^{μ} has $\begin{Bmatrix}\bar{\mathcal{P}}\\ \mathcal{P}_\perp\\ in\cdot\partial\end{Bmatrix} + g\begin{Bmatrix} n\cdot A_n\\ A_n^{\perp}\\ \bar{n}\cdot A_n\end{Bmatrix}$

$\bar{n}\cdot i\mathcal{D}_n = \bar{\mathcal{P}}$

$i\mathcal{D}_n^{\perp\mu} = \mathcal{P}_\perp^{\mu} + g\,\mathcal{B}_{n\perp}^{\mu}$, $in\cdot\mathcal{D}_n = in\cdot\partial + \underbrace{g n\cdot\mathcal{B}_n}_{}$

analogous to defn $\mathcal{B}_{n\perp}^{\mu}$

derivatives $\bar{\mathcal{P}}\,\chi_{n,\omega} = \omega\, \chi_{n,\omega}$ can be absorbed in coefficients

$in\cdot\partial\, \chi_n = -(g n\cdot\mathcal{B}_n)\chi_n - i\not{\mathcal{D}}_{n\perp} \frac{1}{\bar{\mathcal{P}}} i\not{\mathcal{D}}_{n\perp} \chi_n$ equation of motion for χ_n

$\uparrow$ remove $in\cdot\partial$'s

$in\cdot\partial\, \mathcal{B}_{n\perp}^{\mu} = \ldots$ eqtn of motion

Note: $i\mathcal{D}_{n\perp}^{\mu} = W_n^\dagger(\mathcal{P}_\perp^{\mu} + gA_{n\perp}^{\mu})W_n = \mathcal{P}_\perp^{\mu} + [W_n^\dagger iD_\perp^{\mu} W_n] = \mathcal{P}_\perp^{\mu} + \left[\frac{1}{\bar{\mathcal{P}}}\bar{\mathcal{P}}\, W^\dagger iD_\perp^{\mu} W\right]$

$= \mathcal{P}_\perp^{\mu} + \left[\frac{1}{\bar{\mathcal{P}}} W_n^\dagger i\bar{n}\cdot D_n D_{n\perp}^{\mu} W_n\right] = \mathcal{P}_\perp^{\mu} + \left[\frac{1}{\bar{\mathcal{P}}} W_n^\dagger [i\bar{n}\cdot D_n, iD_\perp^{\mu}] W_n\right]$

② $w\,(g\, n \cdot \mathcal{B}_n)_w \;=\; 2\, \mathcal{P}^{\perp}_{\nu}\, g\, \mathcal{B}^{\nu}_{\perp, w} \;+\; \ldots$

also part of gluon e.o.m.

<u>All</u> other collinear operators , $W_n^{\dagger}\,[\,i\mathcal{D}_{n\perp}^{\mu}, i\mathcal{D}_{n\perp}^{\nu}\,]\,W_n$, ...
are reducible to $\{\, \chi_n, \mathcal{B}_{\perp}^{\mu}, \mathcal{P}_{\perp}^{\nu} \,\}$

③ Do need usoft derivatives, Field strengths, q_{us}, etc
Statement of RPI becomes
$$i\mathcal{D}_{n\perp}^{\mu} + i D_{us\perp}^{\mu} \;,\qquad \bar{\mathcal{P}}_n + i\bar{n}\cdot D_{us}$$

(equiv. to earlier, but W's around collinear D^{μ} here, rather than usoft)

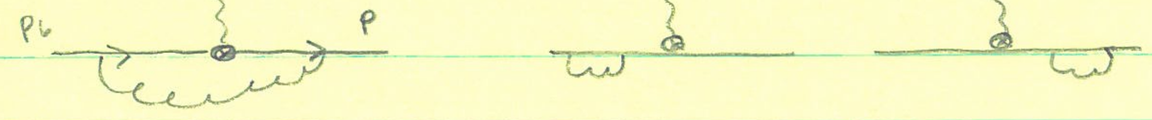
<u>Loops, IR divergences, Matching & Running</u>

Consider heavy-to-light current for $b \to s\gamma$

$J^{QCD} \;=\; \bar{s}\,\Gamma\, b \qquad\qquad \Gamma = \sigma^{\mu\nu} P_R F_{\mu\nu}$

$J^{SCET}_{LO} \;=\; (\bar{\xi} W)\,\Gamma\, h_v\, C(\bar{\mathcal{P}}^{+}) \qquad$ [pre Y-field redefn]

QCD graphs at one-loop , take $p^2 \neq 0$ to regulate
use Feyn. Gauge IR of collin quark



$$= -\,\bar{u}_s \Gamma u_b\, \frac{\alpha_s C_F}{4\pi}\left[\, \ln^2\!\left(\frac{-p^2}{m_b^2}\right) + 2\ln\!\left(\frac{-p^2}{m_b^2}\right) + \ldots \right]$$

$\uparrow$ $f(p\cdot p_b/m_b^2)$, IR finite

$$Z_{\psi_b} = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} + 3\ln\frac{\mu^2}{m_b^2} + \ldots\right]$$

full z's (not $\overline{MS}$) match for S-matrix

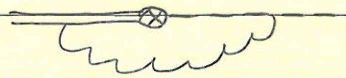
$$Z_{\psi_s} = 1 - \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} - \ln \frac{p^2}{\mu^2}\right]$$

$$Z_{tensor} = 1 + \frac{\alpha_s C_F}{4\pi}\, {}^{1}\!/_{\epsilon}$$ $\leftarrow$ Tensor current in QCD not conserved

$$\text{sum} = \bar{u}_s \Gamma u_b \left[1 - \frac{\alpha_s C_F}{4\pi} \left\{ \ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2} \ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}} + \dots \right\} \right]$$

$\boxed{\text{SCET}_I}$ Use Feyn. Gauge everywhere

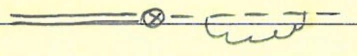
Usoft-loops



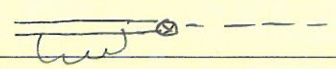
$$\int \frac{\text{đ}^d k \quad n\cdot v}{(v\cdot k + i0)(k^2+i0)(n\cdot k + p^2/\bar{n}\cdot p + i0)}$$

$$= -\bar{u}_n \Gamma u_v \frac{\alpha_s C_F}{4\pi} \left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} \ln\left(\frac{\mu\, \bar{n}\cdot p}{-p^2 - i0}\right) + 2 \ln^2\left(\frac{\mu\, \bar{n}\cdot p}{-p^2 - i0}\right) + \frac{3\pi^2}{4} \right]$$

Note: $p^2/\bar{n}\cdot p \sim \lambda^2$ so logs $\mathcal{O}(1)$ for $\mu \sim \lambda^2$ usoft scale

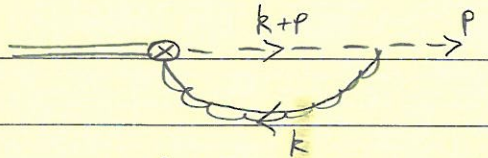


$\propto n_\mu n^\mu = 0$ in Feyn Gauge, $Z_\xi^{us} = 0$



$$Z_{HQET} = 1 + \frac{\alpha_s C_F}{4\pi} \left[\frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}} \right]$$

Collinear graphs



$$= \bar{u}_n \Gamma u_v \sum_{\substack{k_\ell \neq 0 \\ k_\ell \neq -p_\ell}} \int \frac{\text{đ}^d k_r \quad n\cdot\bar{n} \quad \bar{n}\cdot(p+k)}{(\bar{n}\cdot k)(k^2)(k+p)^2}$$

all $+i0$

$n\cdot k_r,\ \bar{n}\cdot k_\ell,\ k_\ell^\perp$

$n\cdot p_r,\ \bar{n}\cdot p_\ell,\ p_\ell^\perp$

each momentum has $k = (k_\ell, k_r)$, label & residual

Label & residual ensure we have LO piece (important for mixed collinear & usoft graphs)

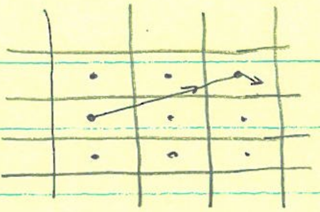
- But now we want to turn $\sum_{k_\ell} \int \text{đ}^d k_r$ back into $\int \text{đ}^d k_\ell$ to do loop integration

Claim if we ignore $\begin{matrix} k_\ell \neq 0 \\ k_\ell \neq -p_\ell \end{matrix}$ restrictions we just get

$$\int \frac{\text{đ}^d k \quad n\cdot\bar{n} \quad \bar{n}\cdot(p+k)}{(\bar{n}\cdot k)(k^2)(k+p)^2}$$

k_r^+ is only + loop momentum. Worry about: $k_\ell^\perp, k_r^\perp$ & k_ℓ^-, k_r^-

call grid



was like Wilsonian EFT (with finite edges)

Continuum EFT: each grid point specifies an ∞-space of residual momenta ($k_r^\mu \in \mathbb{R}$), subject to rules

Ignore $k_\ell \neq 0$, $k_\ell \neq -p_\ell$,

i) $$\sum_{k_\ell} \int dk_r = \int dk_\ell$$ for $-$ & $\perp$ momenta (use 1-dim notation for simplicity)

ii) $$\sum_{k_\ell} \int dk_r \, \underset{\uparrow}{F(k_\ell)} = \sum_{k_\ell} \int dk_r \, F(k_\ell + k_r) = \int dk_\ell \, F(\underset{\uparrow}{k_\ell})$$

constant throughout each box

continuous dummy var.

- This is the (simplified version of) main rule for obtaining $\int dk_\ell$. For each label loop momentum k_ℓ, there will always be a corresponding k_r that we can absorb in this fashion.
- Recall that grid facilitated multipole expansion. For a purely collinear loop there is often no physical $p_r^\perp$, p_r^- flowing through it. In this case answer must reduce to what we get from considering $\int d^d k_n$

iii) $$\sum_{k_\ell} \int dk_r \, (k_r)^j \, F(k_\ell) = 0$$ for $j > 0$ integer

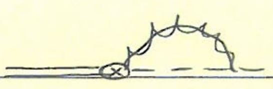
dim-reg type rule which maintains Lorentz-Inv in residual Space

iv) Ultrasoft external particles or loops give non-trivial ℓ_r^μ & hence residual momenta that we can not absorb

eg. $$\sum_{k_\ell} \int dk_r \int d\ell_r \, F(k_\ell, \underset{\uparrow}{\ell_r}) = \int dk_\ell \int d\ell_r \, F(k_\ell, \ell_r)$$

usoft propagator (say)

ignoring restrictions



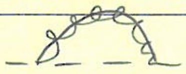
$$\sum_{k_\ell} \int \frac{d^dk_r \;\; n\cdot\bar{n} \;\; \bar{n}\cdot(p_\ell+k_\ell)}{\bar{n}\cdot k_\ell \,(k_\ell^- k_r^+ + k_\ell^{\perp 2})\,\big((k_\ell^- + p_\ell^-)(k_r^+ + p_r^+) + (k_\ell^\perp + p_\ell^\perp)^2\big)}$$

$$= \int \frac{d^dk \;\; n\cdot\bar{n} \;\; \bar{n}\cdot(p+k)}{\bar{n}\cdot k \;\; k^2 \;(k+p)^2}$$

do as standard loop integral

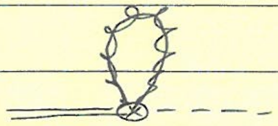
$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-p^2} + \ln^2\left(\frac{\mu^2}{-p^2}\right) + 2\ln\left(\frac{\mu^2}{-p^2}\right) + 4 - \frac{\pi^2}{6}\right]$$

logs minimized for $\mu^2 \sim p^2$, collinear scale



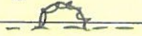
collinear w.fn. renormalization, same as massless QCD

$$Z_{\xi} = 1 + \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{UV}} + \ln\frac{\mu^2}{-p^2}\right]$$



$\propto \bar{n}^2 = 0$ (Feyn.),

scaleless power-divergent $\to 0$

(cancels unphysical sing. for $\bar{n}\cdot(p+k) \to 0$, $k_\perp$ fixed in )

<u>Matching</u> Compare QCD & SCET, kinematics in $b \to s\gamma$ sets $p^- = m_b$

$$(\text{sum QCD})^{\text{ren}} = -\frac{\alpha_s C_F}{4\pi}\left[\underline{\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}}} + 2\ln\left(\frac{\mu^2}{m_b^2}\right) + \cdots\right]$$

$$(\text{sum SCET})^{\text{bare}} = -\frac{\alpha_s C_F}{4\pi}\left\{\underline{\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{IR}}}\right.$$

$$\underbrace{-\frac{1}{\epsilon^2} - \frac{5}{2\epsilon} - \frac{2}{\epsilon}\ln\left(\frac{\mu}{m_b}\right)} \underbrace{- 2\ln^2\left(\frac{\mu^2}{m_b^2}\right) - \frac{3}{2}\ln\frac{\mu}{m_b}} + \cdots$$

$\underline{\hspace{3em}}$ match these IR divergences

want to take care of this with UV renormalization of SCET,

Have to know $1/\epsilon$'s are UV.

difference gives matching for one-loop $C(m_b, \mu)$

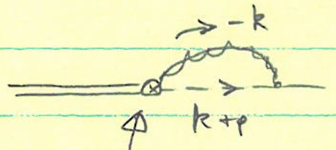
↓ discuss later

Note ① $\sum_\omega C(\omega,\mu)\, \underset{\sim}{\bar\chi_{n,\omega}}\, \Gamma\, h_v$

$(\bar\xi_n W)\, \delta_{\omega,\bar p^+}$ total momentum of $\bar\xi_n$ & W fixed as ω

so its always $\omega = p^-$ above

- non-trivial example



sum $= \bar n\cdot(k+p) + \bar n\cdot(-k) = \bar n\cdot p$

② Should be careful with $k_\ell \neq 0$, $k_\ell \neq -p_\ell$ (Zero-Bin's)
Collinear momenta have non-zero labels
When $k_\ell = 0$ gluon is usoft ($k_\ell = -p_\ell$ quark is usoft)
These restrictions avoid double counting in SCET fields and hence also in results for loop integrations

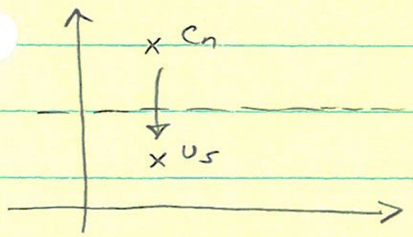
Rule ii) above with restrictions (encoded via propagators) is

$$\sum_{k_\ell\neq 0} \int dk_r\, F(k_\ell) = \sum_{k_\ell}\int dk_r\, F(k_\ell) - \int dk_r\, F^{k_\ell\to 0}(0)$$
$$= \sum_{k_\ell}\int dk_r\, F(k_\ell+k_r) - \int dk_r\, F^{k_\ell\to 0}(k_r)$$
$$= \int dk \Big[F(k) - \underset{\uparrow}{F^{k_\ell\to 0}(k)} \Big]$$

zero-bin subtraction term

$F^{k_\ell\to 0}(k)$ is defined by taking scaling limit $k_n^\mu \to k_{us}^\mu$
ie $k_n^\mu \sim \lambda^2$

and expanding to keep all subtractions that are same order in λ (dropping power suppressed terms, a "minimal subtraction")



subtraction ensures "C_n" has no non-trivial support in ultrasoft "us" region r eg.

$$\int d^dk \left[\frac{n\cdot\bar{n} \quad \bar{n}\cdot(k+p)}{\bar{n}\cdot k \,(k+p)^2\, k^2} - \frac{n\cdot\bar{n} \quad \bar{n}\cdot p}{\bar{n}\cdot k \,(\bar{n}\cdot p\, n\cdot k + p^2)\, k^2} \right]$$

↑ scaling limit

$$= \frac{i}{16\pi^2}\left[\left(\frac{2}{\epsilon_{IR}\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} \ln\frac{\mu^2}{-p^2} + \ln^2\frac{\mu^2}{-p^2} + \left(\frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}}\right)\ln\frac{\mu}{\bar{n}\cdot p} + \dots \right) - \left(\underbrace{\left(\frac{2}{\epsilon_{IR}} - \frac{2}{\epsilon_{UV}}\right)}_{\text{zero in pure-dim reg.}} \left(\frac{1}{\epsilon_{UV}} + \ln\frac{\mu^2}{-p^2} - \ln\frac{\mu}{\bar{n}\cdot p}\right) \right) \right]$$

subtraction:
- cancels $\bar{n}\cdot q \to 0$ IR singularity of first term,
- UV divergences come from $\bar{n}\cdot q \to \infty$ & are indep. of IR regulator
- here $\epsilon_{IR} = \epsilon_{UV}$ and ignoring subtraction gives correct answer

- for other less inclusive calculations (eg. jet algorithms) or other regulators (eg. $\Omega_\perp^2 \le k_\perp^2 \le \Lambda_\perp^2$, $\Omega_-^2 \le (k^-)^2 \le \Lambda_-^2$) the subtraction is crucial to avoid double counting (get correct IR structure) & have UV div. indep of IR regulator.

Renormalization in SLET & Summing Sudakov Logs

our eg.

$$C^{bare} = C + (Z_C - 1)\,C$$

need counter term $\quad Z_C = 1 - \frac{\alpha_s(\mu)\, C_F}{4\pi}\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu}{\omega} + \frac{5}{2\epsilon}\right)$

where current with Wilson Coeff was

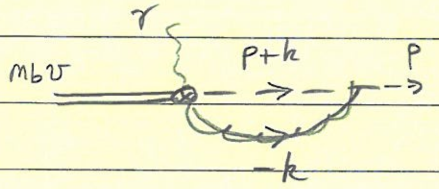
$$\int d\omega\, C(\omega)\, O(\omega) = \int d\omega\, C(\omega)\, \underbrace{\bar{\chi}_{n,\omega}}_{(\bar{\xi} W_n)\,\delta(\omega - \bar{P}^+)} \Gamma\, h_v$$

Running

In general we must be careful with integral over ω, which is the momentum of the product $(\bar{\xi}_n W)$.

