Singularities and Mode Factorization in Field Theories

"The Zero-Bin"

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SLAC seminar, Nov. 2006

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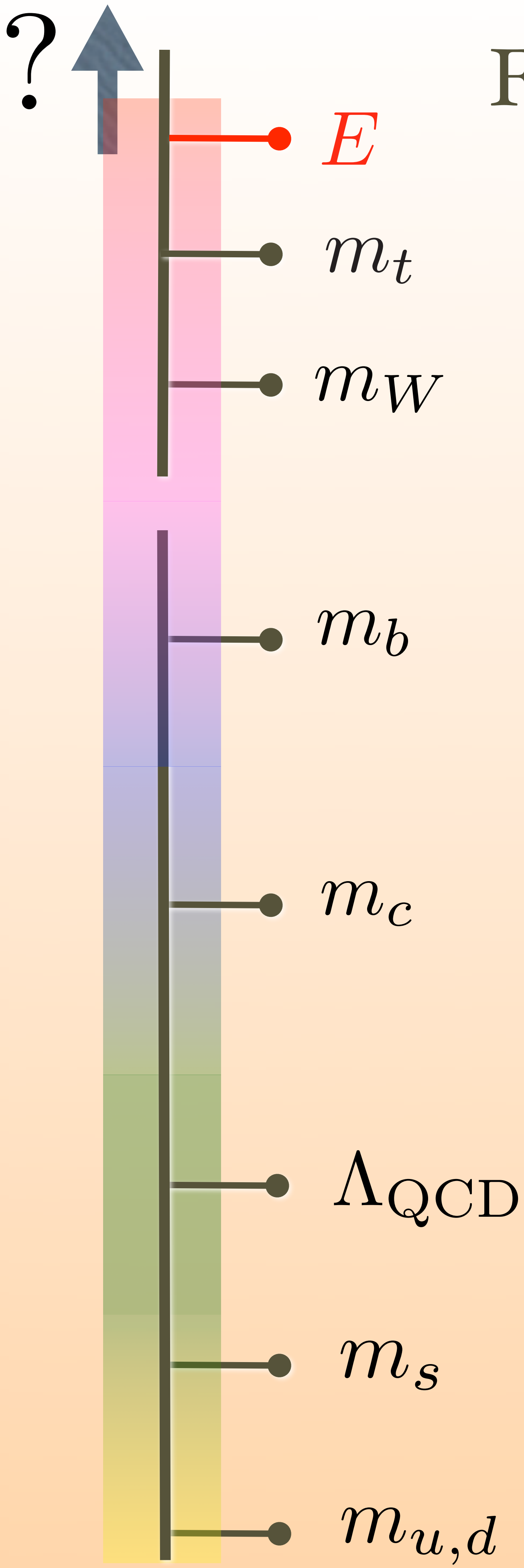
Outline

- Singularity problems in field theory
- Singularities and momentum space modes
- Wilsonian vs. Continuum EFT ;
  " Differential EFT's " and a tiling formula

Applications:

- Confirmation with non-relativistic systems
- Understanding collinear singularities in jets
- Rapidity factorization with singular convolutions

eg. annihilation effects in $B \to K\pi$



Factorization: separation of short distance (perturbative) physics from long distance dynamics

$\alpha_s(\mu)$

Hard processes with large momentum transfer:

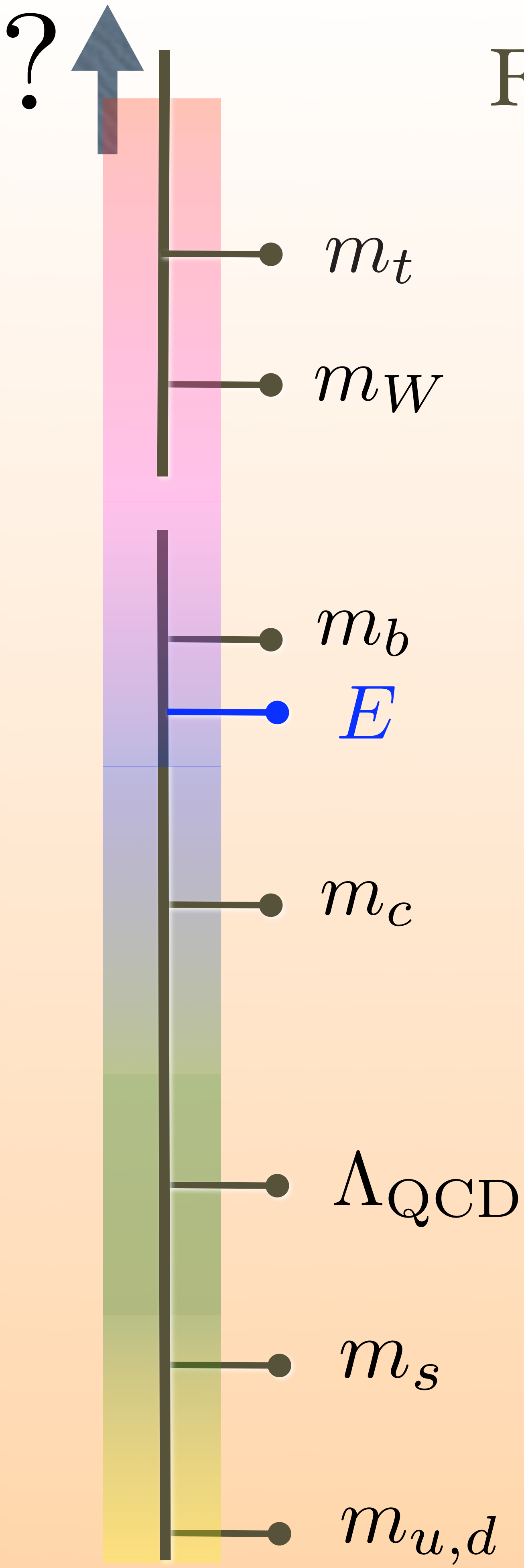
$$e^- p \to e^- X \qquad p\bar{p} \to X \ell^+ \ell^- \qquad \Upsilon \to X\gamma$$

$$e^+ e^- \to \text{jets} \qquad \gamma^* M \to M' \qquad p\bar{p} \to J/\Psi X$$

at the LHC:

$$pp \to HX$$

$$\sigma = \sum_{ij} \int dx_1 dx_2 \hat{\sigma}(ij \to H + X, \mu) f_i^p(x_1, \mu) f_i^p(x_2, \mu)$$



Factorization: separation of short distance (perturbative) physics from long distance dynamics

$\alpha_s(\mu)$

B - decays by weak interactions:

$B \to X_u \ell \bar{\nu}$ $\quad B \to D\pi$ $\quad B \to \pi \ell \bar{\nu}$ $\quad B \to K^* \gamma$

$B \to \gamma \ell \bar{\nu}$ $\quad B \to \rho\rho$ $\quad B \to \pi\pi$ $\quad B \to \rho\gamma$

$B \to K\pi$ $\quad B \to D^* \eta'$

$B \to X_s \gamma$

$$\frac{d\Gamma}{dE_\gamma} = \left|C(m_b,\mu)\right|^2 \int dk^+ \, {\rm Im} J_P(k^+,\mu) S(2E_\gamma - m_b + k^+,\mu)$$

$$E_\gamma \gg \Lambda_{\rm QCD}$$

Does factorization always work?

Given an experimentally measurable observable:

$$\sigma = \sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)} + \ldots \qquad \sigma^{(k)}, \Gamma^{(k)} \sim \left(\frac{\Lambda_{\rm QCD}}{Q}\right)^k$$

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \ldots$$

Can we separate each term into well defined short & long distance parts?

$\surd$ OPE (as in DIS, $B \to X_c e \bar{\nu}$, $\ldots$)

? Other processes treated on a case by case, order by order basis

➡ Effective theories (SCET, ...) allow us to formulate the long & short distance split with operators and Wilson coefficients order by order. Obtain relations between how factorization works in different processes.

What can go wrong? convolution singularities

$$\int_0^1 dx\, C(x)\phi_\pi(x) = \int_0^1 dx\, \frac{\phi_\pi(x)}{x^2} \sim \int_0 dx\, \frac{1}{x} =?$$

here $\phi_\pi(x)$ is the twist-2 pion distribution function

(A common property of these singularities is that they are not regulated by dimensional regularization.)

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A common property of these singularities is that they are not regulated by dimensional regularization.

Need to formulate EFT's like SCET without using dim.reg.

In a large number of cases, this type of singularity does not occur, the convolutions are finite.

We can proceed to sum logs with anomalous dimensions,
compute perturbative matching corrections,
fit the nonperturbative functions,
and do phenomenology.

These are not the cases we will focus on today.

Singular Examples: tend to occur at subleading power or in more differential observables

- large Q^2 pion form factor at subleading twist
- the $\gamma^* \rho \to \pi$ form factor, $F_{\rho\pi} \sim 1/Q^4$
- large Q^2 Pauli nucleon form factor, F_2
- Drell-Yan at low transverse momentum
- semi-inclusive DIS, $e^- p \to e^- \pi(k_\perp) X$
- $B \to \pi e \bar{\nu}$ form factor
- annihilation power corrections in $B \to K\pi$
- $\bar{B}^0 \to D_s K^-$ decays with $m_b, m_c \gg \Lambda_{\rm QCD}$
- $\ldots$

A representative list for $B \to \pi \ell \bar{\nu}$:

Szczepaniak, Henley, Brodsky ('90)
Burdman, Donoghue ('92)
Belyaev, Khodjamirian, Ruckl ('93)
Charles et.al. ('98)
Bagan, Ball, Braun ('98)
Beneke, Feldmann ('01)
Bauer, Pirjol, I.S. ('01)
Kurimoto, Li, Sanda ('02)

More recently:

Bauer, Pirjol, I.S. ('02)
Becher,Hill,Lange,Neubert ('03)
Beneke, Feldmann ('03)

- 't Hooft model (large N_c) form factors at LO in $1/Q^2$

Einhorn ('76)

Lets look at two specific examples.

i) $k_\perp$-dependent parton distribution functions

ii) $B \to \pi \ell \bar{\nu}$

Parton Distribution Functions in DIS

Standard (integrated) p.d.f.

$$W_n(y^-) = P\exp\Big(ig\int_0^\infty ds\ \ n\cdot A(ns+ny^-)\Big)$$

$$f(x,\mu) = \int \frac{dy^-}{4\pi} e^{-ixp^+y^-} \langle p|\bar\psi(0,y^-,0_\perp)W_n(y^-,0)\gamma^+\psi(0)|p\rangle_{\rm ren}$$

$$W_n(y^-,0) = W_n^\dagger(y^-)W_n(0)$$

$$n^2 = 0$$

or $$f(x,\mu) = \langle p|\bar\chi\,\delta\Big(x - \frac{\mathcal{P}^\dagger}{p^+}\Big)\frac{\gamma^+}{p^+}\,\chi|p\rangle$$

$$\chi(k^+) = \int dy^- e^{-iy^-k^+}W_n^\dagger(y^-)\psi(y^-)$$

$\chi_{n,k^+} = (W_n^\dagger\xi_n)_{k^+}$ in SCET

$k_\perp$ dependent p.d.f.

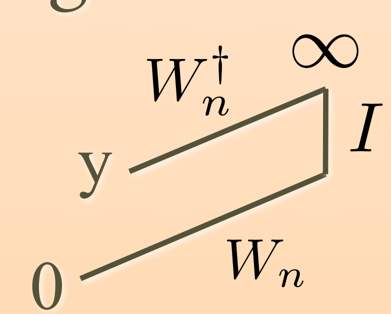
Collins (hep-ph/0304122)

also Brodsky et al, hep-ph/0003082

$$f(x,k_\perp,\mu)=\int\frac{dy^-dy_\perp^2}{16\pi^3}\, e^{-ixp^+y^-+ik_\perp\cdot y_\perp} f(y^-,y_\perp,\mu)$$

$$f(y^-,y_\perp,\mu) \stackrel{?}{=} \langle p|\bar\psi(0,y^-,y_\perp)(\cdots)\gamma^+\psi(0)|p\rangle_{\rm ren}$$

Consider quark m.elt. in light-cone gauge, or with W_n lines in Feyn. gauge:



tree $$f^0(x,k_\perp) = \delta(1-x)\delta^{d-2}(k_\perp)$$

one-loop

singular

$$\int dx\, d^dk_\perp\, t(x,k_\perp) f^1(x,k_\perp)_{qq} = \frac{g^2}{16\pi^3}\int_0^1 dx\, d^dk_\perp\,\big[t(x,k_\perp)-t(1,0_\perp)\big]\Big\{\frac{4}{(1-x)}\frac{1}{k_\perp^2+m_g^2x+m^2(1-x)^2}+\ldots\Big\}$$

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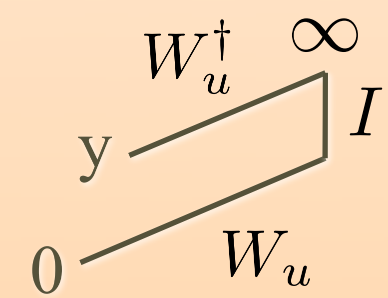
$$f(x,k_\perp,\mu) = \int\frac{dy^-dy_\perp^2}{16\pi^3}\,e^{-ixp^+y^-+ik_\perp\cdot y_\perp}f(y^-,y_\perp,\mu)$$

$$f(y^-,y_\perp,\mu) \stackrel{?}{=} \langle p|\bar\psi(0,y^-,y_\perp)(\cdots)\gamma^+\psi(0)|p\rangle_{\rm ren}$$