But in our example ω is fixed by external kinematics
- it does not involve loop momenta

non-trivial example



$$C(\bar{n}\cdot(p+k)+\bar{n}\cdot(-k)) = C(\bar{n}\cdot p)$$

- $m_b v = p_\gamma + p = E_\gamma \bar{n} + p$ so $\bar{n}\cdot p = m_b = \omega$

Anom dim $\mu \, d/d\mu \, C^{\text{bare}} = 0 \Rightarrow \mu \frac{d}{d\mu} C(\omega,\mu) = \gamma_C(\omega,\mu)\, C(\omega,\mu)$

where

$$\gamma_C = -Z_C^{-1} \mu \frac{d}{d\mu} Z_C = \mu \frac{d}{d\mu} \frac{C_F \alpha_s(\mu)}{4\pi}\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu}{\omega} + \frac{5}{2\epsilon}\right)$$

$$= \frac{C_F \alpha_s(\mu)}{4\pi}\Big(\underbrace{\cancel{-\frac{2}{\epsilon}} - 4\ln\frac{\mu}{\omega} - 5}_{\text{from } \mu\, d/d\mu\, \alpha_s = -2\epsilon\alpha_s + \mathcal{O}(\alpha_s^2)} + \cancel{\frac{2}{\epsilon}}\Big)$$

$$= -\frac{\alpha_s(\mu)}{4\pi}\Big(\underset{\substack{\uparrow \\ \text{LL from} \\ \text{"cusp anom. dim"}}}{4C_F \ln\frac{\mu}{\omega}} + \underset{\substack{\uparrow \\ \text{part of NLL}}}{5C_F}\Big)$$

LL RGE

$$\mu \, d/d\mu \, C(\mu,\omega) = -\frac{\alpha_s(\mu) C_F}{\pi} \ln\left(\frac{\mu}{\omega}\right) C(\mu,\omega)$$

or
$$\frac{d \ln C(\mu,\omega)}{d\ln\mu} = -\frac{\alpha_s(\mu) C_F}{\pi} \ln\left(\frac{\mu}{\omega}\right)$$

Soln take boundary condition $C(\mu=\omega,\omega)=1$

"QED" α_s = fixed, $C_F=1$, Sudakov Exponential

$$C(\mu,\omega) = \exp\left[\frac{-\alpha}{2\pi}\ln^2\left(\frac{\mu}{\omega}\right)\right]$$

related to restrictions we've placed on radiation with our operators (& to probability of evolving without branching in a parton shower)

QCD $$d\ln\mu = \frac{d\alpha_s}{\beta[\alpha_s]} = \frac{-2\pi}{\beta_0}\frac{d\alpha_s}{\alpha_s^2} + \dots$$

$$\ln(\mu/\omega) = \frac{-2\pi}{\beta_0}\int_{\alpha_s(\omega)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\alpha_s^2}$$

$$\ln C(\mu,\omega) = -\frac{C_F}{\pi}\left(\frac{2\pi}{\beta_0}\right)^2\int_{\alpha_s(\omega)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\alpha_s^2}\,\alpha_s\int_{\alpha_s(\omega)}^{\alpha_s}\frac{d\alpha}{\alpha^2}$$

$$C(\mu,\omega) = \exp\left[\frac{-4\pi C_F}{\beta_0^2\,\alpha_s(\omega)}\left(\frac{1}{z}-1+\ln z\right)\right], \quad z=\frac{\alpha_s(\mu)}{\alpha_s(\omega)}$$

↑ running coupling effects

To discuss the order we're working to, look at series in

$$\ln C(\mu,\omega) \sim \underset{LL}{\alpha_s^k\ln^{k+1}} + \underset{NLL}{\alpha_s^k\ln^k} + \underset{NNLL}{\alpha_s^k\ln^{k-1}} + \dots$$

What Coefficients do we need to compute?

	tree-level	one-loop	2-loop	3-loop
LL	matching	$1/\epsilon^2$	—	—
NLL	matching	$1/\epsilon$	$1/\epsilon^2$	—
NNLL		matching	$1/\epsilon$	$1/\epsilon^2$

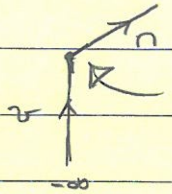
↑ differs from our earlier single log resummation case

- Where is the "cusp" in "cusp anomalous dimension"?

$$J_{SCET} = (\bar{\xi}_n W_n)\, \Gamma\, (Y_n^\dagger h_v)$$

Here h_v has $\mathcal{L}_{HQET} = \bar{h}_v\, i v \cdot D h_v$ and coincides with a timelike Wilson line. $h_v = Y_v h_v^{(0)}$, $\mathcal{L}_{HQET} = \bar{h}_v^{(0)} i v \cdot \partial h_v^{(0)}$

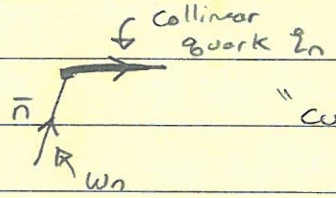
$Y_n^\dagger Y_v$ is



cusp is kink in Wilson line path

With light-like particle give a single $\ln(\mu/\omega)$ in anom. dimensions.

Also

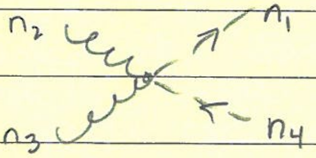


"cusp" where one part is not a Wilson line

- When will ω's be fixed by external kinematics?

If our operator only involves one building block (χ_n or $\mathcal{B}_{\perp n}^\mu$) for each collinear direction

eg $\int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \ \bar{\chi}_{n_1,\omega_1} \Gamma\, \mathcal{B}_{\perp n_2,\omega_2}^{\mu}\, \mathcal{B}^{\perp}_{n_3,\omega_3\,\mu}\, \chi_{n_4,\omega_4}\, C(\omega_1,\dots,\omega_4)$



again w_i's only involve momenta external to collinear loops

eg. where its not true, same n in two χ_n's

$$\int d\omega_1 d\omega_2 \quad \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega_2} \ C(\omega_1, \omega_2)$$

Here the w_i's will involve loop momenta [one combination is not fixed by momentum conservation] and we'll get anom. dimension equations with integrals

$$\mu \frac{d}{d\mu} C(\mu,\omega) = \int d\omega' \gamma(\mu,\omega,\omega')\, C(\mu,\omega')$$

Indeed, the above operator is responsible for several classic evolution equations

DIS	Altarelli – Parisi (DGLAP) evolution for PDF
$\gamma^* \pi^0 \to \pi^0$	Brodsky – Lepage "
$\gamma^* p \to \gamma p'$	Deeply Virtual Compton Scattering

Lets see how this works for the parton dist'n

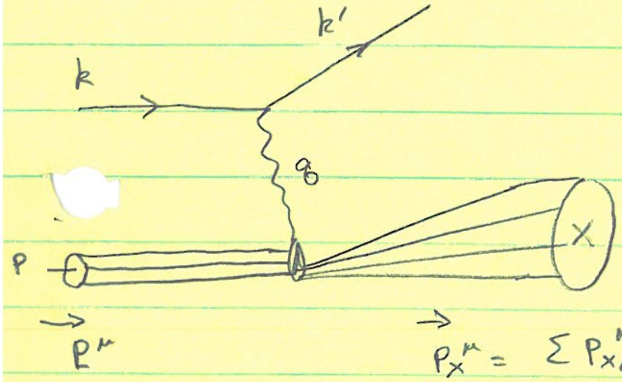
First we'll prove its the right operator by studying DIS factorization

(there is no page (139) in my notes)

$\boxed{\text{DIS}}$ A rich subject, only aspects related to QCD factorization are covered here using SCET

Refs: § 1.8 of Heavy Quark physics
Aneesh M.'s review: hep-ph/9204208
Bob J.'s review: hep-ph/9602236
paper: hep-ph/0202088 (for material below)

$e^- p \to e^- X$



$\vec{P_X}^\mu = \sum_{\text{all hadrons}} P_{X_i}^\mu$

$Q^2 \gg \Lambda^2$

$q^2 = -Q^2$, $x = \dfrac{Q^2}{2P\cdot q}$

$P_X^\mu = P^\mu + q^\mu$

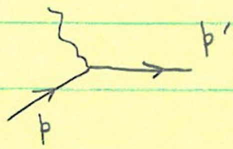
$P_X^2 = \dfrac{Q^2}{x}(1-x) + M_p^2$

regions

P_X^2	$\left(\frac{1}{x}-1\right)$	
$\sim Q^2$	~ 1	inclusive OPE
$\sim Q\Lambda$	$\sim \Lambda/Q$	endpt. region
$\sim \Lambda^2$	$\sim \Lambda^2/Q^2$	resonance region

$e^- p \to e^- p'$
$\uparrow$
eg. excited state

Parton Variables



struck quark carries some fraction ξ of proton momentum

$\bar{n}\cdot p = \xi\, \bar{n}\cdot P$ ← we'll see how to formulate ξ in QCD

$p'^2 \simeq Q^2\left(\dfrac{\xi}{x} - 1\right)$

Frames

Breit Frame

$$q^\mu = \frac{Q}{2}\left(\bar{n}^\mu - n^\mu\right)$$

$$P^\mu = \frac{n^\mu}{2}\,\bar{n}\cdot P + \frac{\bar{n}^\mu M_p^2}{2\,\bar{n}\cdot P} = \frac{n^\mu}{2}\,\frac{Q}{x} + \dots \qquad \underline{\text{collinear}}$$

$$P_X{}^\mu = \frac{\bar{n}^\mu}{2}\,Q + \frac{n^\mu}{2}\,\frac{Q(1-x)}{x} + \dots \qquad \underline{\text{hard}}$$

Proton is made of collinear quarks and gluons

$$\left(\begin{array}{lll} \text{Rest Frame} & P^\mu = \frac{m_p}{2}\left(n^\mu + \bar{n}^\mu\right) & \text{soft} \\ & q^\mu = \frac{\bar{n}^\mu}{2}\,\frac{Q^2}{m_p x} - \frac{n^\mu}{2}\,m_p x + \dots & \\ & P_X{}^\mu = \text{sum} & \text{"collinear"}\ P_X^2 \sim Q^2 \end{array}\right)$$

Like $B \to X_c e \nu$ we can write cross-section in terms of leptonic & hadronic tensors

$$d\sigma = \frac{d^3k'}{2|\vec{k}'|}\,\frac{e^4}{sQ^4}\,L^{\mu\nu}(k,k')\,W_{\mu\nu}(P,q)$$

we'll look at spin-avg. case

$$W_{\mu\nu} = \frac{1}{\pi}\,\text{Im}\,T_{\mu\nu}$$

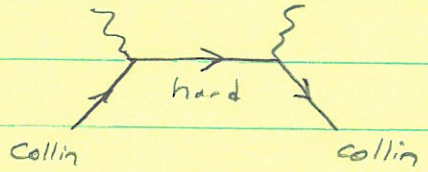
$$T_{\mu\nu} = \frac{1}{2}\sum_{spin}\langle p|\,\hat{T}_{\mu\nu}(q)\,|p\rangle$$

$$\hat{T}_{\mu\nu} = i\int d^4x\; e^{iq\cdot x}\; T\left[\underset{\substack{\uparrow \\ \text{e.m. currents}}}{J_\mu(x)}\, J_\nu(0)\right]$$

$$T_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(x,Q^2) + \left(p_\mu + \frac{q_\mu}{2x}\right)\left(p_\nu + \frac{q_\nu}{2x}\right) T_2(x,Q^2)$$

satifies current conservation, P, C, T, etc.

Want imaginary part of forward scattering



First Match onto SCET ops.

at L.O. :

$$\hat{T}^{\mu\nu} = \frac{g_\perp^{\mu\nu}}{Q}\left(O_1^{(i)} + \frac{O_1^{g}}{Q}\right) + \frac{(n^\mu+\bar{n}^\mu)(n^\nu+\bar{n}^\nu)}{Q}\left(O_2^{(i)} + \frac{O_2^{g}}{Q}\right)$$

$O(\lambda^2)$ operators

flavor = u, d, ...

$$O_j^{(i)} = \bar{\xi}^{(i)}_{n,p'} W \frac{\bar{n}\!\!\!/}{2} C_j^{(i)}(\bar{P}_+, \bar{P}_-) W^\dagger \xi^{(i)}_{n,p}$$

$$O_j^{(g)} = \mathrm{tr}\left[W^\dagger B_\perp^\lambda W C_j^{g}(\bar{P}_+, \bar{P}_-) W^\dagger B_{\perp\lambda} W\right]$$

where $ig B_\perp^\lambda \equiv [i\bar{n}\cdot D_c, iD_{\perp c}^\lambda] \sim \lambda \sim \xi_n$

$\bar{P}_\pm = \bar{P}^\dagger \pm \bar{P}$

$O_j^{(i)}$ will lead to quark, anti-quark p.d.f.'s

O_j^{g} " " gluon p.d.f.'s

Quark contribution in detail:

$$O_j^{(i)} = \int d\omega_1 d\omega_2\, C_j^{(i)}(\omega_+,\omega_-) \left[\underset{\substack{\uparrow \\ \delta(\omega_1 - \bar{P}^\dagger)}}{(\bar{\xi}_n W)_{\omega_1}} \frac{\bar{n}\!\!\!/}{2} \underset{\substack{\uparrow \\ \delta(\omega_2 - \bar{P})}}{(W^\dagger \xi_n)_{\omega_2}} \right]$$

$$\omega_\pm = \omega_1 \pm \omega_2$$

coord space $f_{i/p}(z) = \int dy\; e^{-i 2 z \bar{n}\cdot p y} \langle p | \bar{\xi}(y) W(y,-y) \bar{n}\!\!\!/ \, \xi(y) | p \rangle$

parton distn for quark i in proton p

$\bar{f}_{i/p}(z) = - f_{i/p}(-z)$ for anti-quark

mom. space $\langle p_n | (\bar{\xi}_n W)_{\omega_1} \bar{n}\!\!\!/ \, (W^{\dagger} \xi_n)_{\omega_2} | p_n \rangle = 4 \bar{n}\cdot p \int_0^1 d\xi \; \delta(\omega_-)$

$$* \left[\delta(\omega_+ - 2\xi \bar{n}\cdot p) f_{i/p}(\xi) - \delta(\omega_+ + 2\xi \bar{n}\cdot p) \bar{f}_{i/p}(\xi) \right]$$

recall positive $\omega_1 = \omega_2$ gives particles; negative $\omega_1 = \omega_2$ gives anti-particles

$\underbrace{(\bar{\xi}_n W)_\omega}_{\text{a parton}} \bar{n}\!\!\!/ \, (W^{\dagger} \xi_n)$ is a number operator for collinear quarks with momentum ω

[If we tried to couple usoft or soft gluons to this op. its a singlet so they decouple, more later]

Charge Conjugation

$$C_j^{(i)}(-\omega_+, \omega_-) = - C_j^{(i)}(\omega_+, \omega_-)$$

$\omega_1 \Leftrightarrow -\omega_2$

- relates Wilson-Coeff for quarks & anti-quarks at operator level
- Only need matching for quarks
- δ-functions set $\omega_- = 0$, $\omega_+ = 2\xi \bar{n}\cdot p = 2Q \frac{\xi}{x}$

Relate basis

$$\frac{1}{\pi}\,\mathrm{Im}\,T_1 = \int [d\omega] \;\; \frac{-1}{Q}\left(\frac{1}{\pi}\,\mathrm{Im}\,C_1(\omega)\right) \langle O^{(i)}(\omega) \rangle$$

$$\frac{1}{\pi}\,\mathrm{Im}\,T_2 = \int [d\omega] \;\left(\frac{4x}{Q}\right)^2 \frac{1}{Q}\,\frac{1}{\pi}\,\mathrm{Im}\left(C_2(\omega) - \frac{C_1(\omega)}{4}\right) \;\; \langle O^{(i)}(\omega) \rangle$$

Define $H_j(z) = \frac{\mathrm{Im}}{\pi}\; C_j(\underset{\omega_+}{2Qz},\, \underset{\omega_-}{0},\, Q^2,\, \mu^2)$ $\quad z > 0$

(use charge conj for $H_j(z<0)$) $\quad$ do $\omega_\pm$ with δ-functions

$$T_1(x, Q^2) = \frac{-1}{x} \int_0^1 d\xi \;\; H_1^{(i)}\!\left(\frac{\xi}{x}\right) \left[f_{i/p}(\xi) + \bar{f}_{i/p}(\xi) \right]$$

$$T_2(x, Q^2) = \frac{4x}{Q^2} \int_0^1 d\xi \left(4 H_2^{(i)}\!\left(\frac{\xi}{x}\right) - H_1^{(i)}\!\left(\frac{\xi}{x}\right)\right) \left[f_{i/p}(\xi) + \bar{f}_{i/p}(\xi) \right]$$

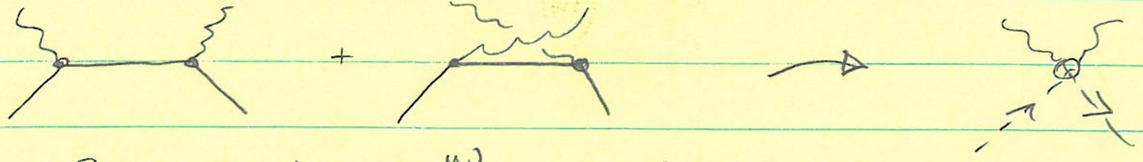
- this is factorization for DIS (to all order in α_s) into
  computable coefficients H_i
  universal non-pert. functions $f_{i/p}$, $\bar{f}_{i/p}$
  (show up in many processes)
- Coefficients C_j were dimensionless and can only have $\alpha_s(\mu)\ \ln(\mu/Q)$ dependence on Q
  → Bjorken scaling

[Analysis valid to LO in $\frac{\Lambda^2}{Q^2}$]

- $H_i(\mu)$ $f_{i/p}(\mu)$ traditionally this μ-dependence is called the "factorization-scale" $\mu = \mu_F$
  & one also has "renorm. scale" $\alpha_s(\mu = \mu_R)$

In SCET the μ is just the ren. scale in SCET. We have new UV. divergences associated with running of p.d.f., along with running for $\alpha_s(\mu)$.

- Tree Level Matching
  (upon which a lot of intuition is based)



find just $g_\perp^{\mu\nu}$ ie $C_2 = 0$

↳ Callan-Gross relation that $W_1 / W_2 = Q^2/4x^2$

$$C_1(\omega_+) = 2e^2 \underset{\overset{\uparrow}{\text{charges}}}{Q_i^2} \left[\frac{Q}{(\omega_+ - 2Q) + i\epsilon} - \frac{Q}{-(\omega_+ + 2Q) + i\epsilon} \right]$$

$$H_1\left(\frac{\xi}{x}\right) = -e^2 Q_i^2 \, \delta\left(\frac{\xi}{x} - 1\right)$$ gives parton-model interpretation $\xi = x$

One-Loop Renormalization of PDF

one δ-function $\searrow$ $\qquad$ $\swarrow$ proton state, momentum P_n^-

$$f_q(\xi) = \langle P_n | \bar{\chi}_n(0) \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega}(0) | P_n \rangle \qquad \text{where } \xi = \frac{\omega}{P_n^-}$$

mass dimension $-1 \quad + \frac{3}{2} \quad + \frac{3}{2} - 1 \quad -1 \quad = 0$

λ dimension $-1 \quad +1 \quad +1 \quad -1 \quad = 0$

$$\frac{d^3p}{2E_p} = \frac{dp^-}{2p^-} d^2p_\perp \qquad \text{states: } \underbrace{\langle P_n(p)}_{\lambda^{-1}} | \underbrace{P_n(p')\rangle}_{\lambda^{-1}} = \underbrace{2p^- \delta(p^- - p'^-)}_{\lambda^0} \underbrace{\delta^2(p_\perp - p_\perp')}_{\lambda^{-2}}$$

$$p^- = \underbrace{(p_\perp^2 + p_z^2)^{1/2}}_{E_p} + p_z$$

Loops can change ω (or ξ). $f_q(\xi)$ mixes with $f_q(\xi')$ which in general is what we expect for operators with same quantum #'s. Loops also mix parton types $i = q, g$

$$f_i^{\text{bare}}(\xi) = \int d\xi' \; Z_{ij}(\xi, \xi') \; f_j(\xi', \mu)$$

$\uparrow$ μ independent $\qquad$ $\uparrow$ $\frac{1}{\epsilon_{UV}}$ & $\alpha_s(\mu)$ in $\overline{MS}$ $\qquad$ $\uparrow$ UV finite, but IR div. encodes Λ_{QCD} effects

gives

$$\mu \, d/d\mu \; f_i(\xi, \mu) = \int d\xi' \; \gamma_{ij}(\xi, \xi') \; f_j(\xi', \mu)$$

$$\gamma_{ij} \equiv - \int d\xi'' \; Z_{ii'}^{-1}(\xi, \xi'') \, \mu \frac{d}{d\mu} Z_{i'j}(\xi'', \xi')$$

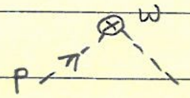
like matrix product in ξ vars. too.

1-loop: $\quad Z_{ii'}^{-1}(\xi, \xi'') = \delta_{ii'} \, \delta(\xi - \xi'')$

$$\gamma_{ij}^{\text{1-loop}} = - \mu \, d/d\mu \left[Z_{ij}(\xi, \xi') \right]^{\text{1-loop}}$$

Calculations

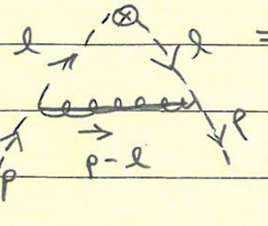
tree level



$$= \sum_{\text{spin}} \bar{u}_n \frac{\bar{n}\!\!\!/}{2} u_n \, \delta(\omega - p^-) = p^- \delta(\omega - p^-) = \delta(1 - \omega/p^-)$$

one-loop, use offshellness $p^2 = p^+ p^- \neq 0$ to regulate IR

ⓐ

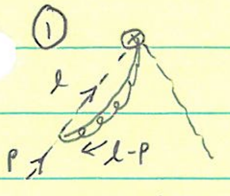


$$= -i g^2 C_F \int \frac{d^d\ell \; p^- (d-2) \, \ell_\perp^2}{[\ell^2 + i0]^2 [(\ell - p)^2 + i0]} \delta(\ell^- - \omega) \, \mu^{2\epsilon} \frac{e^{\epsilon \gamma_E}}{(4\pi)^\epsilon} \quad \leftarrow \text{after simplifying numerator}$$

$$= \frac{2 g^2 C_F}{(4\pi)^2} (1-\epsilon)^2 \, \Gamma(\epsilon) \, e^{\epsilon \gamma_E} \, (1-z) \, \theta(z) \, \theta(1-z) \left(\frac{A}{\mu^2} \right)^{-\epsilon}$$

$$= \frac{\alpha_s C_F}{\pi} (1-z) \, \theta(z) \, \theta(1-z) \left[\frac{1}{2\epsilon} - 1 - \frac{1}{2} \ln \frac{A}{\mu^2} \right], \qquad A \equiv -p^+ p^- z(1-z), \quad z = \omega/p^-$$

①



+ symmetric graph

two contractions $$\bar{\xi}_n \underbrace{W \overset{\text{cut}\,\downarrow}{\delta} \overbrace{W^{\dagger} \xi_n}^{\text{virtual}}}_{\text{real}}$$

$$= 2 i g^2 C_F \int \frac{\bar{d}^d l \;\; \bar{u}_n \frac{\bar{n}\!\!\!/}{2} \frac{n\!\!\!/}{2} \bar{n}\cdot l \, \bar{n}\!\!\!/ u_n}{(l^- - p^-)(l^2)(l-p)^2} \Big[\overset{\text{real}}{\delta(l^- - \omega)} - \overset{\text{virtual}}{\delta(p^- - \omega)} \Big]$$

$$= \frac{C_F \alpha_s(\mu)}{\pi} e^{\epsilon \gamma_E} \Gamma(\epsilon) \Big[\frac{z\, \theta(z)\, \theta(1-z)}{(1-z)^{1+\epsilon}} \Big(\frac{-p^- p^+ z - i0}{\mu^2} \Big)^{-\epsilon}$$
$$- \delta(1-z) \Big(\frac{-p^+ p^- - i0}{\mu^2} \Big)^{-\epsilon} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)} \Big]$$

Distribution Identity

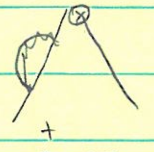
$$\frac{\theta(1-z)}{(1-z)^{1+\epsilon}} = -\frac{\delta(1-z)}{\epsilon} + \mathcal{L}_0(1-z) - \epsilon\, \mathcal{L}_1(1-z) + \cdots$$

<u>plus</u>-<u>functions</u> $\mathcal{L}_n(x) = \Big[\frac{\theta(x) \ln^n x}{x} \Big]_+$

$$\int_0^1 dx\, \mathcal{L}_n(x) = 0 \;, \quad \int_0^1 dx\, \mathcal{L}_n(x)\, g(x) = \int_0^1 dx \frac{\ln^n x}{x} \big[g(x) - g(0) \big]$$

- $1/\epsilon^2$ terms in real & virtual terms cancel
- remaining $1/\epsilon$ is UV

$$= \frac{C_F \alpha_s(\mu)}{\pi} \Big[\big\{ \delta(1-z) + z\, \theta(z)\, \mathcal{L}_0(1-z) \big\} \Big(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^+ p^- z - i0} \Big) - z\, \mathcal{L}_1(1-z)\, \theta(z)$$
$$+ \delta(1-z) \Big(2 - \frac{\pi^2}{6} \Big) \Big]$$



+ symm.

$$= \delta(1-z)\,(Z_\psi - 1) = \frac{\alpha_s C_F}{\pi} \Big[-\frac{1}{4\epsilon} - \frac{1}{4} - \frac{1}{4} \ln \Big(\frac{\mu^2}{-p^+ p^- - i0} \Big) \Big] \delta(1-z)$$

[We'll ignore

which mixes O^f_{glue} & O^f_{quark}

this mixing is needed if O_{quark} is flavor singlet, but not for non-singlet like $\bar{u}_n(\cdots) d_n$

sum of SCET graphs $= f^{\text{bare}}_{q/q}(z)^{\substack{\text{up to}\\ \text{1-loop}}} = \delta(1-z)$

$$+ \frac{C_F\,\alpha_s(\mu)}{\pi}\left[\left\{\frac{3}{4}\delta(1-z) + z\,\theta(z)\,\mathcal{L}_0(1-z) + \frac{(1-z)}{2}\theta(z)\theta(1-z)\right\}\left(\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{-p^+p^-}\right)\right)\right.$$

$$\left. + \text{finite function of } z\right]$$

$$= \delta(1-z) + \frac{C_F\,\alpha_s(\mu)}{\pi}\left[\frac{1}{2}\left(\frac{1+z^2}{1-z}\right)_+ \frac{1}{\epsilon} + \cdots\right]$$

$$= \int d\xi'\; Z_{qq}(\xi,\xi')\; f_j(\xi',\mu)$$

$$= \underbrace{\int d\xi'\;\frac{1}{\xi'}\,Z_{qq}\!\left(\frac{\xi}{\xi'}\right)}\; f_j(\xi',\mu) \qquad \xi = z \text{ for quark state}$$

RPI-III invariant (ratios)
& indep of proton momentum p^- (renormalization indep. of state)

$$= \delta(1-z) + \int\frac{d\xi'}{\xi'}\left[Z^{(1)}_{qq}\!\left(\frac{\xi}{\xi'}\right) f^{(0)}_q(\xi',\mu) + Z^{(0)}_{qq}\!\left(\frac{\xi}{\xi'}\right) f^{(1)}_q(\xi',\mu)\right]$$

$$= \delta(1-z) + \underbrace{Z^{(1)}_{qq}(z)}_{1/\epsilon\text{ part}} + \underbrace{f^{(1)}_q(z,\mu)}_{\text{rest}}$$

$$\gamma_{qq}(\xi,\xi') = -\mu\frac{d}{d\mu}\,\frac{1}{\xi'}\,\frac{C_F\,\alpha_s(\mu)}{2\pi}\left(\frac{1+z^2}{1-z}\right)_+ , \qquad \mu\frac{d}{d\mu}\alpha_s = -2\epsilon\,\alpha_s + \cdots$$

$$= \frac{C_F\,\alpha_s(\mu)}{\pi}\,\frac{\theta(\xi'-\xi)\,\theta(1-\xi')}{\xi'}\left(\frac{1+z^2}{1-z}\right)_+ \qquad z = \frac{\xi}{\xi'}$$

Quark Splitting Function, One-loop PDF anom. dim.

$\boxed{\text{SCET}_{\text{I}}}$ hard $p^\mu \sim (Q, Q, Q)$

collin $(Q\lambda^2, Q, Q\lambda)$

usoft $(Q\lambda^2, Q\lambda^2, Q\lambda^2)$

↑ non-trivial communication between sectors

$\boxed{\text{SCET}_{\text{II}}}$ (still to come)

hard (Q, Q, Q)

hard-collin $(Q\lambda, Q, \sqrt{Q\lambda})$

collin $(Q\lambda^2, Q, Q\lambda)$

soft $(Q\lambda, Q\lambda, Q\lambda)$

Results for observables which tie d.o.f. together are "Factorization Theorems"

They can involve convolutions between objects defined by different degrees of freedom (hard, soft, jet, hadron dist'n functions) as long as they have some power counting for the convoluted momenta

Processes

- $\gamma^* \gamma \to \pi^0$ $\pi - \gamma$ form factor at $Q^2 \gg \Lambda^2$ for γ^*

  Breit frame $q^\mu = \frac{Q}{2}(n^\mu - \bar{n}^\mu)$, $p_\gamma^\mu = E\,\bar{n}^\mu$

  $$p_\pi^\mu = \frac{Q}{2} n^\mu + \underbrace{\left(E - \frac{Q}{2}\right)}_{m_\pi^2/2Q} \bar{n}^\mu$$

  pion = collinear in n-direction (SCET$_{II}$)

- $\gamma^* M \to M'$ $M - M'$ (meson) form factor $Q^2 \gg \Lambda^2$ for γ^*

  $M =$ collinear in n

  $M' =$ " " $\bar{n}$ (say) (SCET$_{I}$)

- $B \to D\pi$ Matrix Elt. of 4-quark operators

  $Q = \{m_b, m_c, E_\pi\} \gg \Lambda$

  B, D are soft $p^2 \sim \Lambda^2$, π-collinear (SCET$_{II}$)

- DIS Structure Functions at $Q^2 \gg \Lambda^2$

  $e^- p \to e^- X$ and $1 - x \gg \Lambda/Q$ (ie not near endpts in Bjorken x)

  Breit frame: proton n-collinear, X - hard

  $\left(\begin{matrix}\text{SCET}_{II} \\ \text{or} \\ \text{SCET}_{I}\end{matrix}\right)$

- Drell-Yan $\frac{d\sigma}{dQ^2}$ Q^2 = inv. mass of $l^+ l^- \gg \Lambda^2$

  $p\bar{p} \to l^+ l^- X$

  p - n-collin, $\bar{p}$ - $\bar{n}$-collin, X - hard

- $e^+ e^- \to$ jets

  $p\bar{p} \to$ jets • depends on observable we formulate

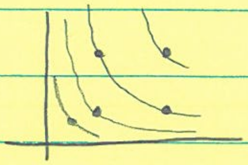
  $pp \to$ jets eg two jets n-collin jet

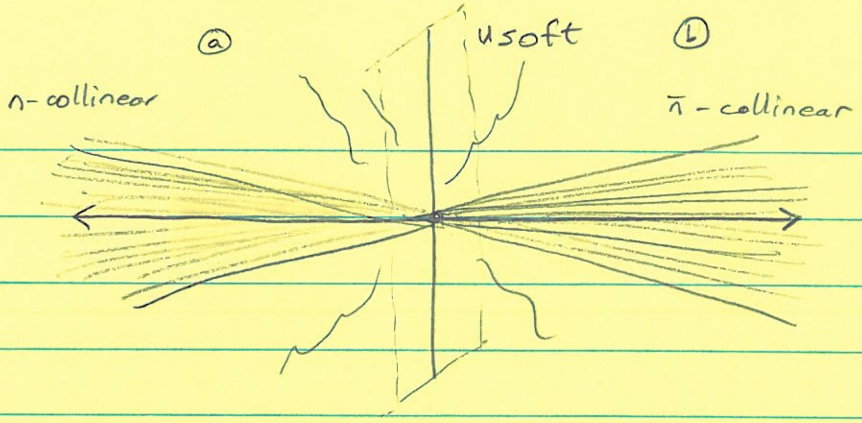
  $\bar{n}$-collin jet

etc.

$\boxed{e^+e^- \to \text{dijets}}$

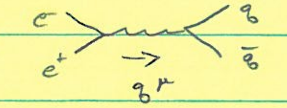
SCET$_{I}$





$e^+e^- \to \gamma^*$ or $Z^* \to X_n X_{\bar n} X_{usoft}$ (e^+e^-) cm frame

Scales



- $q^2 = Q^2$ hard $\mu_h \sim Q$

- Hemisphere invariant mass divide $P_X^\mu = P_{Xa}^\mu + P_{Xb}^\mu$

$M^2 \equiv (P_{Xa}^\mu)^2 = \left(\sum_{i\varepsilon a} (p_i^\mu)\right)^2$, $\bar{M}^2 = \left(\sum_{i\varepsilon b} p_i^\mu\right)^2$

jet $\to$ $M^2 \ll Q^2$

n- collinear $Q(\lambda^2, 1, \lambda)$ $\mu_J \sim M$

n̄- collinear $Q(1, \lambda^2, \lambda)$ $\lambda = M/Q$

- Usoft Radiation
  - uniform in space
  - communication btwn jets
  - eikonal

energy $\sim Q\lambda^2 = M^2/Q$

$\mu_S \sim M^2/Q$

$\nearrow$ $M^2/Q \gg \Lambda_{QCD}$ perturbative

"tail region"

$\mu_h \gg \mu_J \gg \mu_S \gg \Lambda_{QCD}$

$\searrow$ $M^2/Q \sim \Lambda_{QCD}$ non-perturbative

"peak region"

$\mu_h \gg \mu_J \gg \mu_S \sim \Lambda_{QCD}$

In tail region we have power corrections

$\left(\frac{\Lambda_{QCD}}{\mu_S}\right)^K \ll 1$. Leading order cross-section perturbative.