Proposed Definitions

$$\zeta = \frac{(p\cdot u)^2}{u^2} \qquad u^2 \neq 0$$



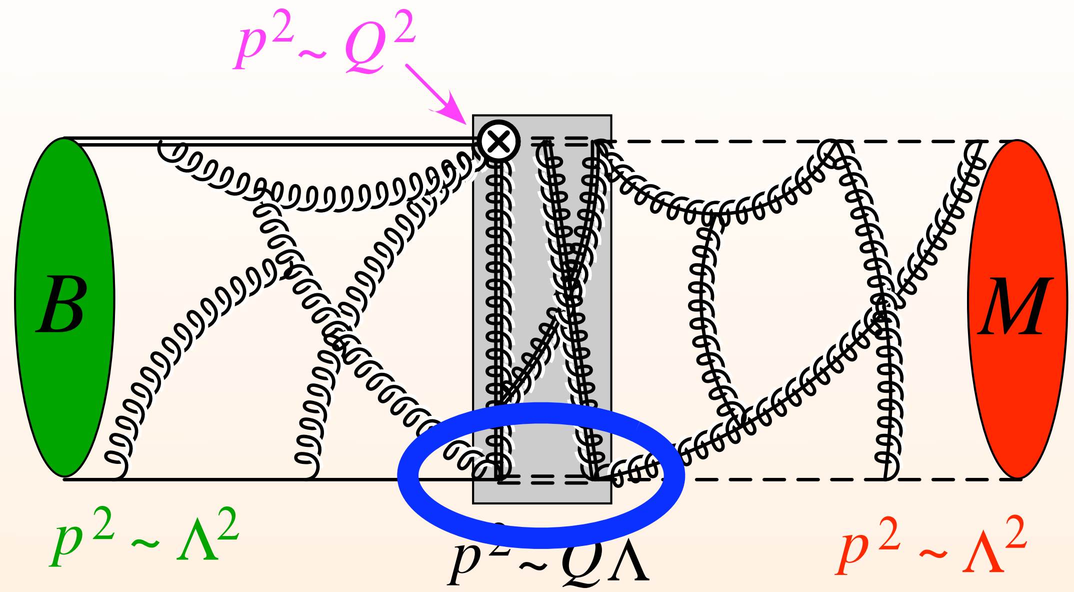
$$f(y^-,y_\perp,\zeta,\mu) = \langle p|\bar\psi(0,y^-,y_\perp)W_u^\dagger(y)I_u(y,0)\gamma^+W_u(0)\psi(0)|p\rangle$$

$$f(y^-,y_\perp,\zeta,\mu) = \frac{\langle p|\bar\psi(0,y^-,y_\perp)W_n^\dagger(y)I_n(y,0)\gamma^+W_n(0)\psi(0)|p\rangle}{\langle 0|W_n^\dagger(y)W_{u'}(y)I_n(y,0)I_{u'}^\dagger(y,0)W_n(0)W_{u'}(0)^\dagger|0\rangle}$$

$B \to \pi \ell \bar{\nu}$

Step 1:

$Q^2 \gg Q\Lambda$



Requires a power suppressed interaction

SCETI

needs time-ordered products

$$Q^{(0)} = \bar{\chi}_{n,\omega} \Gamma \mathcal{H}_v^n$$

$$Q^{(1)} = \bar{\chi}_{n,\omega} ig \mathcal{B}^{\perp}_{n,\omega'} \Gamma \mathcal{H}_v^n$$

with

$$\mathcal{L}^{(1)}_{\xi q} = (\bar{q} Y) ig \mathcal{B}^{\perp}_{n,\omega'} \chi_n \ ,$$

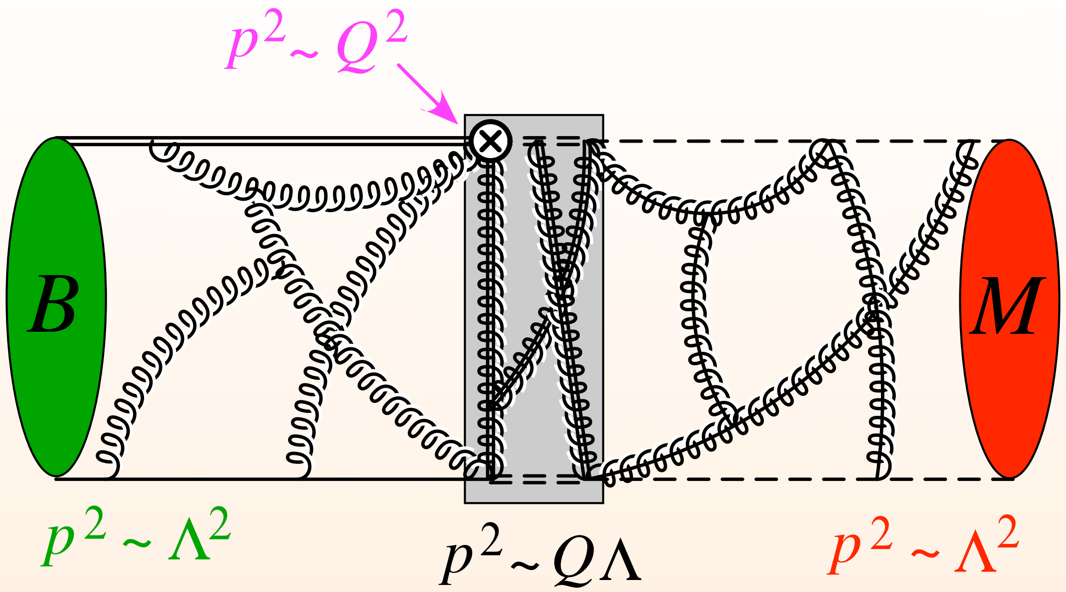
...

$$f(E) = \int dz\, T(z,E)\, \zeta_J^{BM}(z,E) \;+\; C(E)\, \zeta^{BM}(E)$$

no singularity problem here

same functions in $B \to \pi\pi$
universality at $E\Lambda$

Bauer, Pirjol, Rothstein, I.S.

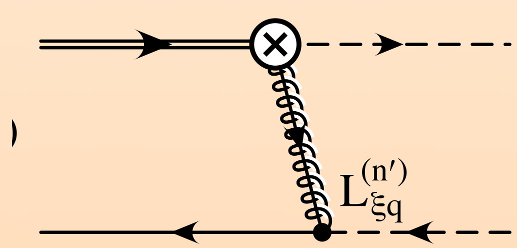


$$f(E) = \int dz\, T(z,E)\, \zeta_J^{BM}(z,E) + C(E)\, \zeta^{BM}(E)$$

Step 2: (further factorization) $Q\Lambda \gg \Lambda^2$ SCETII

ok: $$\zeta_J^{BM}(z) = f_M f_B \int_0^1 dx \int_0^\infty dk^+ J(z,x,k^+,E)\phi_M(x)\phi_B(k^+)$$

but: $\zeta^{BM} = ?$



$$\int_0^1 \mathrm{d}x\, \frac{\phi_\pi(x)}{x^2} = ???$$

endpoint singularity

one x from the Wilson line
one x from the gluon propagator

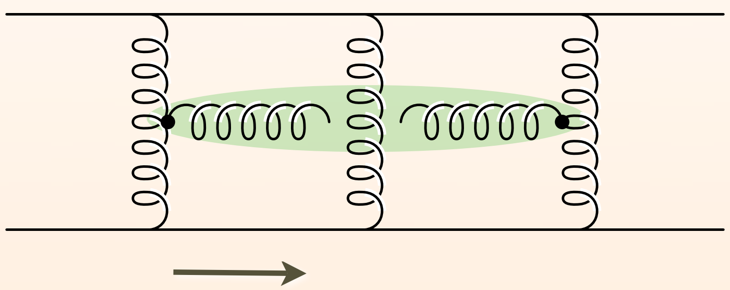
for phenomenology $\zeta^{BM}(E)$ is left unfactorized

Do we know of other examples of singularities of this type?

Three singularities in non-relativistic field theory

1) Static potential in perturbative QCD is IR divergent

Appelquist, Dine, Muzinich ('78)



$E \sim mv$

static potential involves soft gluons

$E \sim \vec{p} \sim mv$

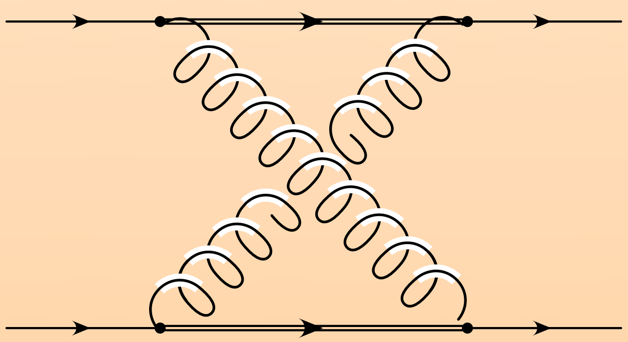
log singularity is cutoff by $E \sim mv^2$ an ultrasoft energy

$$V(q) = -\frac{4\pi\alpha_s(\mu)}{q^2}\left[1 + \frac{\alpha_s(\mu)}{4\pi}\left\{a_1 + \beta_0 \ln\left(\frac{\mu^2}{q^2}\right)\right\} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left\{a_2 + (\beta_1 + 2\beta_0 a_1)\ln\left(\frac{\mu^2}{q^2}\right) + \beta_0^2 \ln^2\left(\frac{\mu^2}{q^2}\right)\right\}\right.$$
$$\left. + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3\left\{8\pi^2 C_A^3 \ln\left(\frac{E_{\rm IR}^2}{q^2}\right) + \ldots\right\}\right]$$

2) Lamb shift in positronium

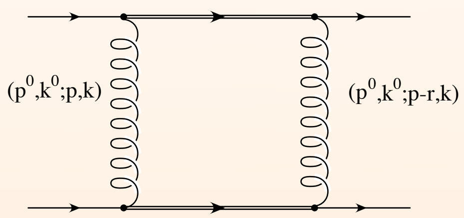
eg. Pineda-Soto '98 in dim.reg.

- both e- can recoil, here soft and ultrasoft regions contribute
- the soft region has an IR divergence, which matches the UV structure of the ultrasoft



3) (Simple!)

pinch singularity with two heavy static (soft) particles



$$\int \frac{dk^0}{(k^0 + i0^+)(-k^0 + i0^+)} f(k^0)$$

$$\frac{1}{k^0 - \frac{\mathbf{k^2}}{2m} + i\epsilon}$$

Several ways out here:

finite T, exp(i T V(r)), calculate V(r)

Gatheral's non-abelian exponentiation theorem

Avoid these poles in the contour integration

eg. static potential for color octet state at two-loops

(Kniehl, Penin, Schroder, Smirnov, Steinhauser '05)

2PI effective action

Non-Relativistic EFT (NRQCD, NRQED)

These singularities come from taking a double limit:

3) $k_0 \gg \dfrac{\mathbf{k}^2}{2m}$, then $k_0 \to 0$

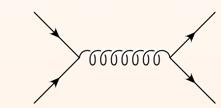
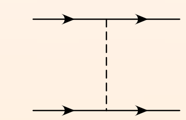
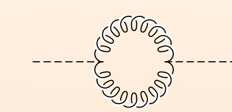
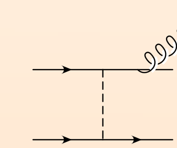
soft overlaps potential region

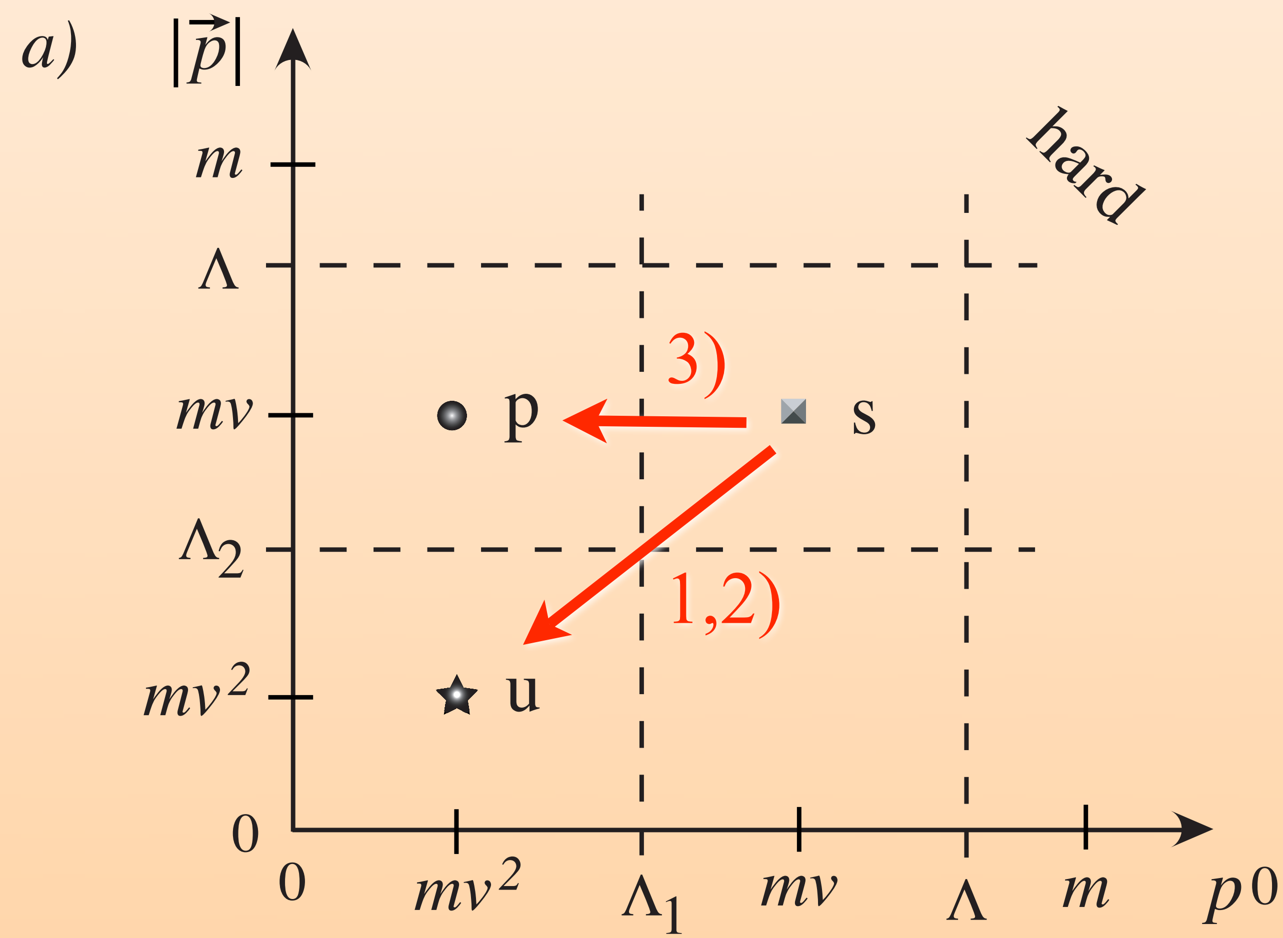
1,2) $k^\mu \gg E$, then $k^\mu \to 0$

soft overlaps ultrasoft region

A different momentum space mode properly describes the the infrared in the region of the singularity.