In peak region $(\Lambda_{QCD}/\mu_S)^K \sim 1$ (any K) $\to$ non-pert. soft function

Other Power Corrections

- μ_S/μ_J "Kinematic" expansion of kinematic variables
- Λ_{QCD}/μ_h hard power corr. (Hwk)
- $\Lambda_{QCD}/\mu_J = \frac{\Lambda_{QCD}}{\mu_S} \frac{\mu_S}{\mu_J}$ not independent

color singlet

$$\underline{\text{Current}} \quad \overset{QCD}{J^\mu = \bar\psi \Gamma^\mu \psi} \to (\bar\xi_n W_n)_\omega \Gamma^\mu (W_{\bar n}^\dagger \xi_{\bar n})_{\bar\omega}$$

$$= (\bar\xi_n W_n)_\omega \Gamma^\mu (Y_n^\dagger Y_{\bar n})_{\bar\omega} (W_{\bar n}^\dagger \xi_{\bar n})$$ field redefn

Kinematics $q^\mu = P_{X_n}^\mu + P_{X_{\bar n}}^\mu + P_S^\mu$

large $\bar n \cdot q = Q = \bar n \cdot P_{X_n} + \overset{\text{small}}{\cdots}$ $\omega = Q$

$n \cdot q = Q = n \cdot P_{X_{\bar n}} + \cdots$ $\bar\omega = Q$

momentum conservation is strong enough that there are no convolutions in $\omega, \bar\omega$

Factorize the Cross-Section

QCD $\sigma = \sum_X^{res} (2\pi)^4 \delta^4(q - P_X) L_{\mu\nu} \langle 0 | J^{\nu\dagger}(0) | X \rangle \langle X | J^\mu(0) | 0 \rangle$

↑ restricted to dijet X states

SCET allows us to move restrictions into operators

$|X\rangle = |X_n\rangle |X_{\bar n}\rangle |X_s\rangle$

$\bar 3$ rep 3-rep

$$\sigma = N_0 \sum_{\vec n} \sum_{X_n, X_{\bar n}, X_S}^{res'} (2\pi)^4 \delta^4(q - P_{X_n} - P_{X_{\bar n}} - P_S) \langle 0 | \overline{Y}_{\bar n} Y_n | X_S \rangle \langle X_S | Y_n^\dagger \overline{Y}_{\bar n}^\dagger | 0 \rangle$$

$$* \; |C(Q)|^2 \langle 0 | \bar{n}\!\!\!/ \chi_{n,Q} | X_n \rangle \langle X_n | \bar\chi_n | 0 \rangle$$

$$\langle 0 | \bar\chi_{\bar n, Q} | X_{\bar n} \rangle \langle X_{\bar n} | n\!\!\!/ \chi_{\bar n} | 0 \rangle$$

all orders in α_s

$+ \ldots$ ← "other" power corr.

res' : we must still measure enough things about X to ensure its a dijet state

Measure hemisphere masses $M^2, \bar{M}^2$

$$1 = \int dM^2 \, d\bar{M}^2 \;\; \delta\big(M^2 - (\underset{\substack{\uparrow\\ \text{n-collinear}\\ \text{total mom. in hemisphere (a)}}}{P_n} + \underset{\substack{\uparrow\\ \text{soft momenta}}}{k_s^a})^2\big) \;\; \delta\big(\bar{M}^2 - (P_{\bar{n}} + k_s^b)^2\big)$$

expand $\delta(M^2 - P_n^2 - P_n^- (k_s^a)^+ + \cdots) = \delta(M^2 - Q(P_n^+ + k_s^{a+})) + \cdots$

$\dfrac{d\sigma}{dM^2 d\bar{M}^2}$ has these δ's under $\sum\limits_X$

- factor measurements:

eg. $\delta(M^2 - Q(P_n^+ + k_s^{a+})) = \int dk^+ d\ell^+ \; \delta(M^2 - Q(k^+ + \ell^+)) \underbrace{\delta(k^+ - P_n^+)}_{\substack{\text{with}\\ \text{n-collinear}\\ \text{matrix elt}}} \underbrace{\delta(\ell^+ - k_s^{a+})}_{\substack{\text{with}\\ \text{soft}}}$

- factor $\delta^4(q - P_{X_n} - P_{X_{\bar{n}}} - P_s)$ too

- write δ's in Fourier space $\delta(k^+ - P_n^+) = \int \frac{dx^-}{2} \, e^{ix^- k^+/2} \underbrace{e^{-ix^- P_n^+/2}}_{\substack{\text{shifts field}\\ \text{to } \chi_{n,Q}(x^-)}}$

etc

After some work we get factorized result

$$\frac{d\sigma}{dM^2 d\bar{M}^2} = \sigma_0 \, |C(Q)|^2 \int dk^+ d\ell^+ dk^- d\ell^- \; \delta(M^2 - Q(k^+ + \ell^+)) \; \delta(\bar{M}^2 - Q(k^- + \ell^-))$$

$$* \sum_{X_n} \frac{1}{2\pi} \int d^4x \; e^{ik^+ x^-/2} \; \mathrm{tr} \, \langle 0 | \frac{\bar{n}\!\!\!/}{4N_c} \chi_{n,Q}(x) | X_n \rangle \langle X_n | \bar{\chi}_n(0) | 0 \rangle$$

$$* \sum_{X_{\bar{n}}} \frac{1}{2\pi} \int d^4y \; e^{ik^- y^+/2} \; \mathrm{tr} \, \langle 0 | \bar{\chi}_{\bar{n},Q}(y) | X_{\bar{n}} \rangle \langle X_{\bar{n}} | \frac{n\!\!\!/}{4N_c} \chi_{\bar{n}}(0) | 0 \rangle$$

$$* \underbrace{\sum_{X_s} \frac{1}{N_c} \, \delta(\ell^+ - k_s^{a+}) \, \delta(\ell^- - k_s^{b-}) \; \mathrm{tr} \, \langle 0 | \bar{Y}_{\bar{n}} Y_n | X_s \rangle \langle X_s | Y_n^\dagger \bar{Y}_{\bar{n}}^\dagger | 0 \rangle}$$

Matrix Elements • $S_{\text{hemi}}(\ell^+, \ell^-)$ soft function

encodes both momentum scales $\ell^\pm \sim \dfrac{M^2}{Q}$ and $\ell^\pm \sim \Lambda_{QCD}$

- $$\sum_{X_n} \mathrm{tr} \langle 0| \frac{\bar{n}\!\!\!/}{4N_c} \chi_{n,Q}(x) |X_n\rangle \langle X_n| \bar{\chi}_n(0)|0\rangle = Q\int \frac{d^4r}{(2\pi)^3} e^{-ir\cdot x} J(Qr^+)$$

$$= \underbrace{\delta(x^+)\,\delta^2(x_\perp)}_{\text{due to collinear multi-pole expn}} \int dr^+ e^{-ir^+x^-/2} \underbrace{J(Qr^+)}_{\text{jet function}}$$

- Same for $\sum_{X_{\bar{n}}} \mathrm{tr}$ ---

- $H(Q) \equiv |C(Q)|^2$ hard function

All Together

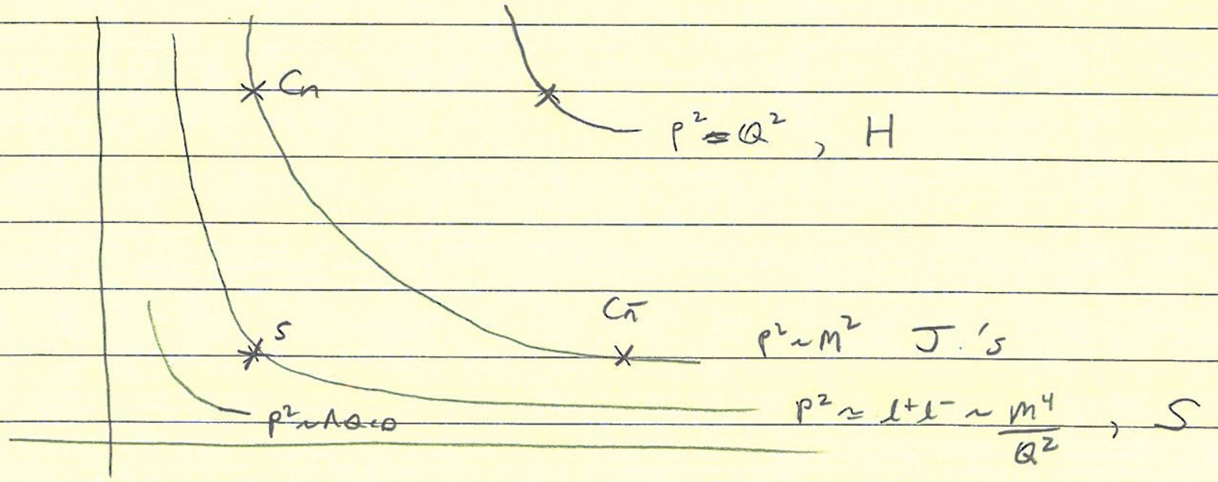
$$\frac{d\sigma}{dm^2 d\bar{m}^2} = \sigma_0 \, H(Q) \int dl^+ dl^- \, J(m^2 - Ql^+)\, J(\bar{m}^2 - Ql^-)\, S(l^+, l^-)$$

using renormalized objects on RHS ($C_i^{\text{bare}} O^{\text{bare}} = C(\mu)\, O(\mu)$)

$$= \sigma_0 \, H(Q,\mu) \int dl^+ dl^- \, J(m^2 - Ql^+, \mu)\, J(\bar{m}^2 - Ql^-, \mu)\, S(l^+, l^-, \mu)$$

dijet factorization theorem for hemisphere masses

Note



The functions H, J, S have α_s expansions without large logs only if each is evaluated at different scale μ

Soft Function OPE

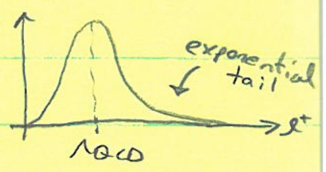
$$S_{hemi}(\ell^+,\ell^-) = \int d\ell'^{\pm}\; S^{pert}_{hemi}(\ell^+-\ell^{+\prime},\ell^--\ell^{-\prime})\; F(\ell^{+\prime},\ell^{-\prime})$$

$\uparrow$

power tail

$$\frac{(\ln \ell^+/\mu)^k}{\ell^+}$$

$\sim \Lambda_{QCD}$ effects



Thrust $\quad T = \max\limits_{\hat t} \dfrac{\sum_i |\vec p_i\cdot\hat t|}{\sum_i |\vec p_i|}$

$\frac{1}{2} \le T \le 1$

$0 \le \tau \le 1/2$

$\tau = 1-T$

for dijets $\quad \tau = \dfrac{M^2+\bar M^2}{Q^2}$ ← symmetric projection

$\downarrow \qquad \downarrow$

$$\frac{d\sigma}{d\tau} = \sigma_0\, H(Q,\mu)\, Q\int d\ell\;\; J_\tau(Q^2\tau - Q\ell,\mu)\; S_\tau(\ell,\mu)$$

$p^2 \sim Q^2 \qquad$ jet $p^2\sim Q^2\tau \qquad$ usoft $p^2 \sim Q^2\tau^2$

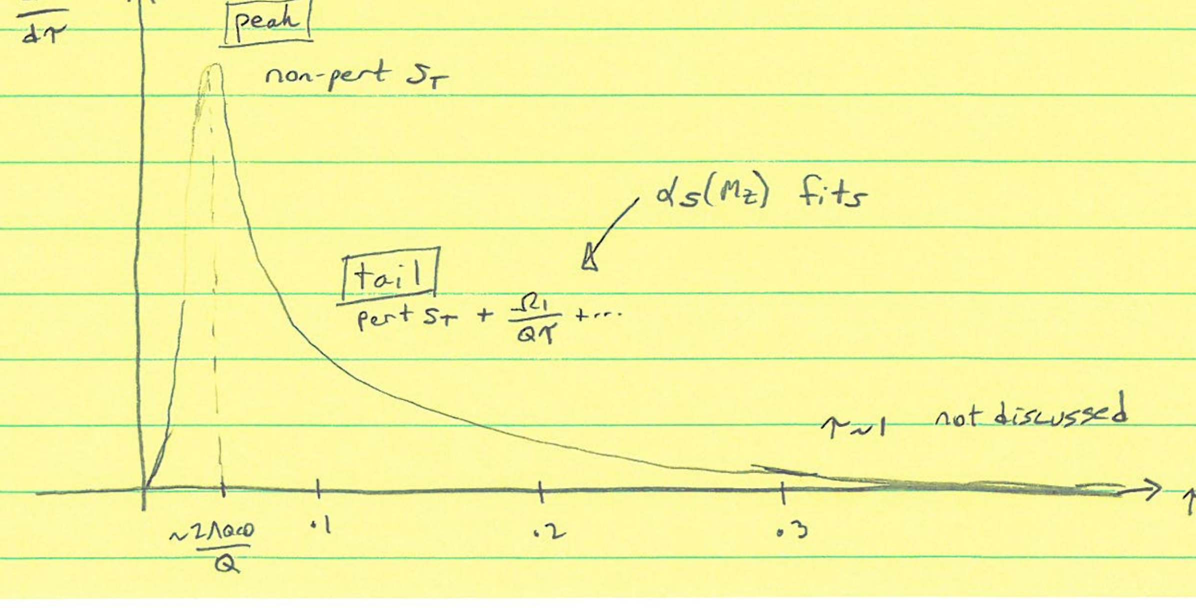
$$Q^2 \gg Q^2\tau \gg Q^2\tau^2$$

$$\mu_h^2 \gg \mu_J^2 \gg \mu_S^2 \;\overset{\gg}{\underset{\sim}{\text{or}}}\; \Lambda_{QCD}^2$$

sum logs

schematically: $\quad \dfrac{d\sigma}{d\tau} \sim \sum\limits_{n,m} \dfrac{\alpha_s^n \ln^m\tau}{\tau}$ + non-perturbative effects in F

+ power corrections



Pert. Results

- match quark form factor

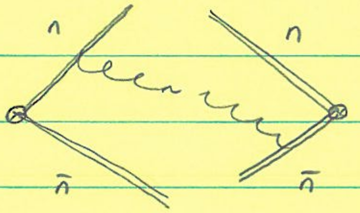
$$C(Q,\mu) = 1 + \frac{C_F \alpha_s(\mu)}{4\pi}\left[3\ln^2\left(\frac{-Q^2}{\mu^2}\right) - \ln\left(\frac{-Q^2}{\mu^2}\right) - 8 + \frac{\pi^2}{6}\right]$$

$$H = |C|^2$$

- (Renormalized) Jet Function

$$J_n(s,\mu) = \delta(s) + \frac{\alpha_s(\mu)\, C_F}{4\pi}\left[\# \delta(s) + \# \left[\frac{\mu^2\,\theta(s)}{s}\right]_+ + \# \left[\frac{\mu^2 \ln(\mu^2/s)\,\theta(s)}{s}\right]_+\right]$$

- Pert. Soft Fn



etc.

$$S^{pert}(\ell^+,\ell^-) = \left\{ \delta(\ell^+) + \frac{\alpha_s C_F}{4\pi}\left[\# \delta(\ell^+) + 0 \left[\frac{\mu}{\ell^+}\theta(\ell^+)\right] + \# \left[\frac{\mu}{\ell^+}\ln\left(\frac{\ell^+}{\mu}\right)\right]_+ \right]\right.$$

$$\left. * \left\{ \delta(\ell^-) + \frac{\alpha_s C_F}{4\pi}\left[\text{ditto } \ell^+ \to \ell^- \right]\right\}\right\}$$

C renormalizes multiplicatively $\quad C^{bare} = Z_C\, C = C + (Z_C - 1)\, C$

$$\mu\, d/d\mu\; C(Q,\mu) = \gamma_C(Q,\mu)\; C(Q,\mu)$$

J, S renormalize like PDF, with convolutions

eg. $\quad J_n^{bare}(s) = \int ds'\; Z_J(s-s')\; J_n(s',\mu)$

$$\mu\, d/d\mu\; J_n(s,\mu) = \int ds'\; \gamma_J(s-s')\; J_n(s',\mu)$$

↑ invariant mass evolution

$$\text{Coefficient Renormalization} = \left(\begin{matrix}\text{Operator}\\ \text{Renormalization}\end{matrix}\right)^{-1}$$ "consistency conditions"

$$|Z_C|^2 \, \delta(s)\,\delta(\bar{s}) = \int ds'\, d\bar{s}' \; Z_J^{-1}(s-s')\, Z_J^{-1}(\bar{s}-\bar{s}')\, Z_S^{-1}\left(\frac{s'}{Q}, \frac{\bar{s}'}{Q}\right)$$

$\boxed{\text{RGE}}$

$$\gamma_J(s,\mu) = -2\,\Gamma^{cusp}[\alpha_s]\, \frac{1}{\mu^2}\left[\frac{\mu^2\theta(s)}{s}\right]_+ + \gamma[\alpha_s]\,\delta(s)$$

all order structure

(γ_S similar, two variables factorize)

Fourier Transform $y = y - i0$

$$\gamma_J(y) = \int ds \; e^{-isy}\, \gamma_J(s)$$

$$J(y) = \int ds \; e^{-isy}\, J(s)$$

$$\mu \, d/d\mu \; J(y,\mu) = \gamma_J(y,\mu)\, J(y,\mu)$$

<u>simple</u>

$$\gamma_J(y,\mu) = 2\,\Gamma^{cusp}[\alpha_s]\, \ln\left(iy\mu^2 e^{\gamma_E}\right) + \gamma[\alpha_s]$$

$$\left[\frac{\ln^k(s/\mu)}{s}\right]_+ \longleftrightarrow \ln^{k+1}\left(iy\mu^2 e^{\gamma_E}\right)$$

$$d\ln\mu = \frac{d\alpha_s}{\beta[\alpha_s]} \quad , \quad \ln \mu/\mu_0 = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta[\alpha_s]}$$

All orders Solution

$$\ln\left[\frac{J(s,\mu)}{J(s,\mu_0)}\right] = \omega(\mu,\mu_0)\, \ln\left(iy\mu_0^2 e^{\gamma_E}\right) + K(\mu,\mu_0)$$

← same structure for H, J, S

$$\omega = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \; \frac{2\,\Gamma^{cusp}[\alpha_s]}{\beta[\alpha_s]}$$

$$K = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta[\alpha_s]}\, \gamma[\alpha_s] + \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\, 2\,\Gamma^{cusp}[\alpha] \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta[\alpha']}$$

determine ω, K order by order

$y \leftrightarrow \tau$

$$\ln \frac{d\sigma}{dy} = \underbrace{(\ln y)(\alpha_s \ln)^K}_{LL} + \underbrace{(\alpha_s \ln)^K}_{NLL} + \underbrace{\alpha_s (\alpha_s \ln)^K}_{NNLL} + \cdots$$

Momentum Space Answer with resummation

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = H(Q,\mu_Q)\, U_H(Q,\mu_Q,\mu_J)\; J_T(Q^2\tau - s') \otimes U_J(s'-Q\ell,\mu_S,\mu_J) \otimes S_T^{pert}(\ell-\ell',\mu_S) \otimes F(\ell')$$

where $\mu_Q \sim Q, \quad \mu_J \sim Q\sqrt{\tau}, \quad \mu_S \sim Q\tau$

$$U_J(s,\mu,\mu_0) = \frac{e^K \left(e^{\gamma_E}\right)^\omega}{\mu_0^2\, \Gamma(-\omega)} \left[\frac{(\mu_0^2)^{1+\omega}\,\theta(s)}{s^{1+\omega}}\right]_+$$

$\uparrow$ boundary at ∞ rather than 1

Consistency says $\qquad \gamma_J[\alpha_s] + \gamma_S[\alpha_s] = -\frac{1}{2}\gamma_H[\alpha_s]$

Soft-Collinear Interactions (SCET$_{II}$)