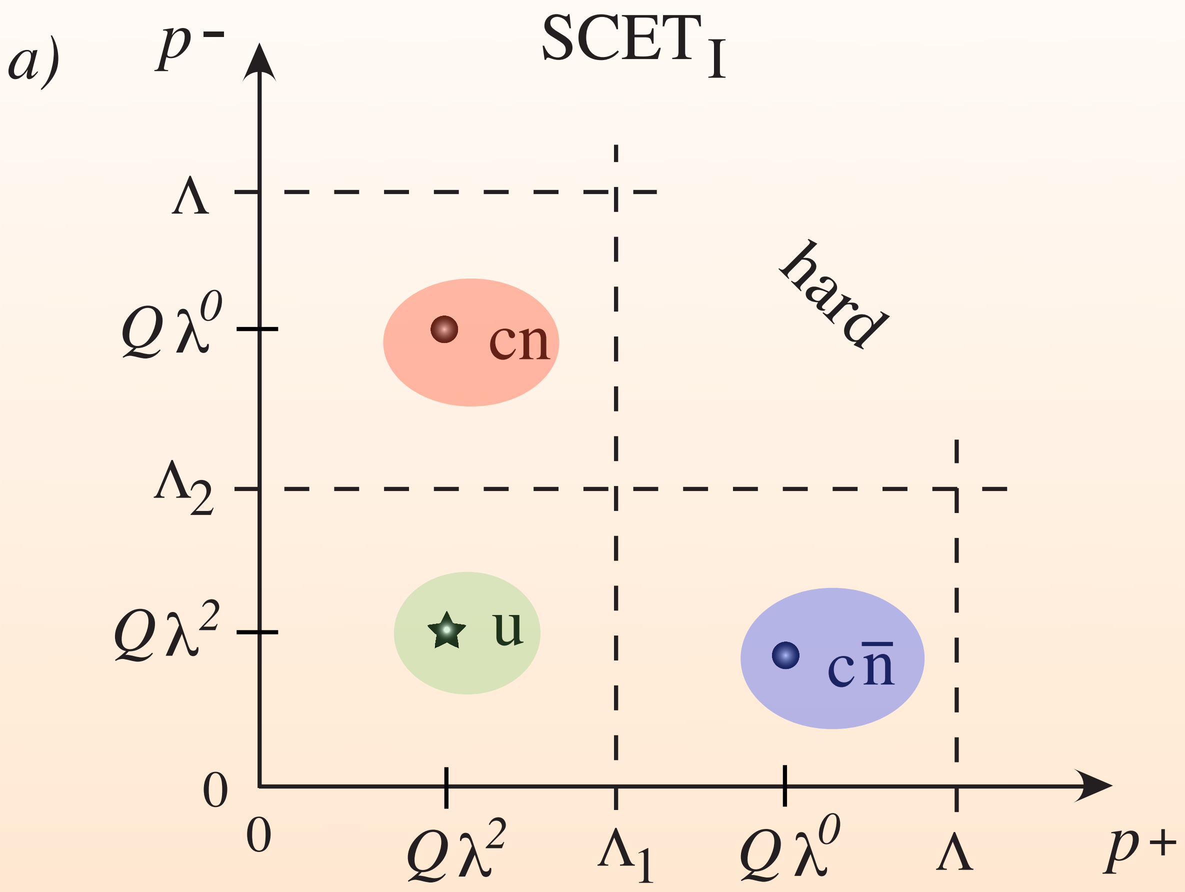
Momentum Regions

	k^0	$\mathbf{k}$	
hard:	m	m	
potential:	mv^2	mv	
soft:	mv	mv	
ultrasoft:	mv^2	mv^2	



Soft - Collinear EFT $_{\mathrm{I}}$

A formalism for jets.



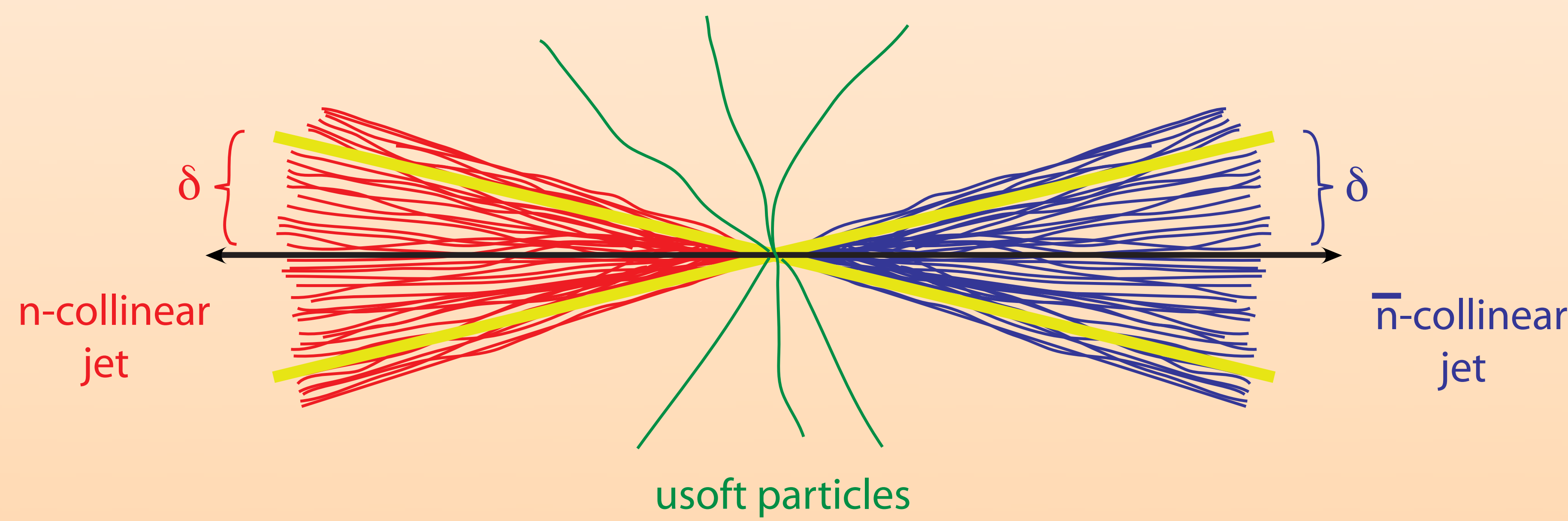
$$p^2 = p^+ p^- + p_\perp^2$$

eg. $e^+e^- \to 2$ jets

ala Korchemsky, Sterman and Bauer et.al.

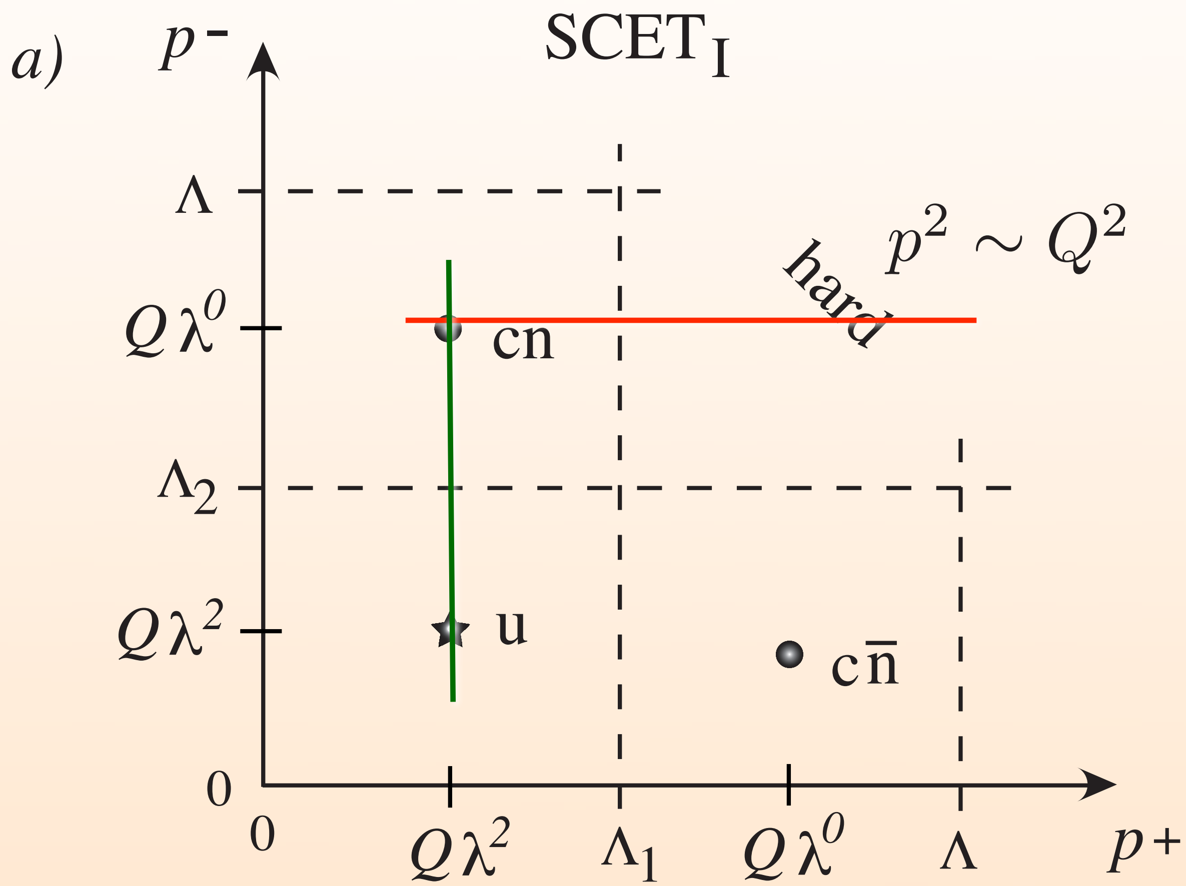
$$\lambda \sim \frac{\Delta}{Q} \qquad m_X^2 \sim \Delta^2$$

$$\Lambda^2 \ll \Delta^2 \ll Q^2$$

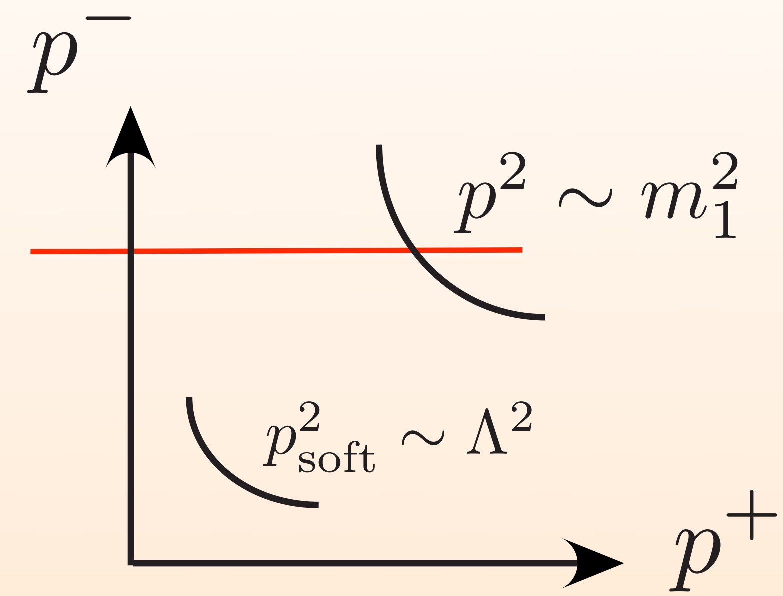


Jet constituents : $p^\mu \sim \left(\frac{\Delta^2}{Q}, Q, \Delta\right) \sim Q(\lambda^2, 1, \lambda)$

Soft - Collinear EFT $_{\mathrm{I}}$



$p^2 = p^+p^- + p_\perp^2$



In SCET a constituent $p^- \sim Q$

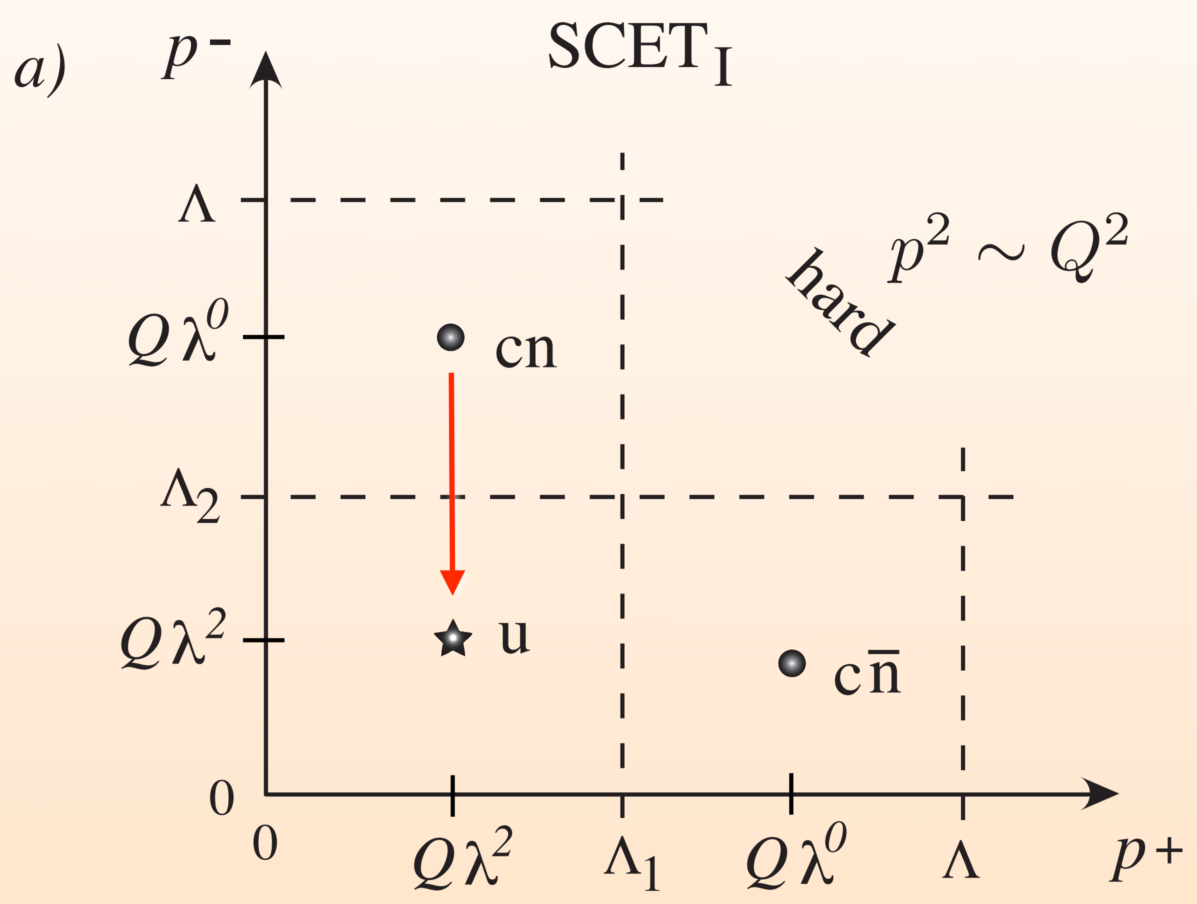
$\int d\omega\ C(\omega)\ O(\omega)$

convolutions

Usually $m_1 \gg \Lambda$

$\sum_{i=1}^{n}\ C_i(\mu, m_1)\ O_i(\mu, \Lambda)$

Soft - Collinear EFT $_{\text{I}}$

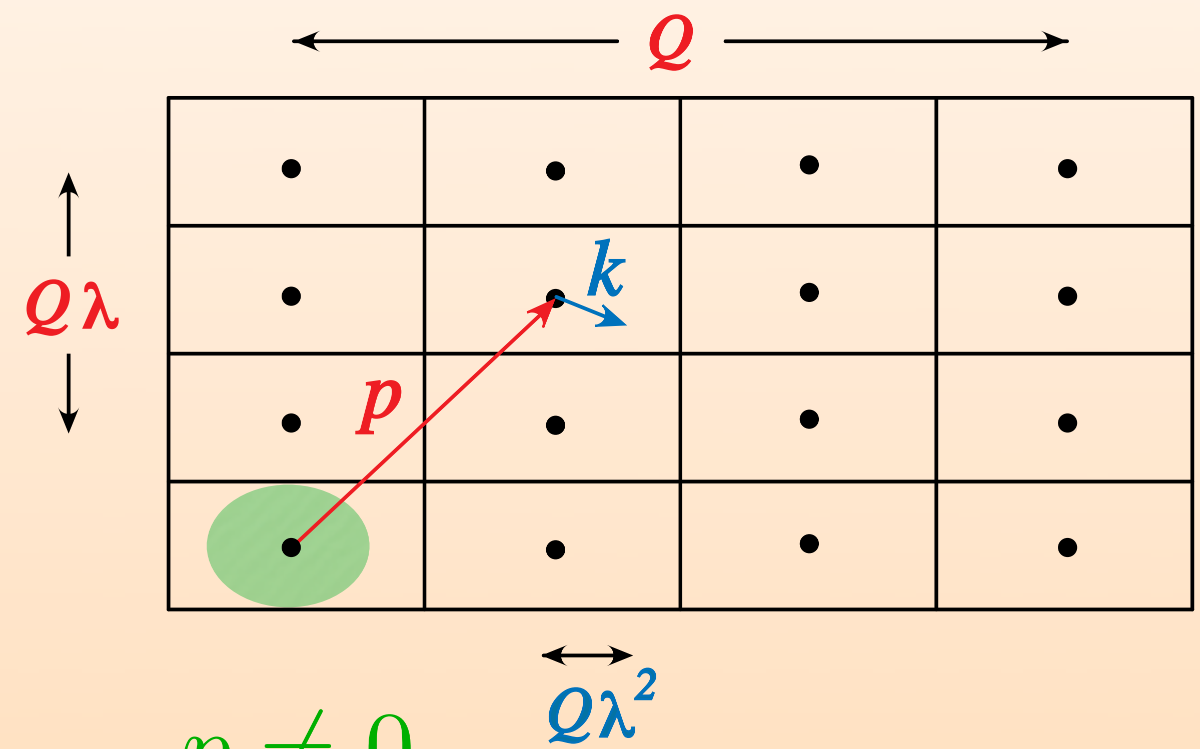


Are there singularities here?

Yes. However, the ones we're interested in occurred for hadron distributions, and will involve a theory SCET$_{\text{II}}$

Separate Momenta (multipole expansion)

		label	residual	
HQET	$P^\mu =$	$m_b v^\mu$	$+\ k^\mu$	$h_v(x)$
SCET	$P^\mu =$	p^μ	$+\ k^\mu$	$\xi_{n,p}(x)$
		$(1,\lambda)$	λ^2	



$p \neq 0$

zero-bin

Collinear Quarks

- $\triangleright\ \psi(x) \to \sum_p e^{-ip\cdot x}\xi_{n,p}(x)$
- $\triangleright\ \not{n}\,\xi_{n,p} = 0$
- $\triangleright\ \partial^\mu\,\xi_{n,p} \sim (Q\lambda^2)\,\xi_{n,p}$

usual derivative

Introduce Label Operator

$$\mathcal{P}^\mu\left(\phi^\dagger_{q_1}\cdots\phi_{p_1}\cdots\right) = \left(p_1^\mu+\ldots-q_1^\mu-\ldots\right)\left(\phi^\dagger_{q_1}\cdots\phi_{p_1}\cdots\right)$$

derivative for labels

$$i\partial^\mu e^{-ip\cdot x}\,\phi_p(x) = e^{-ip\cdot x}\left(\mathcal{P}^\mu + i\partial^\mu\right)\phi_p(x)$$

Separate Momenta (multipole expansion)

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$(1,\lambda)$ $\qquad \lambda^2$

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- $\psi(x) \to \sum_p e^{-ip\cdot x}\xi_{n,p}(x)$
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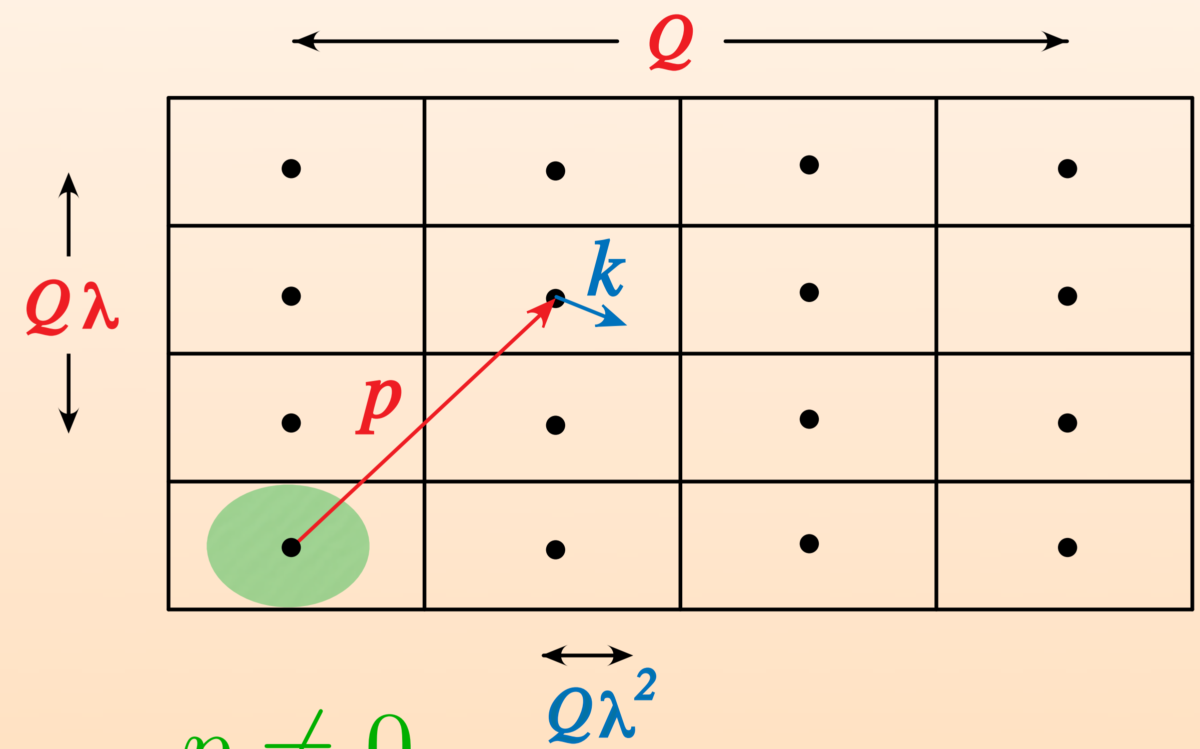


$p \neq 0$

zero-bin

The Naive Replacement

$$\sum_p \int dk \xrightarrow{?} \int dp$$

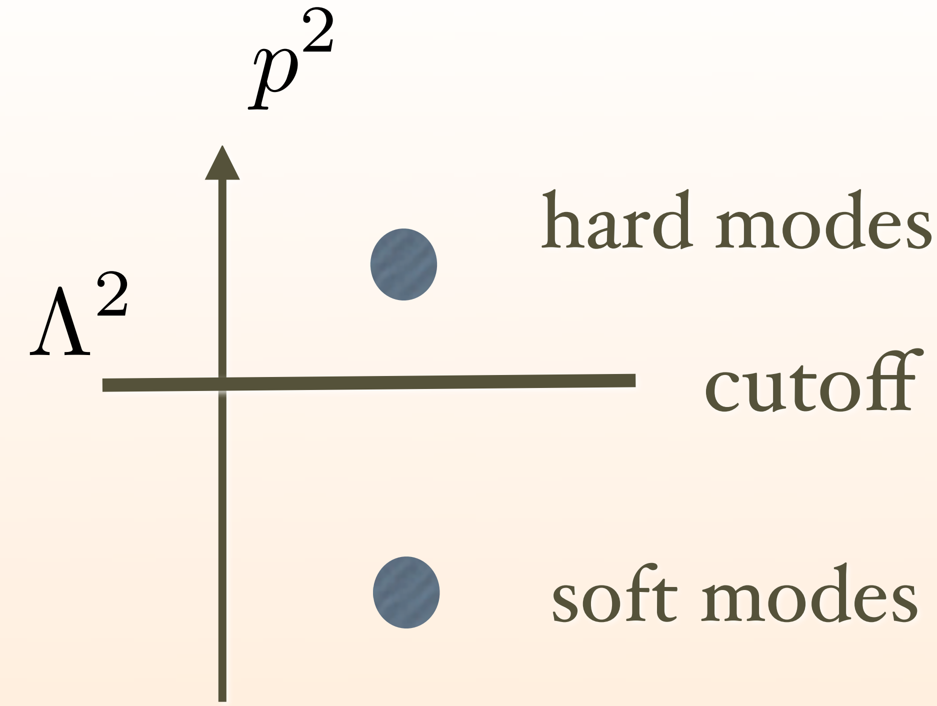
Wilsonian vs. Continuum EFT

Wilson

effective action
for soft modes e^{-S_Λ}

removing modes with $\Lambda - \delta\Lambda < E < \Lambda$

$$e^{-S_{\Lambda-\delta\Lambda}} = \int_{\delta\Lambda} d\phi \; e^{-S_\Lambda}$$

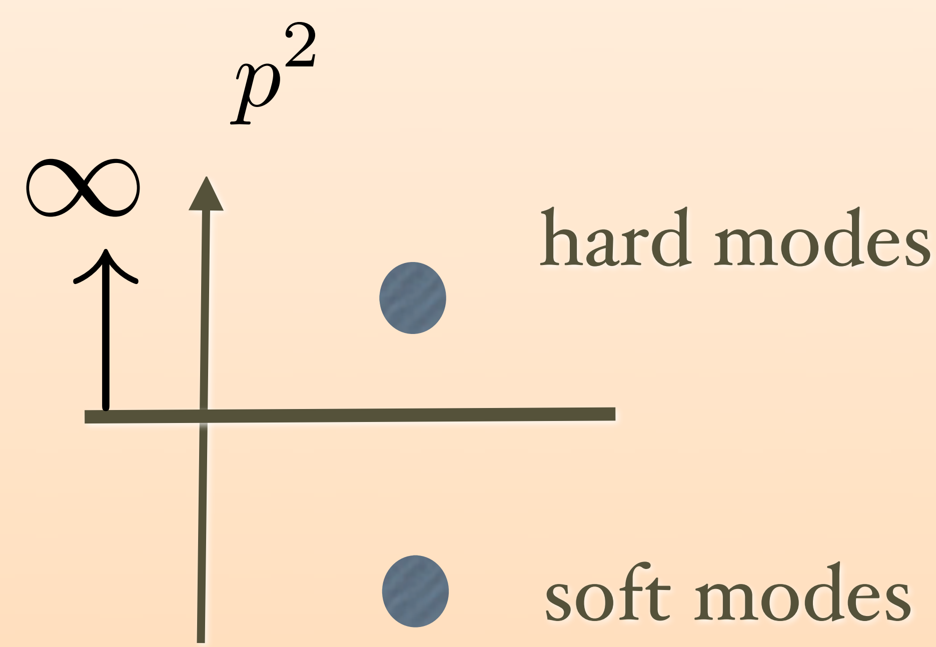


Continuum

$$\mathcal{L}^{\mathrm{EFT}} = C(\mu)\mathcal{O}(\mu)$$

operators for
soft modes $\mathcal{O}(\mu)$

Wilson coefficients
for hard modes $\mathrm{C}(\mu)$



Sending Λ to ∞ includes the hard region in the matrix elements of our operators, but we fix $\mathrm{C}(\mu)$ to correct for this.