Recall $q = q_s + q_c \sim Q\,(\overset{+}{\lambda}, \overset{-}{1}, \overset{\perp}{\lambda})$

$q^2 = Q^2\lambda \gg (Q\lambda)^2$

offshell w.r.t s, c

On-shell modes : $q^\mu \sim Q(\lambda, 1, \sqrt{\lambda})$ are hard-collinear

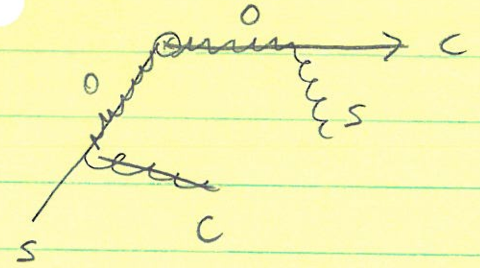
compared to collinear $q^\mu \sim Q(\lambda^2, 1, \lambda)$

Integrating out these fluctuations builds up a soft Wilson line S_n (analogous to $Y(n\cdot A_{us})$ but with soft fields)

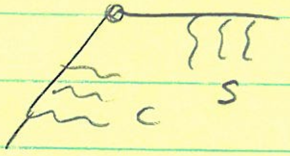
Toy eg. heavy-to-light soft-collin current $\bar{\xi}_n \Gamma h_v$

s = soft, c = collinear

o = offshell



adding more gives

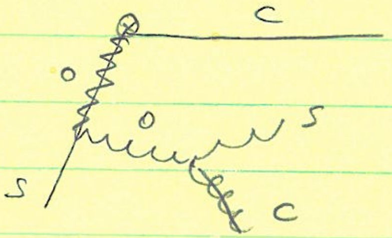


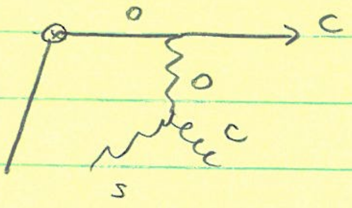
$\bar{\xi}_n S_n^+ \Gamma W h_v$

$S_n^+[n\cdot A_{us}]$

$W[\bar{n}\cdot A_c]$

In QCD need 3-gluon, 4-gluon vertices too; these flip order of S^+ & W

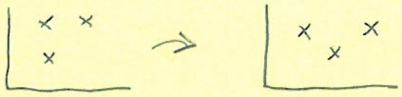




$$\underbrace{(\bar{\xi}_n W)}_{\text{collinear gauge invariant}} \Gamma \underbrace{(S_n^+ h_v)}_{\text{soft gauge invariant}}$$

[can be extended to allorders]

this is <u>soft-collinear factorization</u>



Another Method

- construct SCET$_{II}$ operators using SCET$_I$

i) Match QCD onto SCET$_I$ usoft $p_u^2 \sim \Lambda^2$

collinear $p_c^2 \sim Q\Lambda$

ii) Factorize usoft with field redefinition

iii) Match SCET$_I$ onto SCET$_{II}$ soft $p_s^2 \sim \Lambda^2$

collin $p_c^2 \sim \Lambda^2$

Notes: • this gives us a simple procedure to construct SCET$_{II}$ ops. (even though they're non-local)

• usoft fields in I are renamed soft for II

eg. i) $J^I = (\bar\xi_n W) \Gamma h_v$

ii) $J^I = (\bar\xi_n^{(0)} W^{(0)}) \Gamma (Y^+ h_v)$

iii) $J^{II} = (\bar\xi_n W) \Gamma (S^+ h_v)$ as before

↑ here all T-products in SCET$_I$ & SCET$_{II}$ match up, so matching was trivial

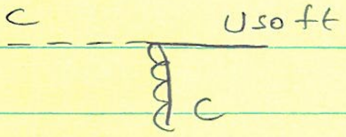
"Thm" • In cases where we have T-products in SCET$_I$ with ≥ 2 operators involving both collin & usoft fields, we can generate a non-trivial coefficient in SCET$_{II}$ (jet-function J)

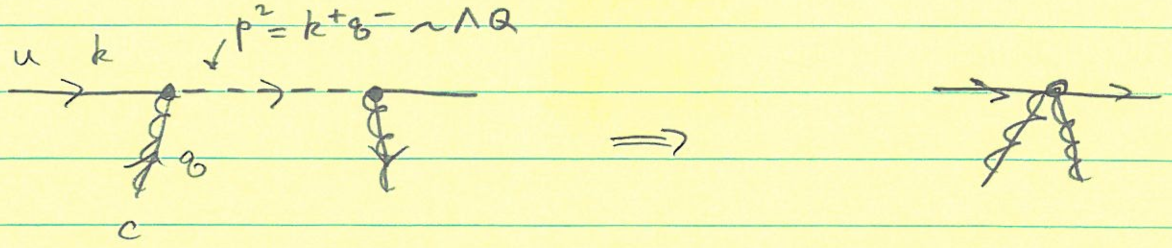
T eg. $$\int dp_- \, dk_+ \, J(p_-, k_+) \overbrace{(\xi W)_{p_-} \Gamma (S^+ q_s)_{k^+} + (\cdots)}^{p^2 \sim \Lambda^2}$$

↑ SCET$_I$ loops $p^2 \sim Q\Lambda$

↑ ind's allow k^+ dependence

eg. two operators





when we lower offshellness of ext. collin fields the intermediate line still has $p^2 \sim Q\Lambda$ and must really be integrated out

P.C. $T^{I} \sim \lambda^{2K} \Rightarrow O^{II} \sim \eta^{K+E}$

where $\lambda^2 = \eta = \frac{\Lambda}{Q}$,

factor $E > 0$ from changing the scaling of ext. fields

eg. $\xi_I \sim \lambda$

$\xi_{II} \sim \eta = \lambda^2$

$\Rightarrow$ No mixed soft-collin $\mathcal{L}$ at leading order

- after field redefn no mixed $\mathcal{L}_I$ ops at LO
- mixed $\mathcal{L}_I^{(1)}$ gives $T\{\mathcal{L}_I^{(1)}, \mathcal{L}_I^{(1)}\} \sim \lambda^2$
  matches onto $O_{II} \sim \eta$ or higher

SCET$_I$ λ^δ

$$\delta = 4 + \underset{\uparrow\ u=1 \text{ no c., else } u=0}{4u} + \sum_k (k-4)\, \overset{\text{rest}}{\overset{\downarrow}{V_k^c}} + (k-8)\, \overset{\text{pure usoft}}{\overset{\downarrow}{V_k^u}}$$

V_k^i = # vertices that are $\mathcal{O}(\lambda^k)$ and type-i

SCET$_{II}$

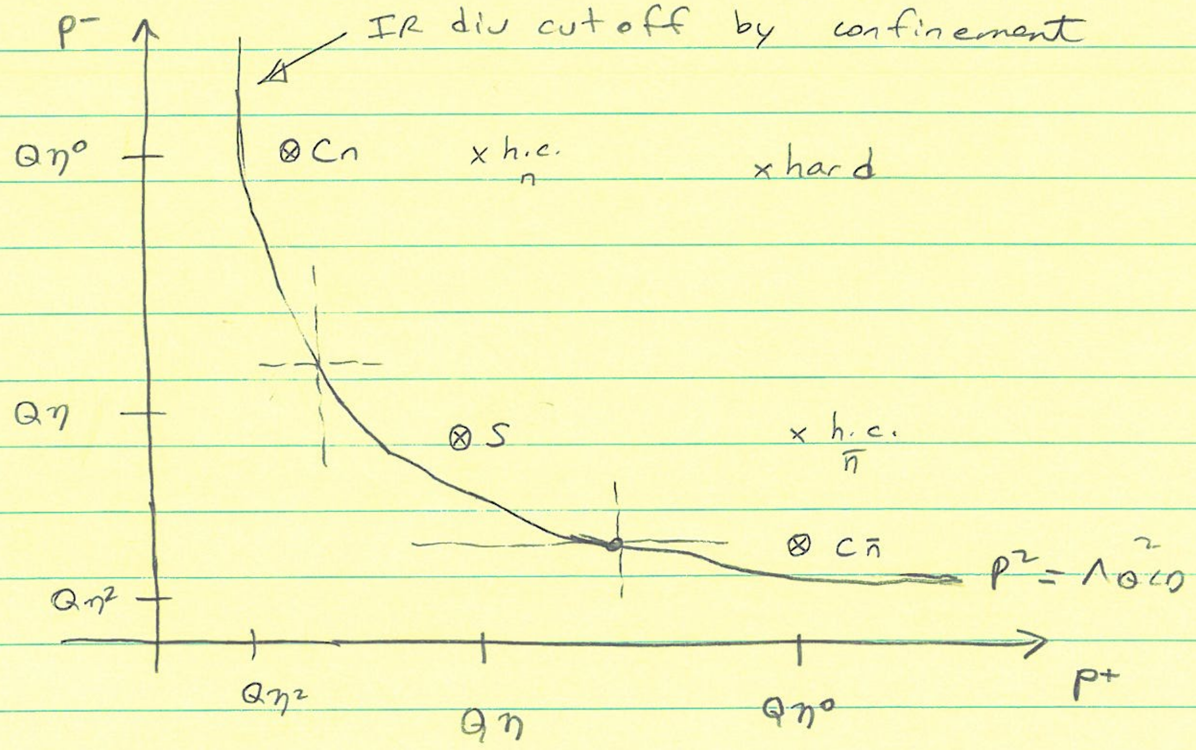
$$\delta = 4 + \sum_k (k-4)\,(\underset{\substack{\uparrow\\ \text{pure}\\ c}}{V_k^c} + \underset{\substack{\uparrow\\ \text{pure}\\ s}}{V_k^s} + \underset{\substack{\uparrow\\ \text{mixed}}}{V_k^{sc}}) + \underset{\substack{\uparrow\\ p\sim(\eta^2,\eta,\eta)\\ \text{loops}}}{L^{sc}}$$

$$\delta = 5 - \underset{\uparrow}{N_c} - \underset{\uparrow}{N_s} + \sum_k (k-4)\,(V_k^S + V_k^c) + (k-3)\,V_k^{sc}$$

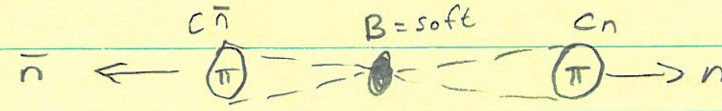
\# connected
soft, collin components

$$\left[\text{in eg. SCET}_I \quad \begin{matrix}\lambda^3\lambda\,\frac{1}{\lambda^2}\,\lambda^3\lambda \sim \lambda^{6-4} \sim \lambda^2 \\ \text{or} \quad \lambda * \lambda \sim \lambda^2\end{matrix} \Rightarrow (\eta^{3/2}\eta)^2\frac{1}{\eta} = \eta^{4-3} = \eta\right]$$

$$\mathcal{L}_{SCET^{II}} = \mathcal{L}^{(0)}_{soft}[q_s, A_s] + \mathcal{L}^{(0)}_{collin\text{-}n}[q_n, A_n] + \mathcal{L}^{(0)}_{collin\text{-}\bar{n}}[q_{\bar{n}}, A_{\bar{n}}]$$



Non-pert d.o.f. in different sectors $B \to \pi\pi$

eg.


Exclusive

eg. $\gamma^* \gamma \to \pi^0$ hard - collin factorization

[Breit frame : soft modes have no active role so this does not really probe difference between $SCET_I$ & $SCET_{II}$]

QCD has

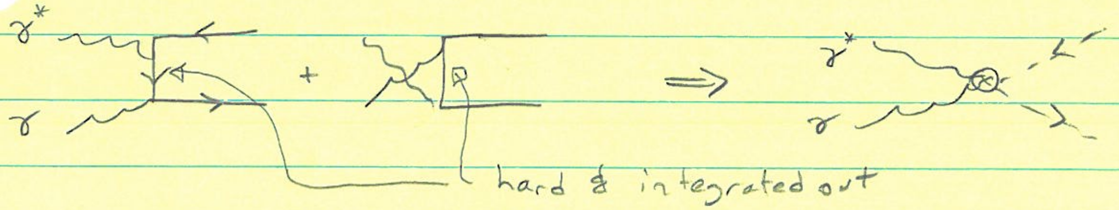
$$\langle \pi^0(p_\pi) | J_\mu(0) | \gamma(p_\gamma, \epsilon) \rangle = ie\, \epsilon^\nu \int d^4z\, e^{-ip_\gamma \cdot z} \langle \pi^0(p_\pi) | T\, J_\mu(0)\, J_\nu(z) | 0 \rangle$$
$$= -ie\, F_{\pi\gamma}(Q^2)\, \epsilon_{\mu\nu\rho\sigma}\, p_\pi^\nu\, \epsilon^\rho\, q^\sigma$$

e.m. current $J^\mu = \bar\Psi \hat{Q} \gamma^\mu \Psi$, $\hat{Q} = \frac{\tau_3}{2} + \frac{\mathbb{1}}{6} = \begin{pmatrix} 2/3 & \\ & -1/3 \end{pmatrix}$

For $Q^2 >> \Lambda^2$ $F_{\pi\gamma}$ simplifies (ala Brodsky - Lepage)

frame $q^\mu = \frac{Q}{2}(n^\mu - \bar{n}^\mu)$, $p_\gamma^\mu = E\, \bar{n}^\mu$

$p_\pi^\mu = p + p_\gamma = \frac{Q}{2} n^\mu + (E - \frac{Q}{2}) \bar{n}^\mu$



SCET Operator at Leading - order (for T-product) is

$$\mathcal{O} = \frac{i\, \epsilon^\perp_{\mu\nu}}{Q} [\bar\xi_{n,p} W]\, \Gamma\, C(\bar{P}, \bar{P}^\dagger, \mu)\, [W^\dagger \xi_{n,p'}]$$

order λ^2 (" twist - 2 ")

- obeys current conservation
- dim analysis fixes $\frac{1}{Q}$ pre-factor for C dimless

- Charge Conj : $T\{J, J\}$ even so $\mathcal{O}$ even

so $C(\mu, \bar{P}, \bar{P}^\dagger) = C(\mu, -\bar{P}^\dagger, -\bar{P})$

- flavor & spin structure $\quad \Gamma = \underbrace{\not{\bar n}\,\gamma_5}_{\text{for pion}} \; 3\sqrt{2} \; \underbrace{\hat{Q}^2}_{\text{2nd order e.m.}}$

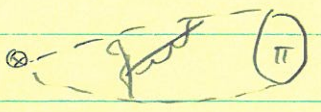
- color singlet, purely collinear (again) so soft gluons decouple

equate $\quad \dfrac{Q^2}{2} F_{\pi\gamma} = \dfrac{i}{Q} \langle \pi^0 | (\bar\xi W)\, \Gamma\, \overset{\swarrow\,\text{SCET}_{II}}{C}\, (W^{+}\xi) |0\rangle$

write $\quad \bar{P}_\pm = \bar{P}^{+} \pm \bar{P}$

now $\bar{P}_-$ gives total mom of $(\bar\xi W)\Gamma(W^{+}\xi)$ operator

ie momentum of pion



$\bar{P}_- = \bar{n}\cdot p_\pi = Q$

→ total mom.

$$F_{\pi\gamma}(Q^2) = \frac{2i}{Q^2} \int d\omega\, C(\omega,\mu)\, \langle \pi^0 | (\bar\xi W)\, \Gamma\, \delta(\omega - \bar{P}_+)(W^{+}\xi) |0\rangle$$

<u>Non-perturbative Matrix Elt</u> $\qquad$ finite Wilson line $\left(P\exp ig\int_x^y ds\ldots\right)$

position space $\qquad$ Fourier Transform of $\bar{n}\cdot p$ label

$$\langle \pi^0(p) |\, \bar\xi_n(y) \frac{\not{\bar n}\gamma_5 \tau^3}{\sqrt{2}} W(y,x)\, \xi_n(x) |0\rangle$$
$$= -i f_\pi\, \bar{n}\cdot p \int_0^1 dz\, e^{i\bar{n}\cdot p\,(zy + (1-z)x)}\, \phi_\pi(\mu,z)$$

$$\int_0^1 dz\, \phi_\pi(z) = 1$$

momentum space

$$\langle \pi^0(p) | (\bar\xi_{n,p} W) \frac{\not{\bar n}\gamma_5 \tau^3}{\sqrt{2}} \delta(\omega - \bar{P}_+)(W^{+}\xi_{n,p_2}) |0\rangle$$

$$= -i f_\pi\, \bar{n}\cdot p \int_0^1 dz\, \delta(\omega - (2z-1)\bar{n}\cdot p)\, \phi_\pi(\mu,z)$$

Plug it into $F_{\pi\gamma}(Q^2)$ and do integral over ω

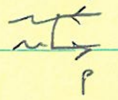
$$F_{\pi\gamma}(Q^2) = \frac{2 f_\pi}{Q^2} \int_0^1 dz \; C\big((2z-1)Q, Q, \mu\big) \; \phi_\pi(z,\mu)$$

- ϕ_π is universal light-cone dist'n for pions

- C is process dependent (all orders factorization in α_s)
- one-dim convolution again

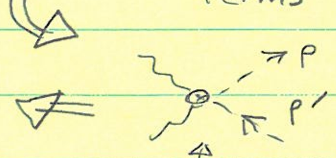
Tree Level Matching

expand

$i\Big($  $+$  $\Big) = \frac{ie}{2} \epsilon_{\mu\nu\rho\beta} \, \epsilon^\nu \bar{n}^\rho n^\beta \left(\frac{\not{\bar{n}}}{2}\gamma_5\right) \hat{Q}^2$

$* \left(\frac{1}{\bar{n}\cdot p} - \frac{1}{\bar{n}\cdot p'}\right) + \ldots$

non-pion terms



match with $W=1$

$$\text{So} \quad C = \frac{1}{6\sqrt{2}}\left(\frac{Q}{\bar{p}^+} - \frac{Q}{\bar{p}}\right)$$

$$C\big(\omega = (2x-1)Q\big) = \frac{1}{6\sqrt{2}}\left(\frac{1}{x} + \frac{1}{1-x}\right)$$