How does EFT matching work? (continuum EFT)

- say we have a full and effective theory specified by:

$$\mathcal{L}_{\rm full} \qquad\qquad \mathcal{L}_{\rm EFT}\,, \quad C^i O^i_{\rm EFT}$$

- introduce UV regulators in the two theories, regulate and renormalize them in some scheme.

- calculate S-matrix elements (observables) in the two theories using the same IR regulator (same states).

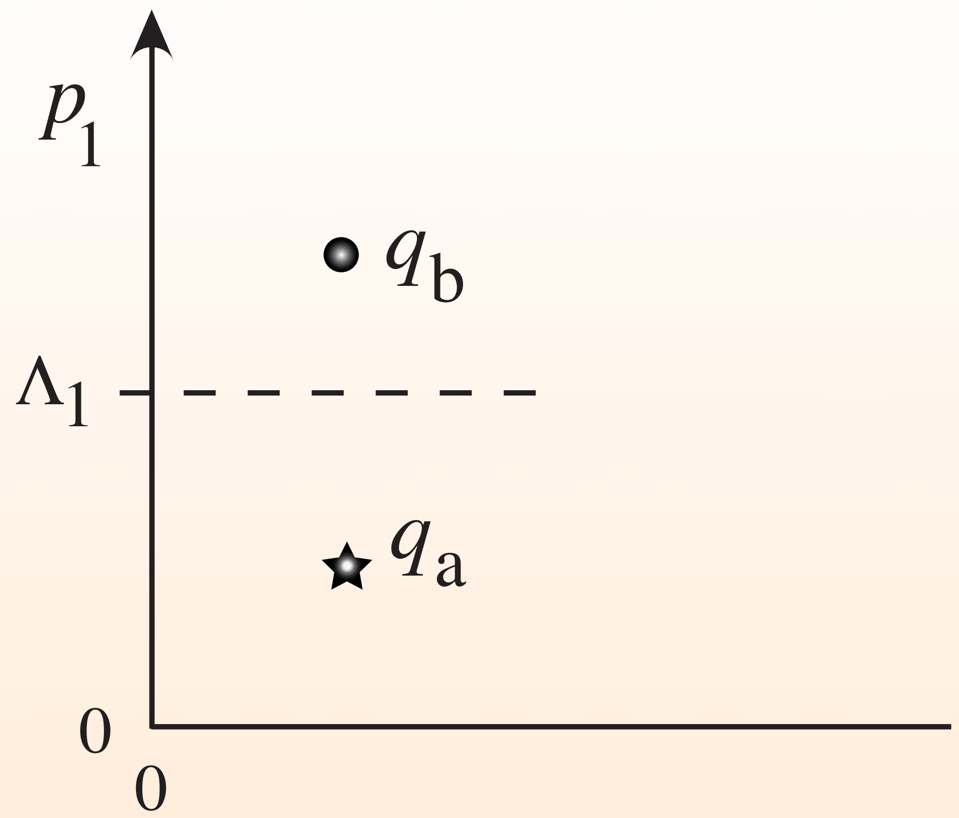
- subtract to determine $C(\mu)$ (that is, we tune C to make the EFT and full theory agree). IR divergences cancel.

Note: • Dependence on μ is from the scheme choice in the EFT (usually $\overline{\rm MS}$). $C(\mu) O_{\rm EFT}(\mu)$ is invariant.

- Any valid IR regulator will give the same $C(\mu)$.

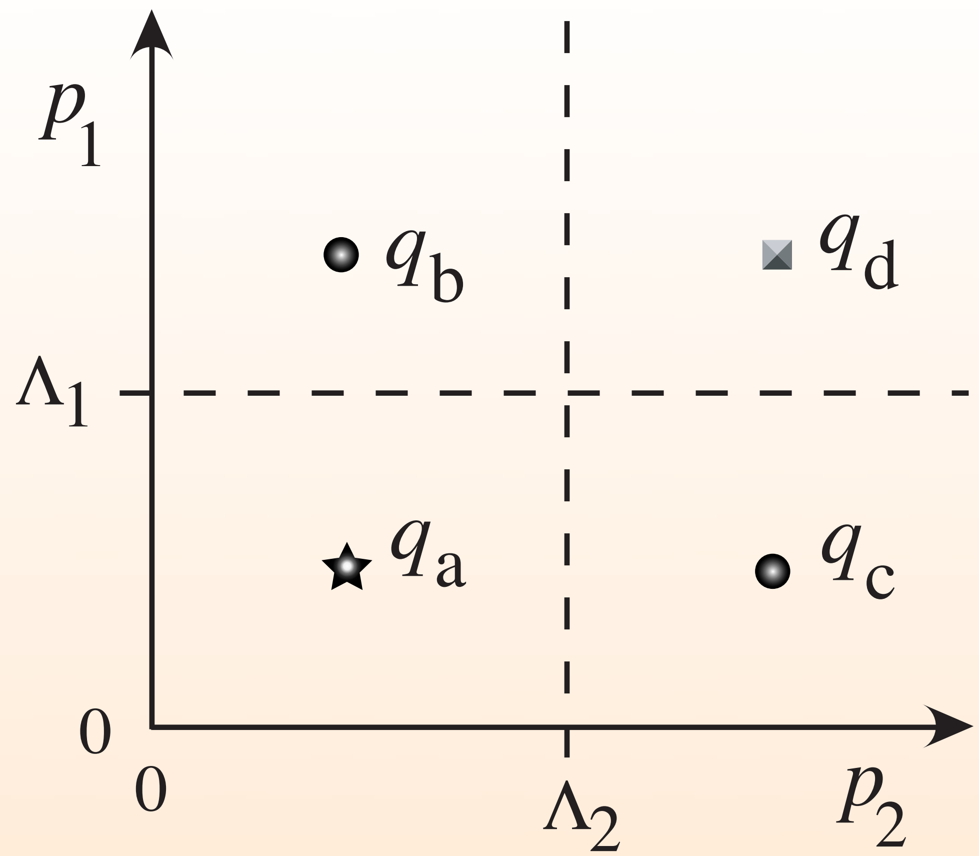
- Wilson EFT is a special case of the continuum EFT procedure if the particle content is left fixed and no approx. are made.

  Simply let $\theta(\Lambda^2 - p^2)$ be the UV regulator and scheme in the EFT.

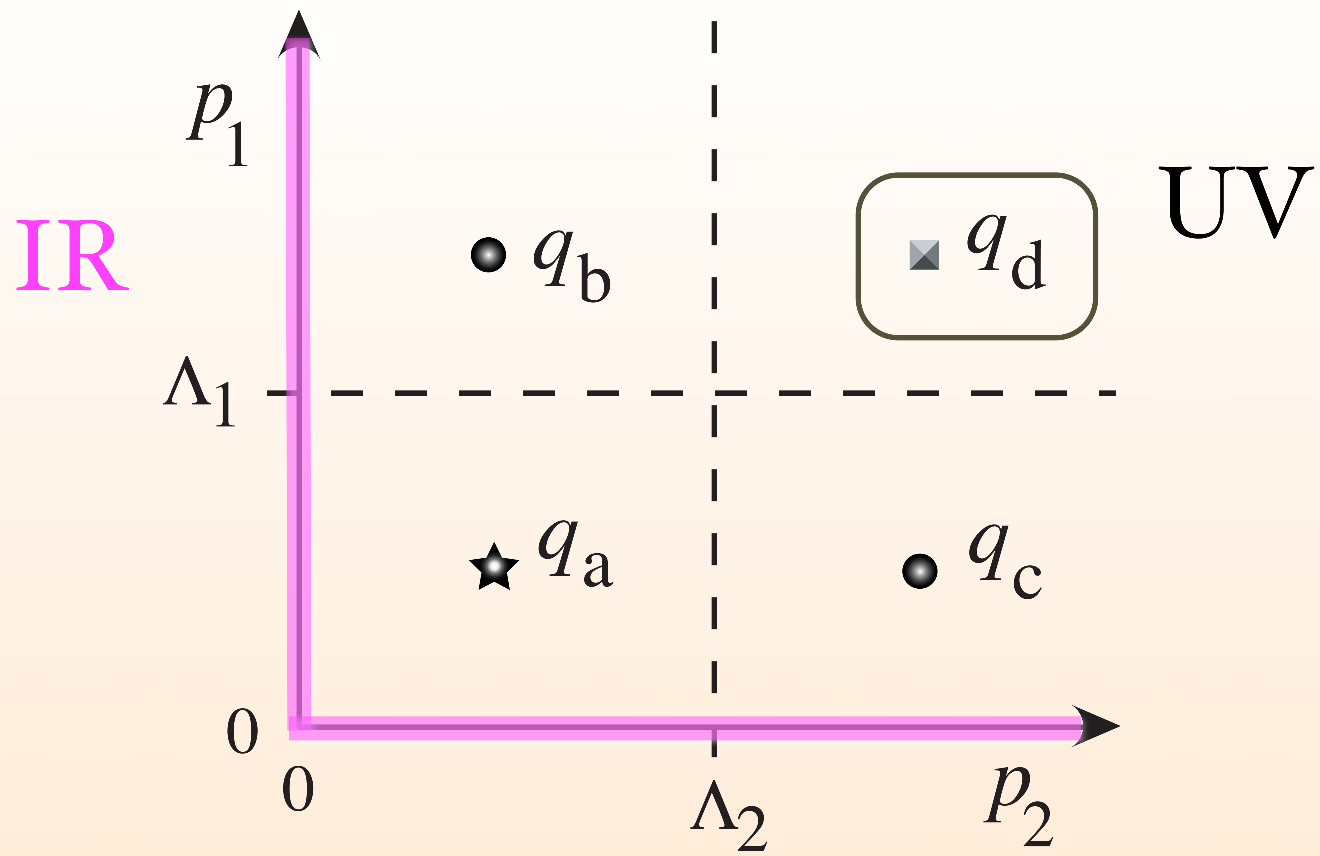


Some EFT’s need another dimension.

I’ll call these “differential EFT’s” .

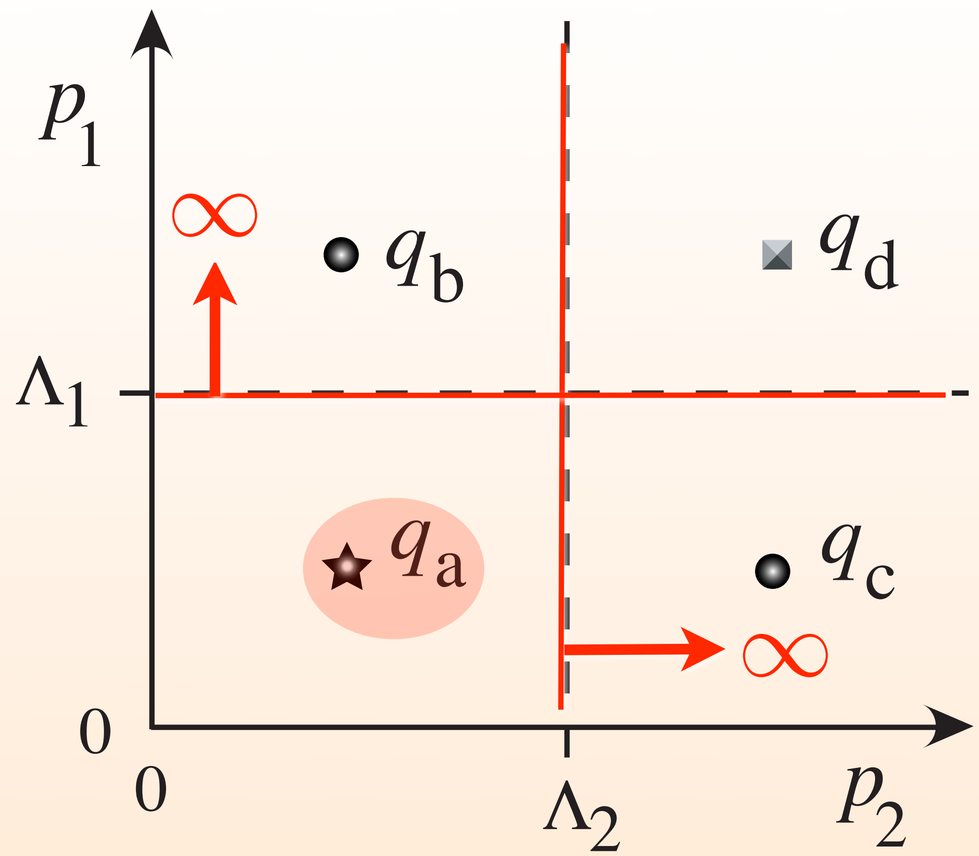


p_1
q_b
q_d
Λ_1
q_a
q_c
0
0
Λ_2
p_2



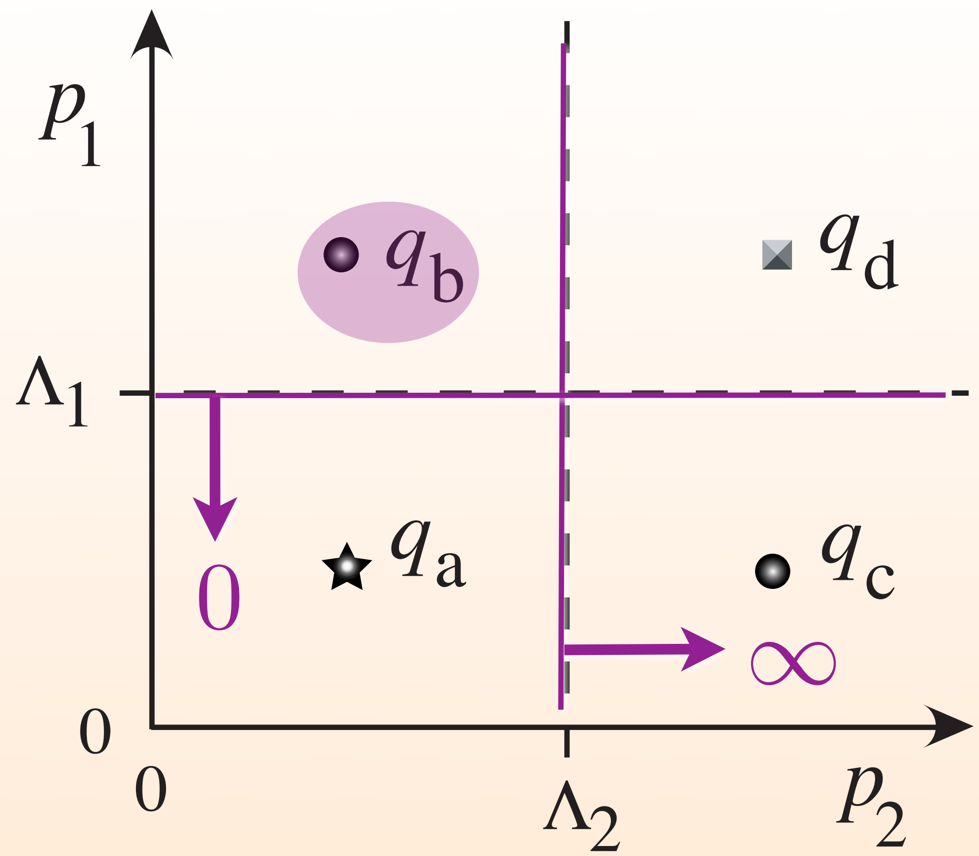
p_1
IR
q_b
q_d
UV
Λ_1
q_a
q_c
0
0
Λ_2
p_2

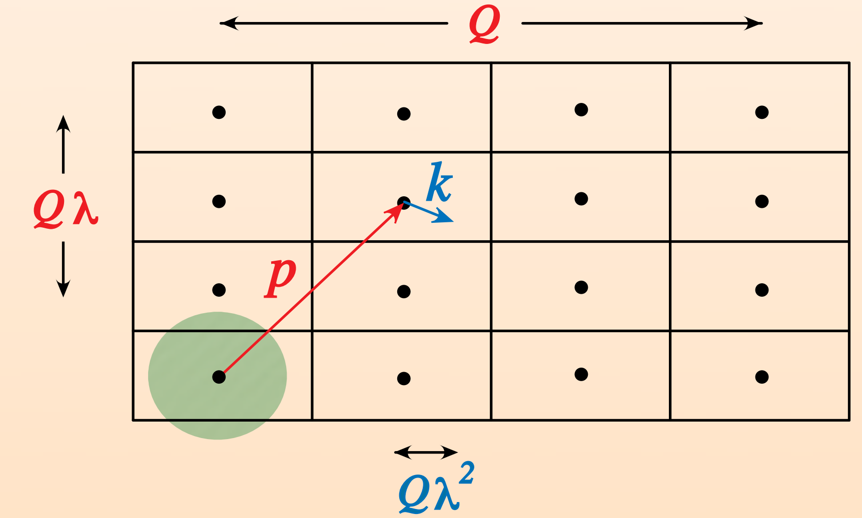
remove



- q_a overlaps only in the UV, fixed by Wilson coefficients

remove





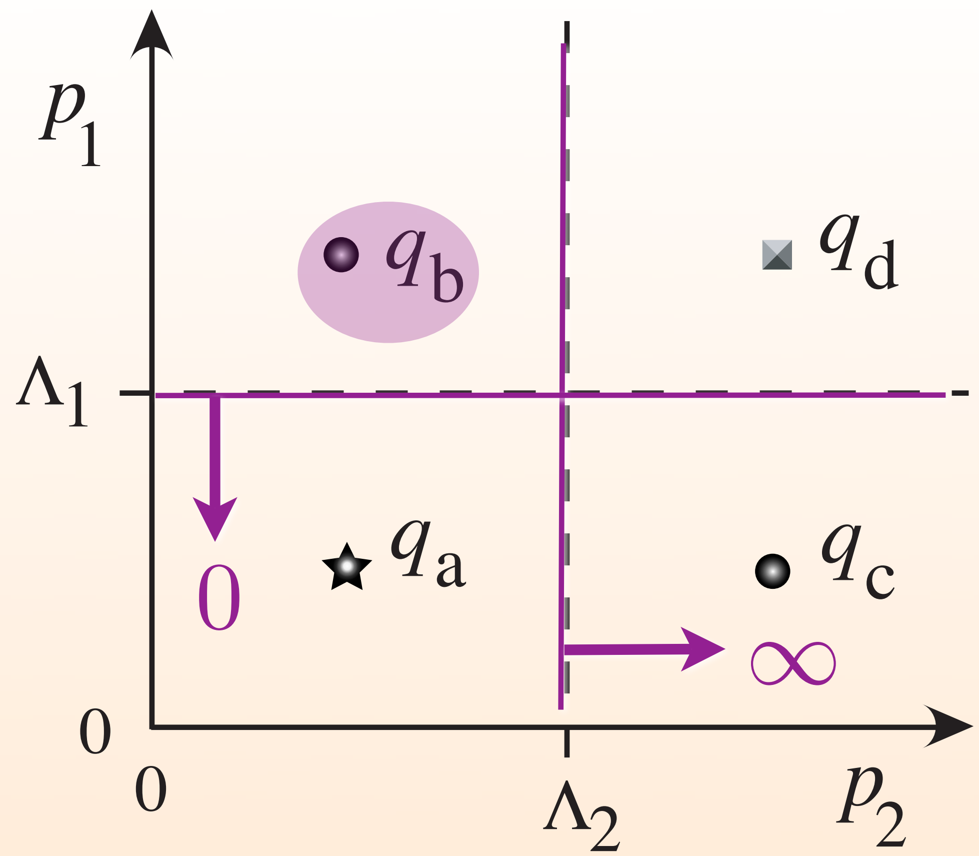
- q_b has label momentum $p_1 \neq 0$

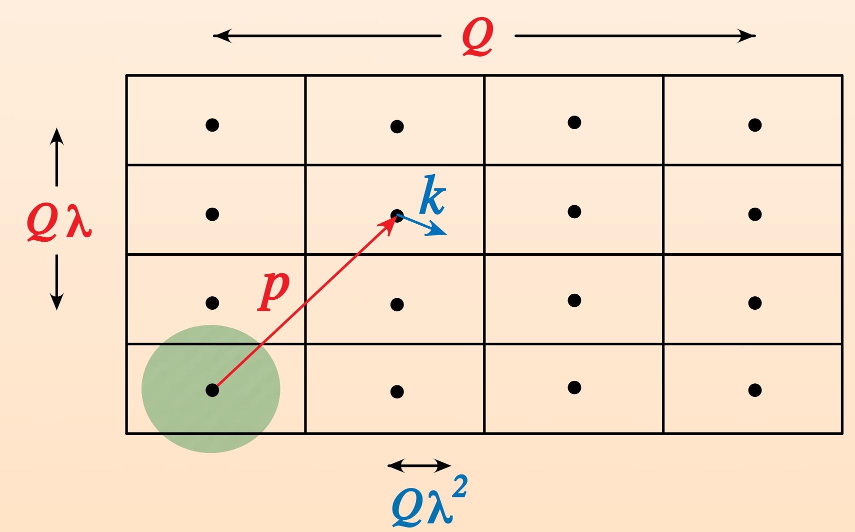
$$\sum_{p_1 \neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \left[F^{(q_b)}(p_1) - F^{(q_b \to q_a)}_{\rm subt}(p_1) \right]$$

tiling formula

- symmetric story for q_c which has label momentum $p_2 \neq 0$

remove





- q_b has label momentum $p_1 \neq 0$

$$\sum_{p_1 \neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \left[F^{(q_b)}(p_1) - F^{(q_b \to q_a)}_{\text{subt}}(p_1) \right]$$

tiling formula

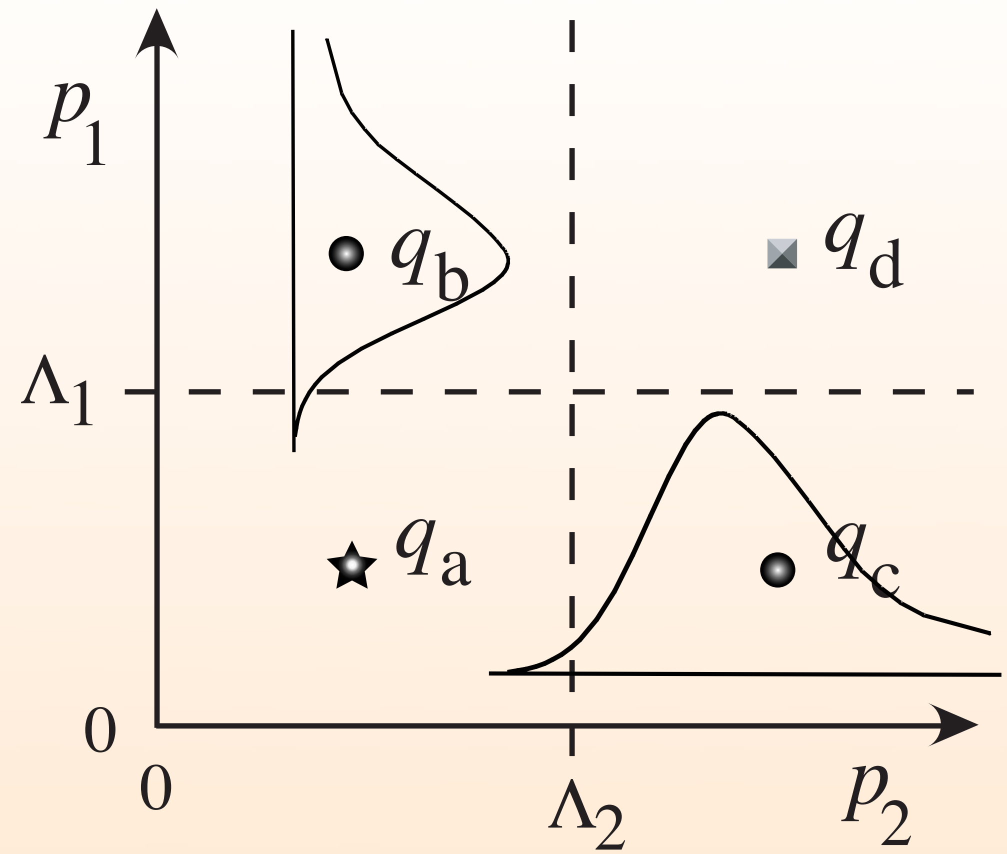
Zero-bin subtractions

This formula describes “differential matching”.

For cases with singularities
the subtractions are needed
to not double count a region!

Beyond this, different ways of implementing the subtractions correspond to a scheme dependence in defining the modes.

eg. Gaussians, hard cutoffs, ...



Lets define a nice, almost "scaleless", scheme like $\overline{\mathrm{MS}}$:

- Take the integrand $F^{(q^b)}(p_1)$ constructed with the p.c. for its region.
- Expand this integrand with p_1 scaling as in region q_a and define $F_{subt}^{(q_b \to q_a)}$ by the terms up to marginal order in the power counting.

$$\sum_{p_1 \neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \Big[F^{(q_b)}(p_1) - F_{\mathrm{subt}}^{(q_b \to q_a)}(p_1) \Big]$$

tiling formula

What has been done in the past?

$$\sum_p \int d^4k \longrightarrow \int d^dp$$ Ok if p=0 is harmless.