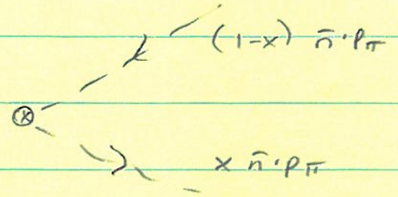
Charge Conj $+1$ for $|\pi^0\rangle$ gives $\phi_\pi(x) = \phi_\pi(1-x)$

So only $\int_0^1 dx \, \frac{\phi_\pi(x,\mu)}{x}$ appears in our prediction

integrate over all x, much different than DIS $\delta(1-\xi/x) \Rightarrow f_{i/p}(x,\mu)$

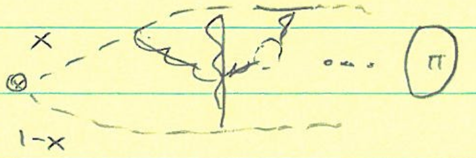
Interpretation:

Naively



mom fraction of quarks in pion

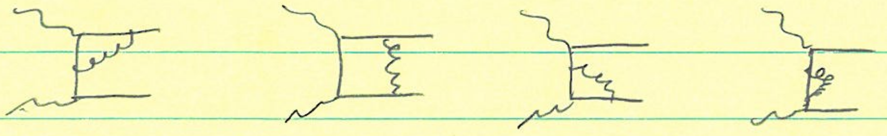
Really



↖ mom. fractions at point where quarks are produced. Hadronization process changes "x" carried by valence quarks which is encoded in $\phi_\pi(x)$

Higher Order Matching

full



\+ w.fn graphs

SCET

\+ w.fn

Difference will be IR finite, and gives C at one-loop

Another Exclusive Example

(hep-ph/0107002)

$B \to D\pi$ $\qquad \underbrace{m_b, m_c, E_\pi}_{Q} \gg \Lambda_{QCD}$

QCD Operators at $\mu \simeq m_b$

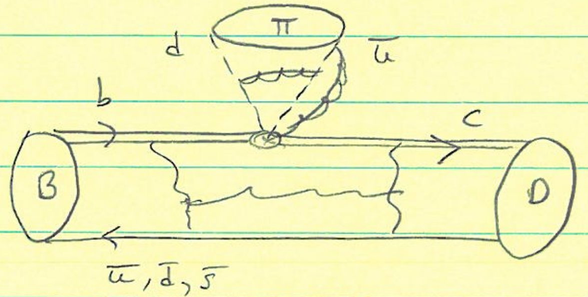
$$H_W = \frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cb} \left[C_0^F O_0 + C_8^F O_8 \right]$$

$$P_L = \frac{1-\gamma_5}{2}$$

where
$$O_0 = [\bar{c}\gamma^\mu P_L b][\bar{d}\gamma_\mu P_L u]$$
$$O_8 = [\bar{c}\gamma^\mu P_L T^a b][\bar{d}\gamma_\mu P_L T^a u]$$

Want to Factorize $\langle D\pi | O_{0,8} | B \rangle$

ie Show at LO



no gluons btwn B, D and quarks in pion

expect $B \to D$ form factor $\xi^{\text{Isgur-Wise}}$

$\phi_\pi(x)$ distn for pion

$$\left.\begin{array}{lll} B, D & \text{soft} & p^2 \sim \Lambda^2 \\ \pi & \text{collinear} & p^2 \sim \Lambda^2 \end{array}\right\} \text{SCET}_{II}$$

use SCET_I as intermediate step

① Match at $\mu^2 \simeq Q^2$

$$\left.\begin{array}{l} O_0 \\ O_8 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} Q_0^{1,5} = [\bar{h}_{v'}^{(c)} \Gamma_h^{1,5} h_v^{(b)}][(\bar{\xi}_n^{(d)} W)\Gamma_\ell C_0(\bar{P}_+) W^\dagger \xi_n^{(u)}] \\ Q_8^{1,5} = [\;\text{''}\; T^A \;\text{''}\;][\;\text{''}\;\;\text{''}\; C_8(\bar{P}_+) T^a \;\text{''}\;] \end{array}\right.$$

↑ usoft SCET$_I$ $\qquad$ ↑ collinear $p^2 \sim Q\Lambda$

$$\Gamma_h^{1,5} = \frac{\not{v}}{2}\{1, \gamma_5\}$$
$$\Gamma_\ell = \frac{\not{\bar{n}}}{4}(1-\gamma_5)$$

② Field redefinitions $\quad \xi_{n,p} = Y \xi_{n,p}^{(0)}, \dots$

in $Q_0^{1,5}$ get $\quad \bar\xi_n^{(0)} W^{(0)} \cancel{Y^\dagger}\cancel{Y} W^{\dagger(0)} \xi_n^{(0)}$

$Q_8^{1,5}$ get $\quad \bar\xi_n^{(0)} W^{(0)} Y^\dagger T^A Y W^{\dagger(0)} \xi_n^{(0)}$

$Y T^a Y^\dagger = \mathcal{Y}^{ba} T^b \qquad\qquad Y^\dagger T^a Y = \mathcal{Y}^{ab} T^b$

↑ adjoint Wilson line

$T^a \otimes Y^\dagger T^a Y = Y T^a Y^\dagger \otimes T^a$

↑ moves usoft Wilson lines next to h_v fields

③ Match $SCET_I$ onto $SCET_{II}$ (trivial here again)

$Y \to S$

$\xi_n^{(0)} \to \xi_n$ in II etc.

$Q_0^{1,5} = [\bar h_{v'}^{(c)} \Gamma_h h_v^{(b)}] [\bar\xi_n^{(d)} W \Gamma_\ell C_0(\bar P_+) W^\dagger \xi_{n,p}^{(a)}]$

$Q_8^{1,5} = [\bar h_{v'}^{(c)} \Gamma_h S T^a S^\dagger h_v^{(b)}] [\bar\xi_n^{(d)} W \Gamma_\ell C_8(\bar P_+) T^a W^\dagger \xi_{n,p}^{(d)}]$

④ Take Matrix Elements

$\langle \pi_n^- | \bar\xi_n W \Gamma C_0(\bar P_+) W^\dagger \xi_n | 0 \rangle = \frac{i}{2} f_\pi E_\pi \int_0^1 dx \, C(2E_\pi(2x-1)) \phi_\pi(x)$

$\langle D_{v'} | \bar h_{v'} \Gamma h_v | B \rangle = N' \xi(w_0, \mu)$

↑ $w_0 = v \cdot v'$

B, D purely soft → no contractions with collinear fields

π " collinear → no " " soft fields

which is why it factors into two matrix elements

F- O8:

$\langle D_{v'} | \underbrace{\bar h_{v'} Y T^a Y^\dagger h_v}_{} | B_v \rangle = 0$

color octet operator between color singlet states

Find Factorization Formula

$$\langle \pi D | H_w | B \rangle = \underset{\substack{\uparrow \\ \text{prefactors}}}{i\,N}\ \xi(w_0, \mu) \int_0^1 dx\, C(2E_\pi(2x-1), \mu)\, \phi_\pi(x, \mu) + \mathcal{O}(\Lambda/Q)$$

- $\xi(w_0, \mu)$ is Isgur-Wise function at max. recoil

  $w_0 = \dfrac{m_B^2 - m_D^2}{2m_B}$ (measured in $B \to De\nu$ recall)

- This applies to type-I (& III) decays

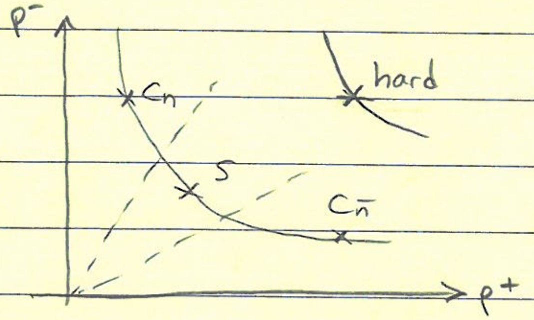
  $\bar{B}^0 \to D^+\pi^-$ $\bar{B}^0 \to D^{*+}\pi^-$, $\bar{B}^0 \to D^+\rho^-$, ...

  $B^- \to D^0\pi^-$ $B^- \to D^{*0}\pi^-$ $B^- \to D^0\rho^-$, ...

  predicts type-II decays are suppressed by Λ/Q

  $\bar{B}^0 \to D^0\pi^0$, ... (we could derive fact. thm. for these too)

$SCET_{II}$ & Rapidity Divergences



In $SCET_I$ we had to worry about double counting: $c_n = c_n - \underbrace{c_0}_{\text{zerobin}}$, us

So far in $SCET_{II}$ we have not had to because the overlaps did not generate log divergences

In general $SCET_{II}$ also has 0-bin's: $\underbrace{C_n}_{k^\mu \sim (\lambda^2, 1, \lambda)} - C_{ns}$ ↑ take $k^\mu \sim (\lambda, \lambda, \lambda)$ in collinear integrand & expand

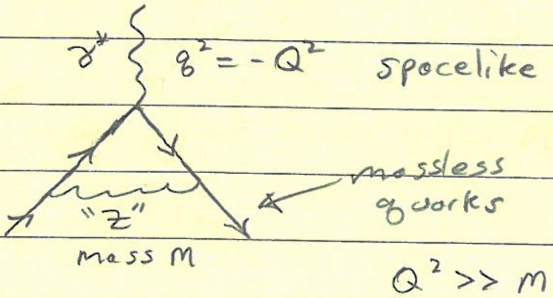
But unlike $SCET_I$ there is another issue. The variable that distinguishes modes is a rapidity y, $e^{2y} = \frac{p^-}{p^+}$

$$e^{2y} \sim \underset{c_n}{\lambda^{-2}}, \underset{s}{\lambda^0}, \underset{c_{\bar n}}{\lambda^2}$$

Since all modes live on the same mass hyperbola p^2, divergences that occur when separating modes can not be regulated by dim.reg! (which is a Lorentz Inv. regulator, $p_E^{-2\epsilon}$, which distinguishs hyperbola's)

Lets explore this with a simple example.

Massive Sudakov Form Factor



$J^\mu = \bar\psi \gamma_\mu \psi$

$Q^2 >> m^2$

$$\langle q(\bar p) | J^\mu | q(p) \rangle = F(Q^2, m^2)\, \bar u(\bar p) \gamma^\mu u(p)$$

$\lambda = \frac{m}{Q}$ Z could be

c_n $Q(\lambda^2, 1, \lambda)$ ← $q(p)$

$c_{\bar n}$ $Q(1, \lambda^2, \lambda)$ ← $q(\bar p)$

s $Q(\lambda, \lambda, \lambda)$

$p^\mu = p^- \frac{n^\mu}{2}$, $\bar{p}^\mu = \bar{p}^+ \frac{\bar{n}^\mu}{2}$ $\qquad Q^2 = -(\bar{p}-p)^2 = p^-\bar{p}^+ \underset{\substack{\uparrow \\ \text{frame choice}}}{=} Q\cdot Q$

Factorize $\qquad J^\mu = (\bar{\xi}_{\bar{n}} W_{\bar{n}})\, S_{\bar{n}}^\dagger S_n \gamma^\mu (W_n^\dagger \xi_n)$

$$F(Q^2,m^2) = \mathcal{H}\, C_{\bar{n}}\, \mathcal{S}\, C_n$$

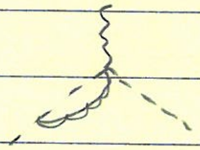
Consider (Scalar) Loop Integral

$$I_{full} = \int đ^d k \frac{1}{(k^2-m^2)(k^2+k^+p^-)(k^2+k^-\bar{p}^+)}$$

← terms with most logs have no k's in numerator

UV & IR finite

$$I_{c_n} = \int đ^d k \frac{1}{(k^2-m^2)(k^2+k^+p^-)(k^-)\,\bar{p}^+} \frac{w^2\nu^\eta}{|k^-|^\eta}$$



$$I_{c_{\bar{n}}} = \int đ^d k \frac{1}{(k^2-m^2)(k^+)(p^-)(k^2+k^-\bar{p}^+)} \frac{w^2\nu^\eta}{|k^+|^\eta}$$

$$I_s = \int đ^d k \frac{1}{(k^2-m^2)(k^+)(k^-)(\bar{p}^+p^-)} \frac{w^2\nu^\eta}{|2p_z|^\eta}$$

do $đ^n k_\perp$ in I_s $\quad \propto \quad \int dk^+dk^- \frac{(k^+k^- - m^2)^{-2\epsilon}}{(k^+)(k^-)} \frac{1}{Q^2}$

diverges as $\frac{k^-}{k^+}$ $\rightarrow 0$ (towards $C_{\bar{n}}$)

$\gg \infty$ (towards C_n)

Need another regulator. One dim-reg like choice is to regulate Wilson lines

$$S_n = \sum_{perms} \exp\left[\frac{-g}{n\cdot p}\, w \frac{\nu^{\eta/2}}{|2p_z|^{\eta/2}}\, n\cdot A_s \right]$$

add red factors above

$p_z = p_- - p_+$ because it does not involve p^0.

(Regulators with p^0 can mess up unitarity / causality)

For collinear W_n , $|2p_z| = |p^-|$ up to power corrections

use $\qquad W_n = \sum_{perms} \exp\left[\frac{-g}{\bar{n}\cdot p}\, w^2 \frac{\nu^\eta}{|\bar{n}\cdot p|^\eta}\, \bar{n}\cdot A_n \right]$

$$w^{bare} = w(\eta,\nu)\,\nu^{\eta} \quad , \quad \nu \frac{d}{d\nu} w(\eta,\nu) = -\frac{\eta}{2} w(\eta,\nu)$$

$$w(0,\nu) \equiv 1$$

$1/\eta$ like $1/\epsilon$ $\quad$ $w(\eta,\nu)$ is dummy coupling to facilitate RGE in ν

$\ln \nu$ like $\ln \mu$

Note: • $1/\eta$ & η^{0} terms are gauge invariant. eg. At one-loop replacing $g^{\mu\nu} \to \left(g^{\mu\nu} + \xi \frac{k^\mu k^\nu}{k^2}\right)$, the $k^\mu k^\nu$ term has no rapidity divergences.

- For any fixed inv. mass we have $1/\eta$ divergences. Proper renormalitation procedure is

  $\eta \to 0$, add $\frac{f(\epsilon)}{\eta}$ counterterm, then $\epsilon \to 0$, find $\frac{1}{\epsilon}$ c.t.'s

For fermion case, including prefactors

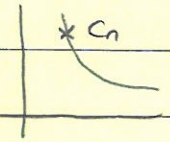
$$I_{cn} = \frac{\alpha_s C_F}{\pi}\left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\frac{\mu}{m}\right)^{2\epsilon}}{2\eta} + \ln\left(\frac{\nu}{p^-}\right)\ln\left(\frac{\mu}{m}\right) + \frac{1}{2\epsilon}\ln\left(\frac{\nu}{p^-}\right) + \frac{1}{2\epsilon} + \ln\left(\frac{\mu}{m}\right) + \text{constant}\right]$$

$$I_{c\bar{n}} = \text{same } p^- \to \bar{p}^+$$

$$I_s = \frac{\alpha_s C_F}{\pi}\left[-\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(\mu/m\right)^{2\epsilon}}{\eta} - 2\ln\left(\frac{\nu}{m}\right)\ln\left(\frac{\mu}{m}\right) + \frac{1}{\epsilon}\ln\left(\frac{\mu}{\nu}\right) + \frac{1}{2\epsilon^2} + \ln^2\frac{\mu}{m} + \text{constant}\right]$$

$$I_{cn} + I_{c\bar{n}} + I_s = \frac{\alpha_s C_F}{\pi}\left[\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu}{Q} + \frac{1}{\epsilon} + \ln^2\frac{\mu}{m} + 2\ln\frac{\mu}{m}\ln\frac{m}{Q} + 2\ln\frac{\mu}{m} + \text{const.}\right]$$

- rapidity divergence cancels between sectors (as expected)
- overall counterterm has only $\ln \mu/Q$, hard scale. same for hard match $\mathcal{H}(\mu, Q)$
- logs in I_{cn} minimized for $\mu \sim m$, $\nu \sim p^- = Q$ which is precisely



the location of Cn

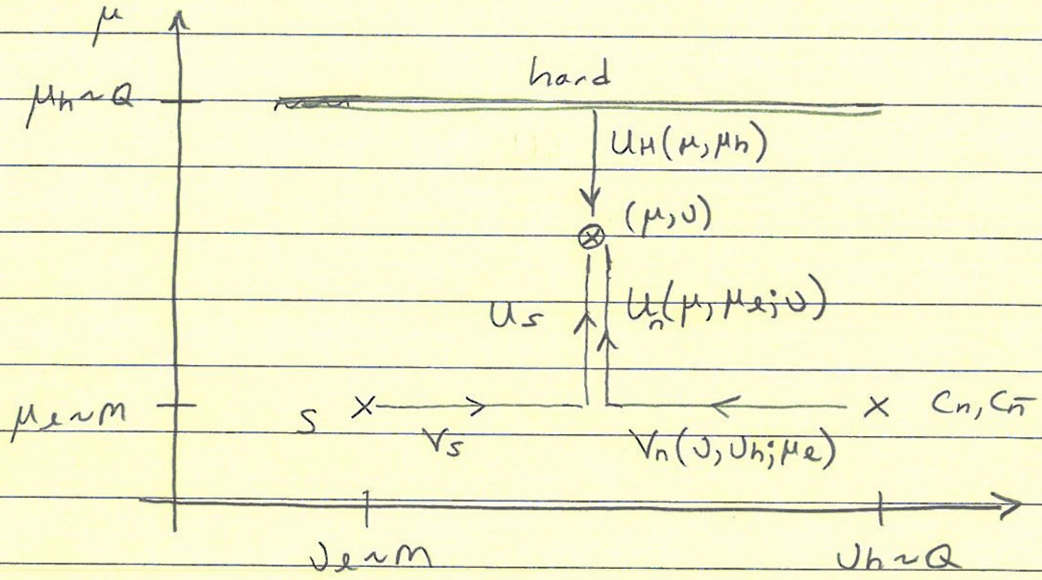
- likewise need $\mu \sim M$, $\nu \sim \bar{p}^+ = Q$ for $C_{\bar{n}}$

  need $\mu \sim \nu \sim M$ for S

$$F(Q^2, M^2) = \mathcal{H}(Q^2,\mu)\, C_n(M,\mu,\nu/Q)\, C_{\bar{n}}(M,\mu,\nu/Q)\, \mathcal{S}(M,\mu,\nu/\mu)$$

renormalized fact. thm. with <u>2-cutoffs</u> μ & ν

- Will have a μ-RGE and ν-RGE to sum logs



choice of (μ,ν) arbitrary (just freedom to run coeffs or operators)

eg. pick $(\mu,\nu) = (\mu_e, \nu_h)$ then just evolution kernels

$$U_H(\mu_e,\mu_h)\, V_S(\nu_h,\nu_e;\mu_e)$$

- Path Independence. μ & ν parameters are independent

$$\mu\frac{d}{d\mu}\,\nu\frac{\partial}{\partial\nu} = \nu\frac{\partial}{\partial\nu}\,\mu\frac{d}{d\mu}$$