In cases where it is not harmless we exploited dimensional regularization:

Method of Regions (Beneke & Smirnov)

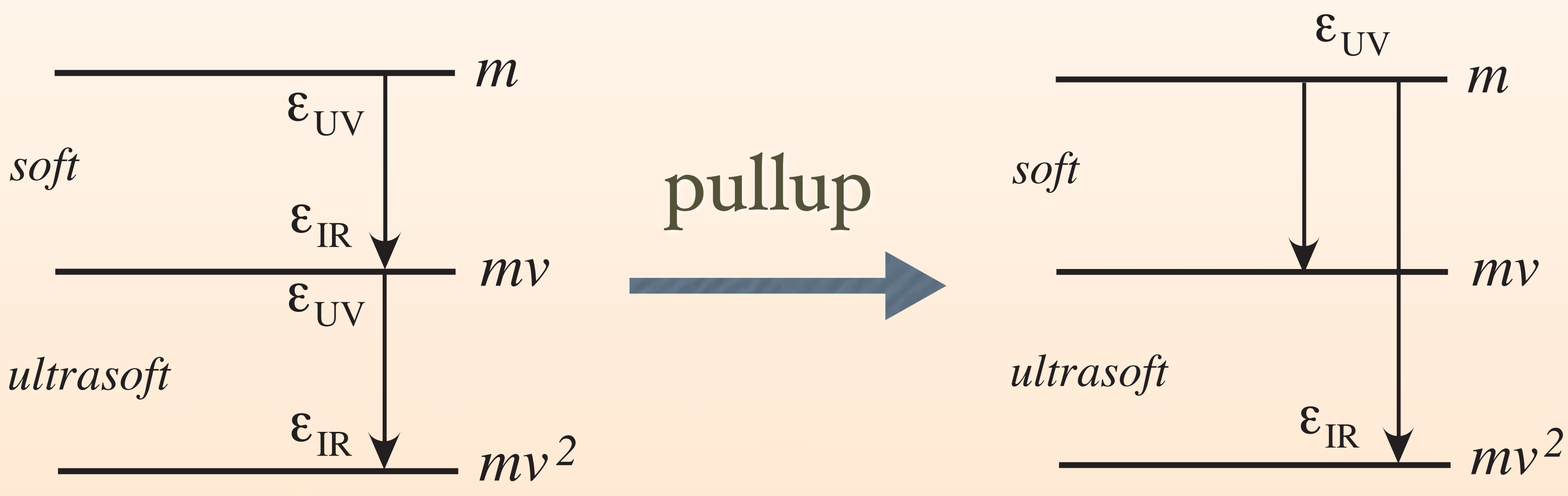
Any full theory loop integral depending on scales p_i satisfies:

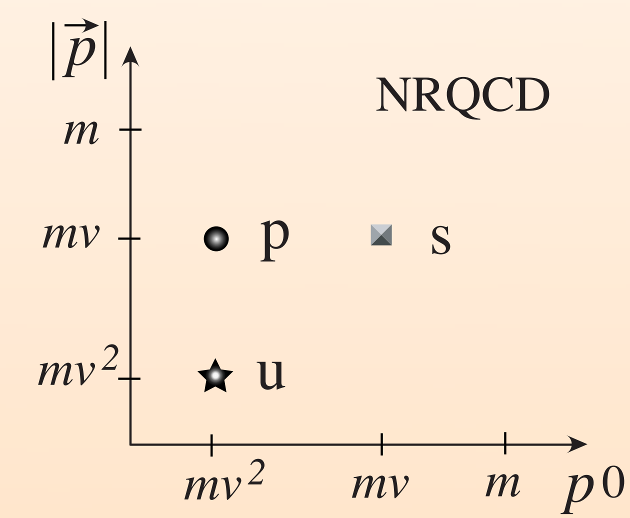
$$\prod_j \int d^dk_j \, F(p_i, k_j) = \sum_{\text{regions } \ell} \prod_j \int d^dk_j \, F^{(\ell)}(p_i, k_j)$$

as long as we set $\epsilon_{\rm IR} = \epsilon_{UV} = \epsilon$ for every region

- Using this, the only errors one makes in defining the EFT modes are proportional to $\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right)$. These can be fixed by hand, “a pullup”, so that there is only one meaning for $\epsilon_{\rm UV}$.

Hoang, Manohar, I.S.





Not elegant, but it works.

- However, dim.reg. does not handle all singularities.
  & we were stuck with not being able to handle other regulators.

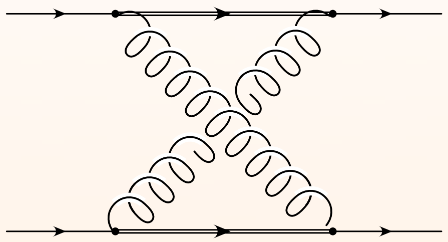
Tiling formula: can use any regulator, nothing to do by hand.

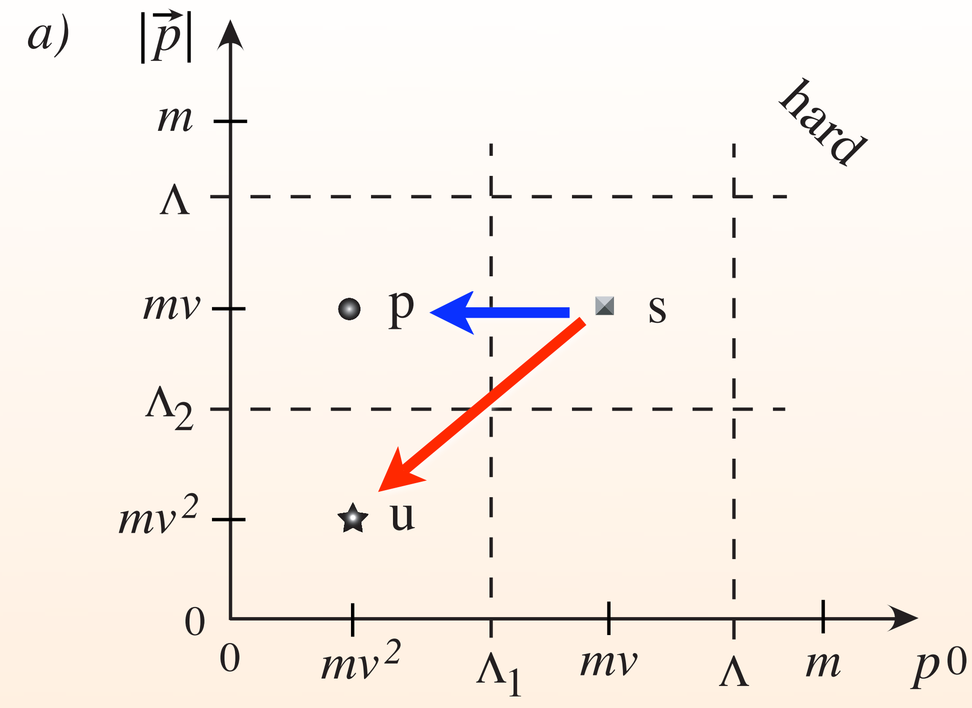
Subtractions reduce to exactly the needed $\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right)$ terms for dim.reg. setup.

We move on to examples, then applications.

NRQCD

soft loop with
ultrasoft sing.





$$I_S^{\rm cross} = \tilde{I}_S^{\rm cross} - I_1^{\rm cross} - I_2^{\rm cross}$$

$$\tilde{I}_S^{\rm cross} = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^0+i0^+}\,\frac{1}{p^0+i0^+}\,\frac{1}{(p^0)^2-\mathbf{p}^2+i0^+}\,\frac{1}{(p^0)^2-(\mathbf{p}-\mathbf{r})^2+i0^+}$$

$$I_1^{\rm cross} = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^0+i0^+}\,\frac{1}{p^0+i0^+}\,\frac{1}{(p^0)^2-\mathbf{p}^2+i0^+}\,\frac{1}{-(\mathbf{r})^2+i\epsilon},$$

$$I_2^{\rm cross} = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^0+i0^+}\,\frac{1}{p^0+i0^+}\,\frac{1}{-\mathbf{r}^2+i0^+}\,\frac{1}{(p^0)^2-(\mathbf{p}-\mathbf{r})^2+i0^+}$$

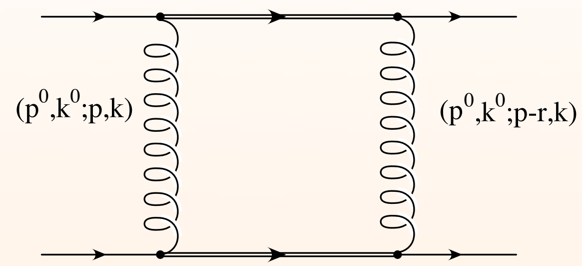
$$I_1^{\rm cross} + I_2^{\rm cross} = -\frac{i}{4\pi^2\mathbf{r}^2}\left[\frac{1}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm UV}}\right],$$

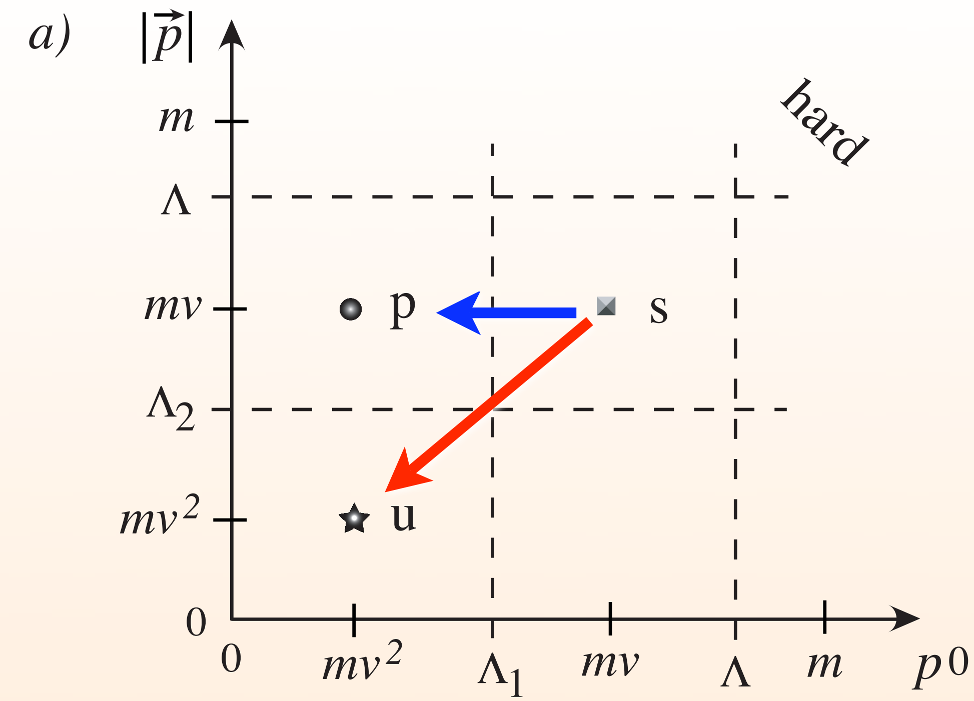
$$I_S^{\rm cross} = \tilde{I}_S^{\rm cross} - I_1^{\rm cross} - I_2^{\rm cross} = -\frac{i}{4\pi^2\mathbf{r}^2}\left[\frac{1}{\epsilon_{\rm UV}} + \ln\left(\frac{\mu^2}{\mathbf{r^2}}\right)\right]$$

Similar for the A.D.M. singularity.

NRQCD

pinch singularity,
overlapping
subtractions





$$I_S^{\rm box} = \tilde{I}_S^{\rm box} - I_1^{\rm box} - I_2^{\rm box} - I_3^{\rm box} + I_4^{\rm box} + I_5^{\rm box}$$

usoft singularity — potential singularity — overlap

$$\tilde{I}_S^{\rm box} = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^0+i0^+} \frac{1}{-p^0+i0^+} \frac{1}{(p^0)^2-\mathbf{p}^2+i0^+} \frac{1}{(p^0)^2-(\mathbf{p}-\mathbf{r})^2+i0^+}$$

$$I_3^{\rm box} = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^0+i0^+} \frac{1}{-p^0+i0^+} \frac{1}{-\mathbf{p}^2+i0^+} \frac{1}{-(\mathbf{p}-\mathbf{r})^2+i0^+}$$

pinch sing. is not regulated by dim.reg. but

$$\tilde{I}_S^{\rm box} - I_3^{\rm box}$$

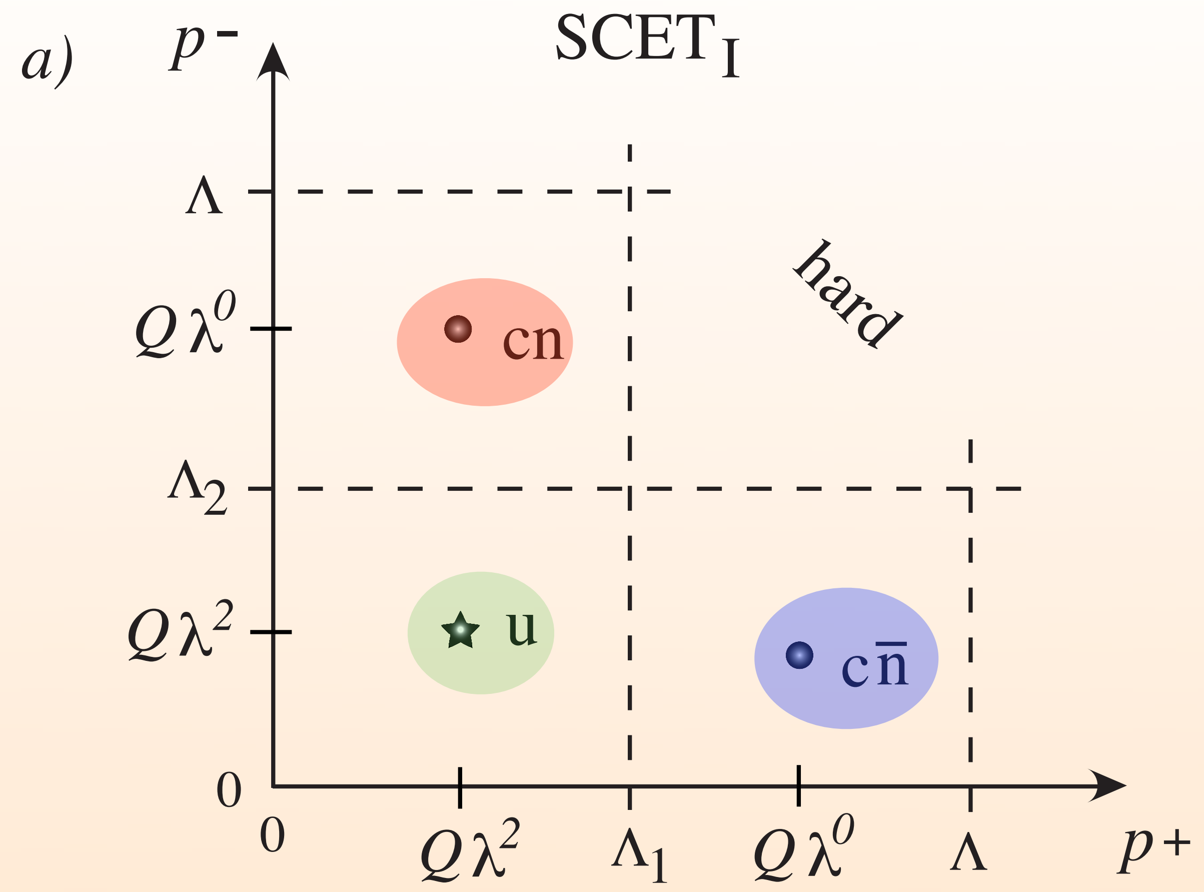
is well defined

Using our formula to define the soft integrals, the singularities do NOT appear

$$\sum_{p_1\neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \left[F^{(q_b)}(p_1) - F^{(q_b\to q_a)}_{\rm subt}(p_1) \right]$$