- Counterterms & Anom. Dims.

$$C_n(m,\mu,\nu/Q) = Z_{\xi_n}^{-1/2}\, Z_n^{-1}\, C_n^{\text{bare}}$$

$$\mathcal{S}(m,\mu,\nu/\mu) = Z_S^{-1}\, \mathcal{S}^{\text{bare}} \qquad Z_{\xi_n} = 1 + \frac{\alpha_s C_F}{4\pi\epsilon}$$

$$Z_S = 1 - \frac{\alpha_s C_F}{\pi} w^2 \left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)(\mu/m)^{2\epsilon}}{\eta} - \frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu}{\nu} \right]$$

$$Z_n = 1 + \frac{\alpha_s C_F}{\pi} w^2 \left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)(\mu/m)^{2\epsilon}}{2\eta} + \frac{3}{8\epsilon} + \frac{1}{2\epsilon}\ln\frac{\nu}{p^-} \right]$$

μ- Anom. Dim

$$\gamma_\mu^S = -Z_S^{-1}\,\mu\frac{d}{d\mu} Z_S = \frac{\alpha_s(\mu)C_F}{\pi}\, 2\ln\frac{\mu}{\nu} \qquad \mu\frac{d}{d\mu} S = \gamma_\mu^S S \text{ etc.}$$

$$\gamma_\mu^n = -Z_n^{-1}\,\mu\frac{d}{d\mu} Z_n = \frac{\alpha_s(\mu)C_F}{\pi}\left[\ln\frac{\nu}{p^-} + \frac{3}{4}\right]$$

$$\gamma_\mu^{\bar n} = \qquad = \frac{\alpha_s(\mu)C_F}{\pi}\left[\ln \nu/\bar{p}^+ + 3/4\right]$$

gives U_S, U_n Kernels

consistency $\quad \gamma_\mu^S + \gamma_\mu^n + \gamma_\mu^{\bar n} = -\gamma_H = \frac{\alpha_s C_F}{\pi}\left(2\ln\frac{\mu}{Q} + \frac{3}{2}\right)$

ν Anom-Dim

$$\gamma_\nu^S = -Z_S^{-1}\,\nu\frac{\partial}{\partial\nu} Z_S = -\frac{\alpha_s(\mu)C_F}{\pi}\, 2\ln\frac{\mu}{m}$$

$$\gamma_\nu^n = -Z_n^{-1}\,\nu\frac{\partial}{\partial\nu} Z_n = \frac{\alpha_s(\mu)C_F}{\pi}\ln\frac{\mu}{m} = \gamma_\nu^{\bar n}$$

$$\nu\frac{\partial}{\partial\nu} S = \gamma_\nu^S S \quad \text{etc.}$$

gives V_S, V_n Kernels

Path Independence : $\quad Z^{-1}\left[\mu\frac{d}{d\mu}, \nu\frac{\partial}{\partial\nu}\right] Z = 0$

so $\quad \mu\frac{d}{d\mu}\gamma_\nu^S = \nu\frac{\partial}{\partial\nu}\gamma_\mu^S$ ✓

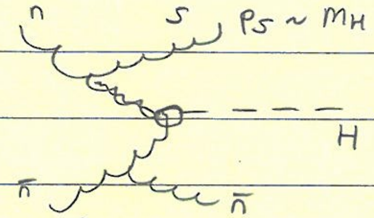
$\quad \mu\frac{d}{d\mu}\gamma_\nu^n = \nu\frac{\partial}{\partial\nu}\gamma_\mu^n$ ✓

eg. $\quad U_S^{LL}(\mu,\mu_S;\nu_S) = \exp\left[-\frac{8\pi C_F}{\beta_0^2}\left(\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu_S)} - \frac{1}{\alpha_s(\nu_S)}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_S)}\right)\right]$

$$V_S^{LL}(\nu,\nu_S;\mu) = \exp\left[\frac{2C_F}{\beta_0}\ln\left(\frac{\alpha_s(\mu)}{\alpha_s(m)}\right)\ln\left(\frac{\nu^2}{\nu_S^2}\right)\right]$$

$SCET_{II}$ examples with rapidity RGE

gg → Higgs p_T distribution



$p_n \sim m_H(\lambda^2,1,\lambda)$ $p_s \sim m_H(\lambda,\lambda,\lambda)$

$\vec{p}_T^{\,H} \sim m_H \lambda$

$p_{\bar n} \sim m_H(1,\lambda^2,\lambda)$ $\alpha_s \ln^2\left(p_T^H/m_H\right)$

Since we only measure $p_T^H \sim \lambda$ we can have soft radiation

Factorize cross-section ($|\mathrm{Ampl}|^2$)

$J_{full} = h\, G^{\mu\nu} G_{\mu\nu}$ h = Higgs field

$$\langle J_{full}(x)\, J_{full}(0)\rangle = H(m_H)\ \langle p_n|\, \mathcal{B}_{n\perp}\, \mathcal{B}_{n\perp}\, |p_n\rangle \quad \leftarrow \text{gluon pdfs}$$

$$\langle p_{\bar n}|\, \mathcal{B}_{\bar n\perp}\, \mathcal{B}_{\bar n\perp}\, |p_{\bar n}\rangle$$

$$\langle 0|\, S_n\, S_{\bar n}\, S_n^T\, S_{\bar n}^T\, |0\rangle$$

↑ adjoint rep for soft Wilson lines

$$\frac{d\sigma}{dp_T^{H\,2}\,dy} = N_0\, H(m_H,\mu) \int d^2p_{1\perp}\, d^2p_{2\perp}\, d^2p_{s\perp}\ \delta\left(p_T^{H\,2} - |\vec{p}_{1\perp}+\vec{p}_{2\perp}+\vec{p}_{s\perp}|^2\right)$$

$$* f_{g/p}^{\mu\nu}\left(\frac{m_H}{E_{cm}}e^{-Y}, \vec{p}_{1\perp}, \mu, \frac{\nu}{m_H e^{-Y}}\right)$$

$$* f_{g/p}^{\mu\nu}\left(\frac{m_H}{E_{cm}}e^{Y}, \vec{p}_{2\perp}, \nu/m_H e^{Y}\right)\ S\left(\vec{p}_{s\perp}^{\,2}, \mu, \nu/\mu\right)$$

↑ p_T dependent soft fn.

↑ transverse momentum dependent PDF

which had rapidity divergences (prior to $1/\eta$ renormalization)

Jet Broadening $\qquad e^+e^- \xrightarrow{Q^2}$ dijets

here only measure $\vec{p}_\perp$ (relative to thrust axis)

Broadening $= B = \sum_i \frac{|\vec{p}_{i\perp}|}{Q} = B_L + B_R = \sum_{i\in L}(\;) + \sum_{i\in R}(\;)$

Again we only measure $\perp$-momenta, $p_\perp \sim \lambda$, $B \sim \lambda$

so have SCET$_{II}$: $c_n, c_{\bar n}, S$

$$\frac{1}{\sigma_0}\frac{d\sigma}{dB_L\,dB_R} = H(Q^2,\mu)\int de_n\, de_{\bar n}\, de_s^L\, de_s^R \int d^2\vec{k}_{1\perp}\, d^2\vec{k}_{2\perp}$$

$$\delta(B_R - e_n - e_s^R)\,\delta(B_L - e_{\bar n} - e_s^L)$$

$$J_n\Big(Q, e_n, \vec{k}_{1\perp}, \mu, \frac{\nu}{Q}\Big)\, J_{\bar n}\Big(Q, e_{\bar n}, \vec{k}_{2\perp}, \mu, \frac{\nu}{Q}\Big)$$

$$* \; S\big(e_s^R, e_s^L, \vec{k}_{1\perp}, \vec{k}_{2\perp}, \mu, \nu/\mu\big)$$

Another inclusive example : $B \to X_s \gamma$ [case where usoft modes matter]

Here we will need both usoft & collinear d.o.f. in $SCET_I$

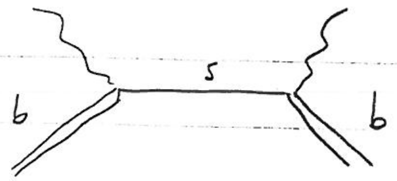
$$H_{eff} = \frac{-4G_F}{\sqrt{2}} V_{tb} V_{ts}^{*} C_7 \mathcal{O}_7 \quad , \quad \mathcal{O}_7 = \frac{e}{16\pi^2} m_b \bar{s} \sigma^{\mu\nu} F_{\mu\nu} P_R b$$

photon $q^\mu = E_\gamma \bar{n}^\mu$

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dE_\gamma} = \frac{4E_\gamma}{m_b^3} \left(\frac{-1}{\pi}\right) \text{Im}\, T$$

$$T = \frac{i}{m_B} \int d^4x \, e^{-iq\cdot x} \langle \bar{B} | T J_\mu^{\dagger}(x) J^\mu(0) | \bar{B} \rangle$$

$$J^\mu = \bar{s}\, i\sigma^{\mu\nu} q_\nu P_R b$$



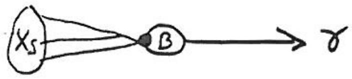
looks like DIS

Consider endpoint region

$$m_B/2 - E_\gamma \lesssim \Lambda_{QCD}$$

$$p_X^2 \simeq m_B \Lambda$$

jet



B-rest frame $\quad p_B = \frac{m_B}{2}(n^\mu + \bar{n}^\mu) \quad = \quad p_X + q$

$$p_X = \frac{m_B}{2} n^\mu + \frac{\bar{n}^\mu}{2} \underbrace{(m_B - 2E_\gamma)}_{\Lambda}$$

collinear

so quarks and gluons in X are collinear with $p_c^2 \sim m_B \Lambda$

B has usoft light d.o.f.

match onto LO SCET operator

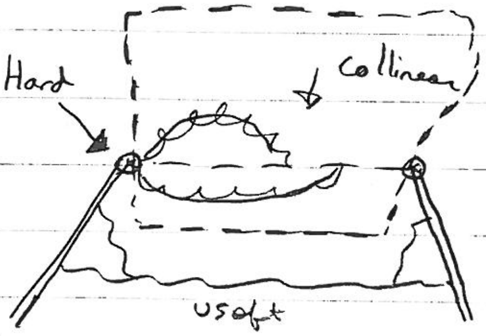
$$J_\mu = -E_\gamma \, e^{i(\bar{P}\frac{n}{2} - m_b v)\cdot x} \quad \bar{\xi} W \gamma_\mu^\perp P_L h_v \quad C(\bar{P}^+,\mu)$$

↑ our heavy-to-light current from earlier $\equiv J_{eff}^\mu$

The coefficient $C(\bar{P}^+)$ has $\bar{P}^+ = m_b$ since this is total momentum of s-quark jet in $\bar{n}\cdot P_X$

Factor with Field redefn

$$J_{eff}^\mu = \bar{\xi}_n^{(0)} W^{(0)} \gamma_\mu^\perp P_L Y^+ h_v$$



$$T_{eff} = i\int d^4x \, e^{i(m_b\frac{\bar{n}}{2} - q)\cdot x} \langle \bar{B} | T J_{eff}^{+\mu}(x) J_{eff\,\mu}(0) | \bar{B}\rangle$$

factored

$$= i\int d^4x \, e^{i()} \quad \langle \bar{B} | T (\bar{h}_v Y)(x) \, (Y h_v)(0) \, | \bar{B}\rangle$$

$$* \quad \langle 0 | T \, (W^{+(0)}\xi^{(0)})(x) \, (\bar{\xi}^{(0)}W)(0) \, | 0\rangle$$

← spin & color indices & structures $\gamma_\mu^\perp P_L$ suppressed

$$= \frac{1}{2}\int d^4x \int \text{đ}^4k \, e^{i(m_b\frac{\bar{n}}{2} - q - k)\cdot x} \quad \langle \bar{B} | T (\bar{h}_v Y)(x) \, (Y^+ h_v)(0) \, | \bar{B}\rangle$$

$$* \quad J_P(k)$$

$$\langle 0 | T \, (W^+\xi)_{P,\omega_\perp} \, (\bar{\xi}W) \, |0\rangle = \frac{i}{P^-}\int \text{đ}^4k \, e^{-ik\cdot x} \, J_P(k) \, \frac{\not{\bar{n}}}{2}$$

↑ minus & ⊥ labels

↑ only depends on k^+! so do $k^-, k^\perp$ integrals

in Teff we then get → $\delta(x^+)\,\delta^2(x_\perp)$ →

$$S(\ell^+) = \frac{1}{2}\int \frac{dx^-}{4\pi} \, e^{-i/2\,\ell^+x^-} \, \langle \bar{B} | T[\bar{h}_v Y)(\tfrac{n}{2}x^-) \, (Y^+h_v)(0) \, |\bar{B}\rangle$$

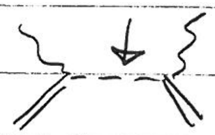
↑ ↑ $Y(\frac{n}{2}x^-, 0)$

$$= \frac{1}{2}\langle \bar{B}_v | \, \bar{h}_v \, \delta(in\cdot D - k^+) \, h_v \, | \bar{B}_v\rangle$$

imaginary part is in jet function

let $J(k^+) \hat{=} -\frac{1}{\pi}\, \mathrm{Im}\, J_p(k^+)$

(tree level $J(k^+) = \delta(k^+)$ from



)

All order's factorization

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_\gamma} = N\, \underset{\underset{p^2\sim m_b^2}{\uparrow}}{C(m_b,\mu)} \int_{2E_\gamma - m_b}^{\bar{\Lambda}} d\ell^+\, \underset{\underset{p^2\sim\Lambda^2}{\uparrow}}{S(\ell^+)}\, \underset{\underset{p^2\sim m_b\Lambda}{\uparrow}}{J(\ell^+ + m_b - 2E_\gamma)}$$

$\uparrow$

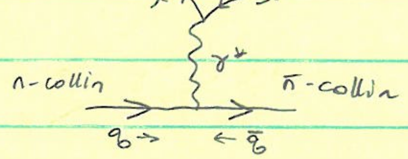
Shape function
is <u>seen</u> in the
data

Final Example: Drell-Yan $pp \to X \ell^+ \ell^-$

- prototype LHC process (pp in, measure leptons, replace $\ell^+\ell^-$ by jets, ..., etc)

$\boxed{\text{Kinematics}}$ $p \; p \to X \underbrace{(\ell^+\ell^-)}$ CM frame

$$p_A + p_B = p_X + q$$

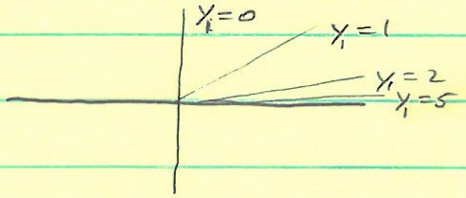


$E_{cm}^2 = (p_A + p_B)^2$ collision energy

q^2 hard scale of partonic collision

$\tau \equiv q^2/E_{cm}^2 \leq 1$

$Y = \frac{1}{2} \ln\left(\frac{p_b \cdot q}{p_a \cdot q}\right)$ total lepton rapidity (angular variable)



$$\left.\begin{aligned} x_a &\equiv \sqrt{\tau}\, e^{Y} \\ x_b &= \sqrt{\tau}\, e^{-Y} \end{aligned}\right\} \text{analogs of Bjorken Var in DIS}$$

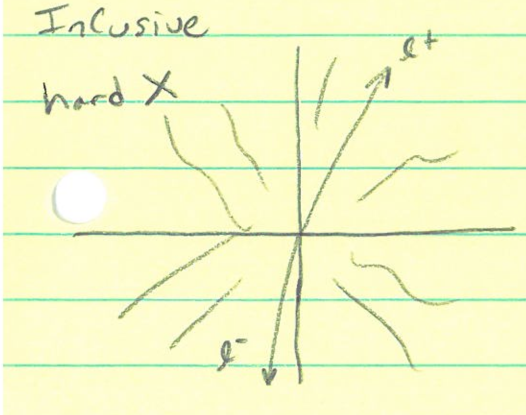
$$\tau \leq x_{a,b} \leq 1$$

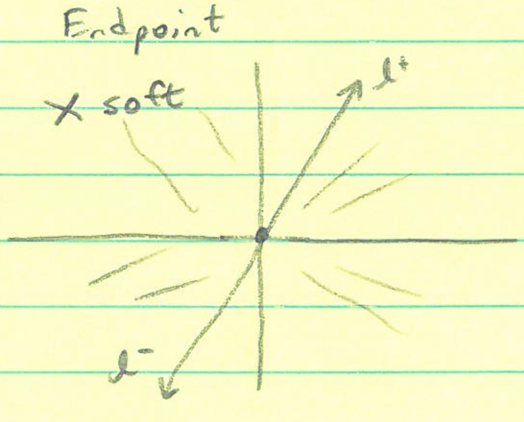
parton fractions ($\xi_a = x_a$ tree level) $x_a \leq \xi_a \leq 1$

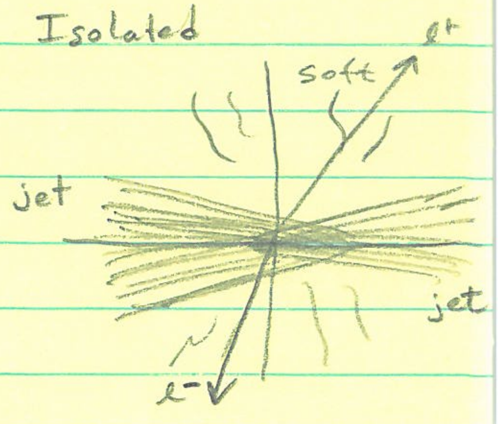
$x_b \leq \xi_b \leq 1$

$$p_X^2 \leq E_{cm}^2 (1 - \sqrt{\tau})^2$$

Cases:	Inclusive	$\tau \sim 1$	$p_X^2 \sim q^2 \sim E_{cm}^2$	$x_{a,b} \sim 1$	$\xi_{a,b} \sim 1$
	Endpoint	$\tau \to 1$	$\underset{\uparrow \text{ usoft}}{p_X^2} \ll q^2 \to E_{cm}^2$	$x_{a,b} \to 1$	$\xi_{a,b} \to 1$
	(Small x	$\tau \to 0$	take $\xi_a, \xi_b \to 0$)		
	"Isolated"	$\tau \sim 1$	$p_X^2 \to$ two ISR jets		$x_{a,b} \sim \xi_{ab} \sim 1$

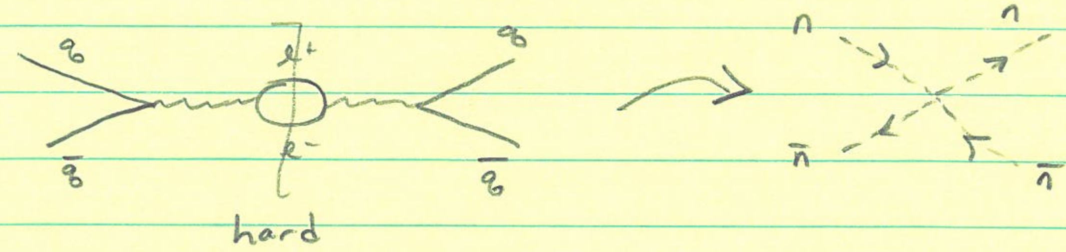






Inclusive $p_n\, p_{\bar n} \to X_{hard}\,(\ell^+\ell^-)$

Factorization: SCET$_I$ problem (hard-collinear Factorization)



4-quark operator in SCET, which after Fierzing is

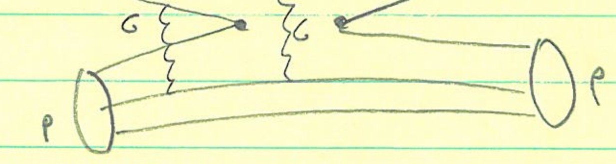
$$\left[(\bar\xi_n W_n)\frac{\bar{n}\!\!\!/}{2}(W_n^\dagger \xi_n)\right]\left[(\bar\xi_{\bar n} W_{\bar n})\frac{n\!\!\!/}{2}(W_{\bar n}^\dagger \xi_{\bar n})\right]$$

- $T^A \otimes T^A$ octet structure vanishes under $\langle p_n | \cdots | p_n\rangle$
- $\xi_n \to Y_n \xi_n$, $\xi_{\bar n} \to Y_{\bar n}\xi_{\bar n}$ etc, no coupling to soft gluons, they cancel out
- $\langle p_n | \bar\chi_{n,\omega} \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega'} | p_n\rangle$ gives PDF
  $\langle p_{\bar n} | \bar\chi_{\bar n,\bar\omega} \frac{n\!\!\!/}{2} \chi_{\bar n,\bar\omega'} | p_{\bar n}\rangle$ " "

$$\frac{1}{\sigma_0}\frac{d\sigma}{dq^2\,dY} = \sum_{i,j}\int_{x_a}^{1}\frac{d\xi_a}{\xi_a}\int_{x_b}^{1}\frac{d\xi_b}{\xi_b}\, H_{ij}^{incl}\left(\frac{x_a}{\xi_a},\frac{x_b}{\xi_b},q^2,\mu\right) f_i(\xi_a,\mu)\, f_j(\xi_b,\mu)$$
$$* \left[1 + \mathcal{O}\left(\frac{\Lambda_{QCD}}{\sqrt{q^2}}\right)\right]$$

- One more (important) caveat, "Glauber Gluons"

$$p_G^\mu \sim (\overset{+}{\lambda^2}, \overset{-}{\lambda^2}, \overset{\perp}{\lambda})$$



These gluons cancel out at Leading Order (Proving this would take us too far afield)

Threshold Limit

only certain terms in H_{ij}^{incl} contribute (most singular in $1-\tau$)

$$H_{ij}^{incl} \to S^{thr}_{q\bar{q}}\left[\sqrt{q^2}\left(1-\frac{\tau}{\xi_a\xi_b}\right),\mu\right] H_{ij}(q^2,\mu)\left[1+\mathcal{O}(1-\tau)^0\right]$$

$\uparrow$ $ij = u\bar{u}, d\bar{d}, \ldots$ quarks
no glue

$\xi_{a,b}\to 1$ so one parton in each proton carries all the momentum (not the dominant LHC region but pdf's may enhance the importance of these terms)

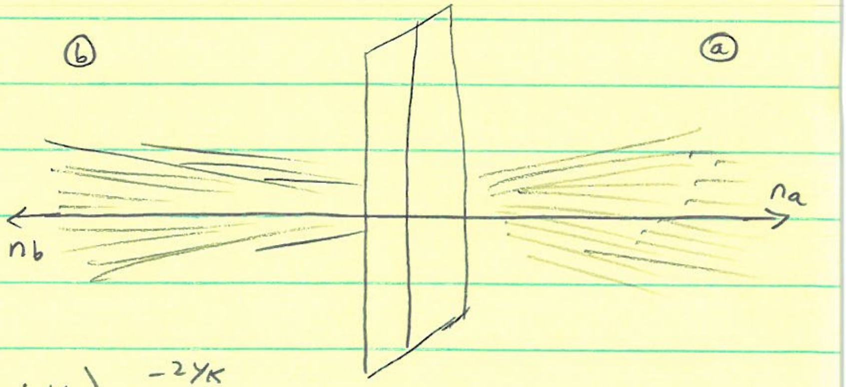
Isolated DY

- allow forward jets to carry away part of Ecm, so $\xi_{a,b}\not\to 1$
- restrict central region to still only have soft radiation (signal region is bkgnd free, no jets, ie jet veto)

need to observe something to guarantee this.

Observable $P_X = B_a + B_b$

- two hemispheres, ⊥ to the beam axis



$$B_a^+ = n_a\cdot B_a = \sum_{k\in a} n_a\cdot p_k$$
$$= \sum_{k\in a} E_k\,(1+\tanh y_k)\,e^{-2y_k}$$

plus momenta for n-collinear radiation should be small

Take $B_a^+ \le Q\,e^{-2y_{cut}} \ll Q$ $Q=\sqrt{q^2}$

$B_b^+ \equiv n_b\cdot B_b \le$ " $\ll Q$

does the trick
(inclusive variable for jet veto)

n-collinear : proton ⓐ <u>and</u> jet ⓐ

we do not simply get a PDF from the hard-collinear-soft factorization

[Glauber's again cancel]

$$\frac{1}{\sigma_0}\frac{d\sigma}{dq^2\,dY\,dB_a^+\,dB_b^+} = \sum_{ij} H_{ij}(q^2,\mu)\int dk_a^+\,dk_b^+\;Q^2\,B_i[\omega_a(B_a^+-k_a^+),x_a,\mu]$$

$$* \; B_j[\omega_b(B_b^+-k_b^+),x_b,\mu]$$

$$* \; S_{ihemi}(k_a^+,k_b^+,\mu)$$

$$* \left[1+\mathcal{O}\left(\frac{\Lambda_{QCD}}{Q},\frac{\sqrt{B_{a,b}^+\,\omega_{a,b}}}{Q}\right)\right]$$

where $\omega_{a,b} = x_{a,b}\,E_{cm}$

B_i = "beam function"

$$B_q(\omega b^+,\omega/P^-,\mu) = \frac{\theta(\omega)}{\omega}\int\frac{dy^-}{4\pi}\,e^{ib^+y^-/2}\langle p_n(P^-)|\,\bar{\chi}_n\!\left(y^-\tfrac{n}{2}\right)\delta(\omega-\bar{P})\frac{\bar{n}\!\!\!/}{2}\chi_n(0)\,|p_n(P^-)\rangle$$

recall jet fn $\langle 0|\,\bar{\chi}_{n,\omega}\!\left(y^-\tfrac{n}{2}\right)\frac{\bar{n}\!\!\!/}{2}\chi_n(0)\,|0\rangle$

PDF $\langle p|\,\bar{\chi}_{n,\omega}(0)\,\frac{\bar{n}\!\!\!/}{2}\chi_n(0)\,|p\rangle$

beam function is mix of both

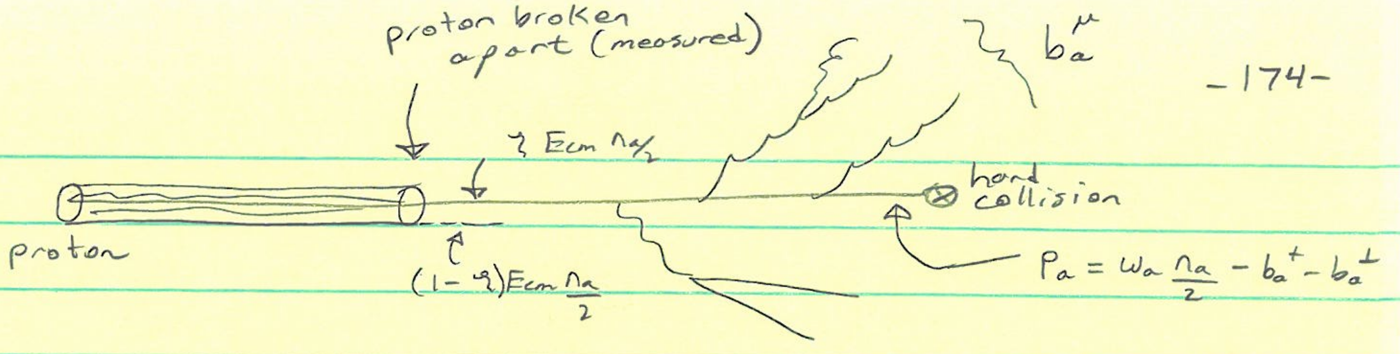
proton = $SCET_{II}$ collinear

jet = $SCET_{I}$ collinear (B_q is in $SCET_I$)

Match $SCET_I \to SCET_{II}$:

$$B_i(t,x,\mu) = \sum_j\int_x^1\frac{d\xi}{\xi}\,\mathcal{I}_{ij}\!\left(t,\frac{x}{\xi},\mu\right)f_j(\xi,\mu)\quad\left[1+\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{t}\right)\right]$$

$\uparrow$ f_g & f_q contribute to B_q (B_g)

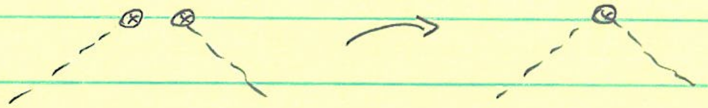


$$b_a^\mu = (\xi - x) E_{cm} \frac{n_a}{2} + b_a^+ \frac{\bar{n}_a}{2} + b_{a\perp}$$

$$p_a^2 = -\underbrace{\omega_a b_a^+}_{t_a \gg \Lambda_{QCD}} - \vec{b}_{\perp a}^{\,2} \leq 0$$

spacelike active parton participates in hard collision

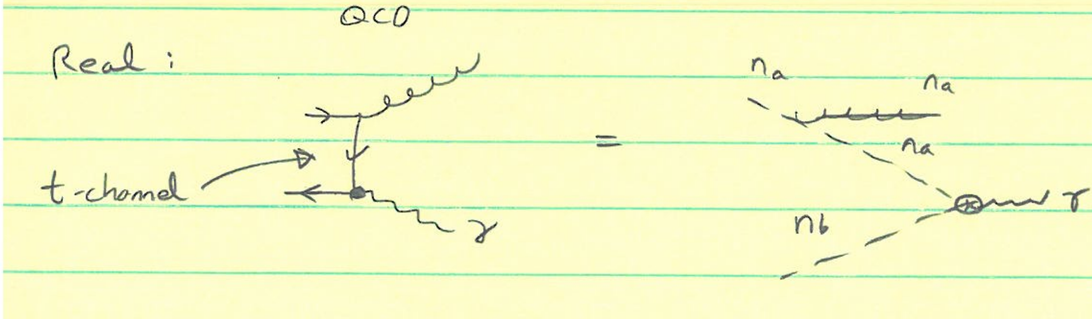
Tree-Level



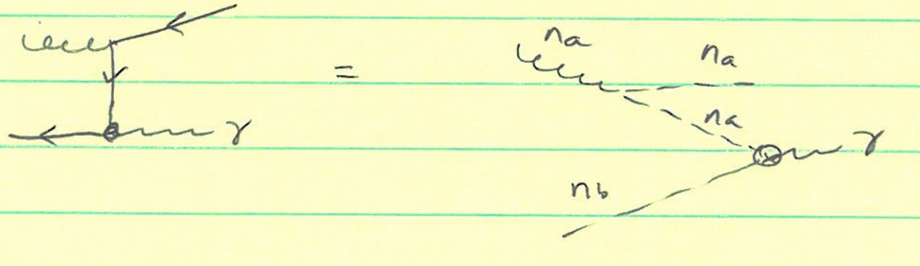
$$B_i(t, x, \mu) = \delta(t) \, f_i(x, \mu)$$

Order α_s Real & Virtual Contractions

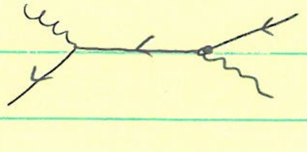
Real:



$\mathcal{I}_{qq}^{(\alpha_s)}$



$\mathcal{I}_{qg}^{(\alpha_s)}$



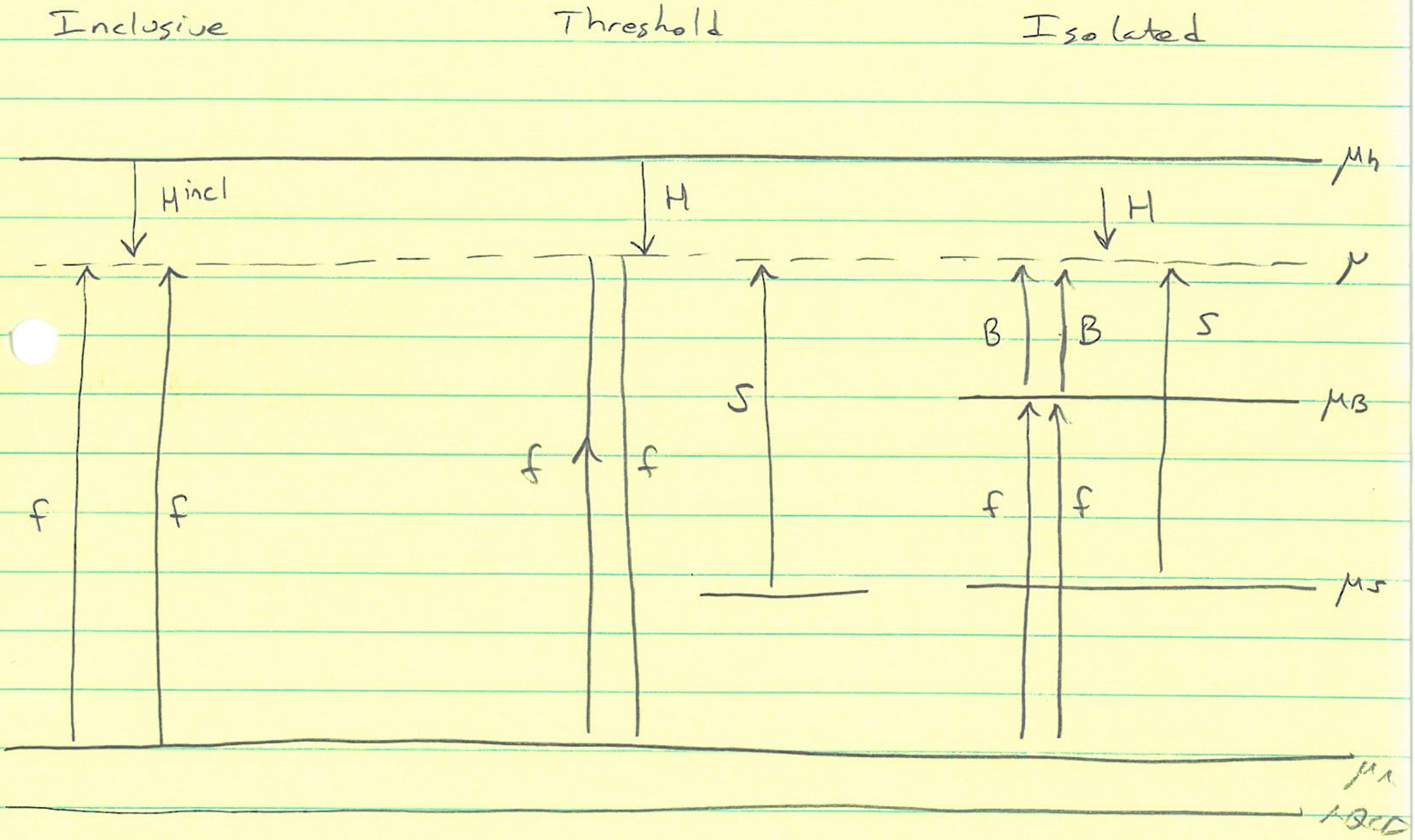
power correction $\sim \frac{t}{s} \sim \frac{\omega B_a^+}{Q^2}$

(would be ~ 1 for inclusive)

$\boxed{\text{RGE}}$ $\quad \mu \frac{d}{d\mu} B_i(t,x,\mu) = \int dt' \, \gamma_i(t-t',\mu) \, B_i(t',x,\mu)$

like the jet function
(invariant mass evolution)

- sums $\ln^2(t/\mu)$
- indep of x & no mixing



consistency of
RGE for isolated case requires B's since
H and S have double logs, but f's do not