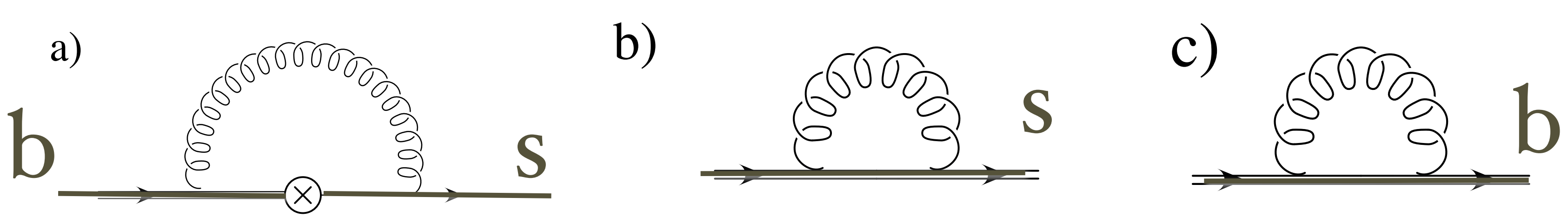
They can now properly be taken care of by other degrees of freedom

SCETI



IR divergences & Matching

QCD $\quad J^{\mathrm{QCD}} = \bar{s}\,\Gamma\, b$ $\qquad \bar{n}\cdot p = m_b$



IR regulator

$p^2 \neq 0$ for s-quark

$1/\epsilon_{\mathrm{IR}}$ for b-quark

SCET$_{\mathrm{I}}$

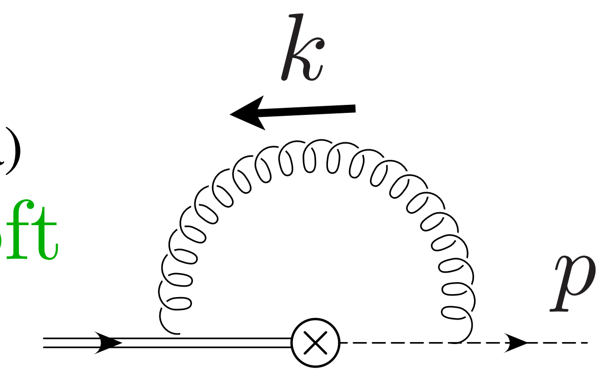
$$J^{\mathrm{SCET}} = (\bar{\xi}_n W)_\omega \Gamma h_v$$

Feyn. Gauge

$\bar{n}\cdot p = m_b$

a)

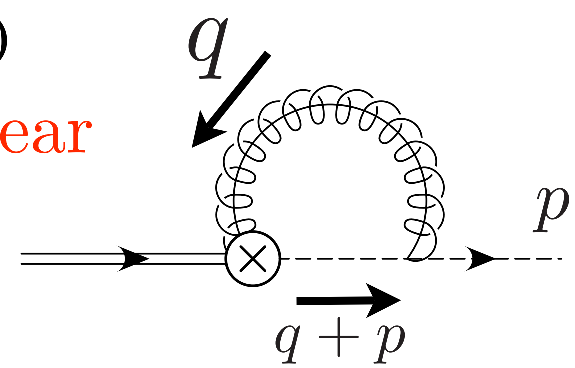
usoft



$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+i0^+)(v\cdot k+i0^+)(n\cdot k + p^2/\bar{n}\cdot p+i0^+)}$$

c)

collinear

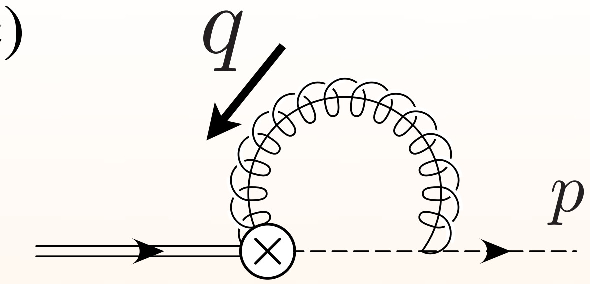


$$\sum_{q\neq 0,\; q\neq -p} \int \frac{d^4q_r}{(2\pi)^4} \frac{2\bar{n}\cdot(q+p)}{(\bar{n}\cdot q+i0^+)((q+p)^2+i0^+)(q^2+i0^+)}$$

$$q = (q, q_r) \quad , \qquad (q+p)^2 = \bar{n}\cdot(q+p)\, n\cdot(q_r+p) - (\vec{q}_\perp+\vec{p}_\perp)^2$$

Comment: The tiling formula applies to each collinear propagator. After using mom. cons. δ-functions, the subtractions are not all at zero.

Apply to

c) 

using dim.reg. in UV

$p^2 \neq 0$ in IR

avoids overcounting the usoft region

$$\sum_{q\neq 0,\, q\neq -p} \int \frac{d^4 q_r}{(2\pi)^4} \frac{2\bar{n}\cdot(q+p)}{(\bar{n}\cdot q+i0^+)((q+p)^2+i0^+)(q^2+i0^+)}$$

$$= \int \frac{d^d q}{(2\pi)^d}\left[\frac{2\bar{n}\cdot(q+p)}{(\bar{n}\cdot q+i0^+)[(q+p)^2+i0^+](q^2+i0^+)} - \underline{\frac{2\bar{n}\cdot p}{(\bar{n}\cdot q+i0^+)[n\cdot q\,\bar{n}\cdot p+p^2+i0^+](q^2+i0^+)}}\right]$$

subtraction

$$= -\frac{i}{16\pi^2}\Big[-\frac{2}{\epsilon_{\rm IR}\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}}\ln\left(\frac{\mu^2}{-p^2}\right) - \ln^2\left(\frac{\mu^2}{-p^2}\right) + \left(\frac{2}{\epsilon_{\rm IR}} - \frac{2}{\epsilon_{\rm UV}}\right)\ln\left(\frac{\mu}{\bar{n}\cdot p}\right) + \ldots$$

$$- \underline{\left(\frac{2}{\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}}\right)\left\{\frac{1}{\epsilon_{\rm UV}} + \ln\left(\frac{\mu^2}{-p^2}\right) - \ln\left(\frac{\mu}{\bar{n}\cdot p}\right)\right\}}\ \Big]$$

$$= -\frac{i}{16\pi^2}\left[-\frac{2}{\epsilon_{\rm UV}^2} - \frac{2}{\epsilon_{\rm UV}}\ln\left(\frac{\mu^2}{-p^2}\right) - \ln^2\left(\frac{\mu^2}{-p^2}\right)\right] + \ldots$$

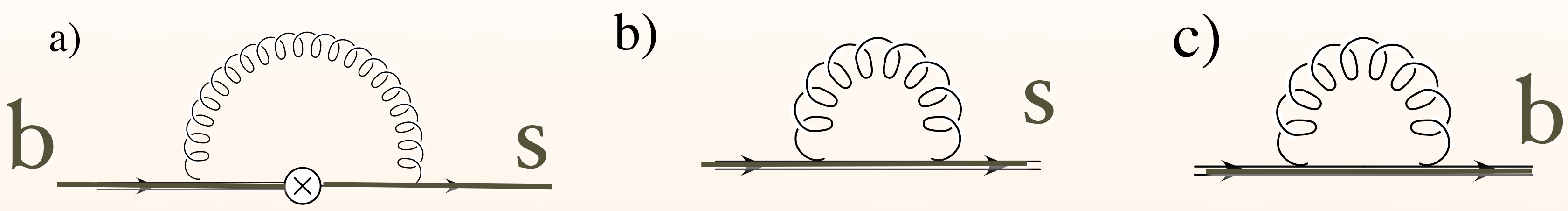
- UV collinear singularity comes from $\bar{n}\cdot q \to \infty$ (in subtraction term)

  This is crucial for it to be independent of the choice of IR regulator.

  Divergences are removed by counterterms as usual.

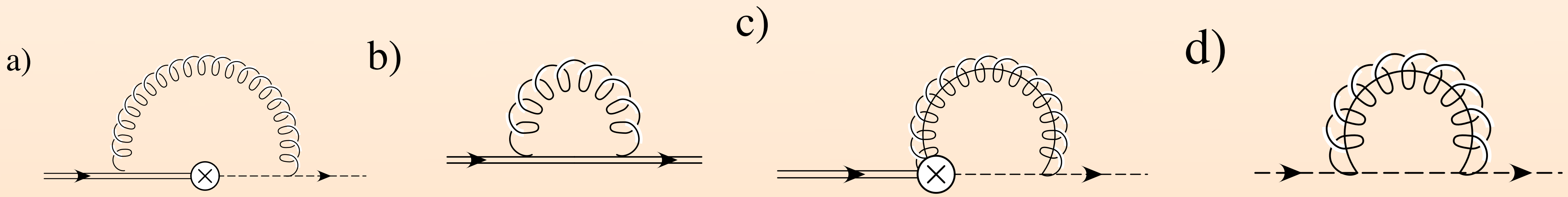
QCD and SCET: same IR divergences

QCD



$$\text{sum} = -\frac{\alpha_s}{3\pi}\left[\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{\rm IR}} + 2\ln\left(\frac{\mu^2}{m_b^2}\right) + \text{constants}\right]$$

SCET



$$\text{sum} = -\frac{\alpha_s}{3\pi}\left[\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm UV}^2} - \frac{5}{2\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm UV}}\ln\left(\frac{\mu}{m_b}\right) - 2\ln^2\left(\frac{\mu}{m_b}\right) - \frac{3}{2}\ln\left(\frac{\mu^2}{m_b^2}\right) + \text{constants}\right]$$

- UV renormalization in SCET sums double Sudakov logs
- difference gives one-loop matching

eg. of another regulator

Cutoffs: $\Omega_\perp^2 \le \vec{q}_\perp^{\,2} \le \Lambda_\perp^2 \qquad \Omega_-^2 \le (q^-)^2 \le \Lambda_-^2$

no constraint on q^+, p^μ on-shell

QCD

$$I_{\rm full}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{p^-}\Big)\ln\Big(\frac{\Omega_- p^-}{\Omega_\perp^2}\Big)\Big] + \ldots$$

SCET

$$I_{\rm us}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{\Lambda_-}\Big)\ln\Big(\frac{\Omega_- \Lambda_-}{\Omega_\perp^2}\Big)\Big]$$

$$I_{\rm C}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Omega_-}{p^-}\Big)\Big] - \underline{\frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Omega_-}{\Lambda_-}\Big)\Big]} = \frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Lambda_-}{p^-}\Big)\Big] + \ldots$$

$$I_{\rm us}^{b\to s\gamma} + I_{\rm C}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{p^-}\Big)\ln\Big(\frac{\Omega_- p^-}{\Omega_\perp^2}\Big) + \ln^2\Big(\frac{\Lambda_\perp}{p^-}\Big) - \ln^2\Big(\frac{\Lambda_\perp}{\Lambda_-}\Big)\Big] + \ldots$$

IR matches again,
zero-bin subtraction is crucial.

RGE, Summing Logs

(SCET Review)

graphs $= -\frac{\alpha_s}{\pi}\left[\ln^2\left(\frac{-p^2}{(\bar{n}\cdot p)^2}\right)+\frac{3}{2}\ln\left(\frac{-p^2}{(\bar{n}\cdot p)^2}\right)+\frac{1}{\epsilon_{\rm IR}}-\frac{1}{\epsilon_{\rm UV}^2}-\frac{5}{2\epsilon_{\rm UV}}-\frac{2}{\epsilon_{\rm UV}}\ln\left(\frac{\mu}{\bar{n}\cdot p}\right)+\dots\right] C(\mu,\bar{n}\cdot p)$

$\bar{n}\cdot p = m_b$

mixes into itself, no convolution

At any order:

$$\mu\frac{d}{d\mu}C(\mu,p^-) = \left[Z^{-1}\mu\frac{d}{d\mu}Z\right]C(\mu,p^-) = \left[\Gamma^{\rm cusp}(\alpha_s)\ln\left(\frac{\mu}{p^-}\right)+\gamma(\alpha_s)\right]C(\mu,p^-)$$

$$= \left[\Gamma^{\rm cusp}\ln\left(\frac{\mu}{\mu_0}\right)+\left\{\Gamma^{\rm cusp}\ln\left(\frac{\mu_0}{p^-}\right)+\gamma(\alpha_s)\right\}\right]C(\mu,p^-)$$

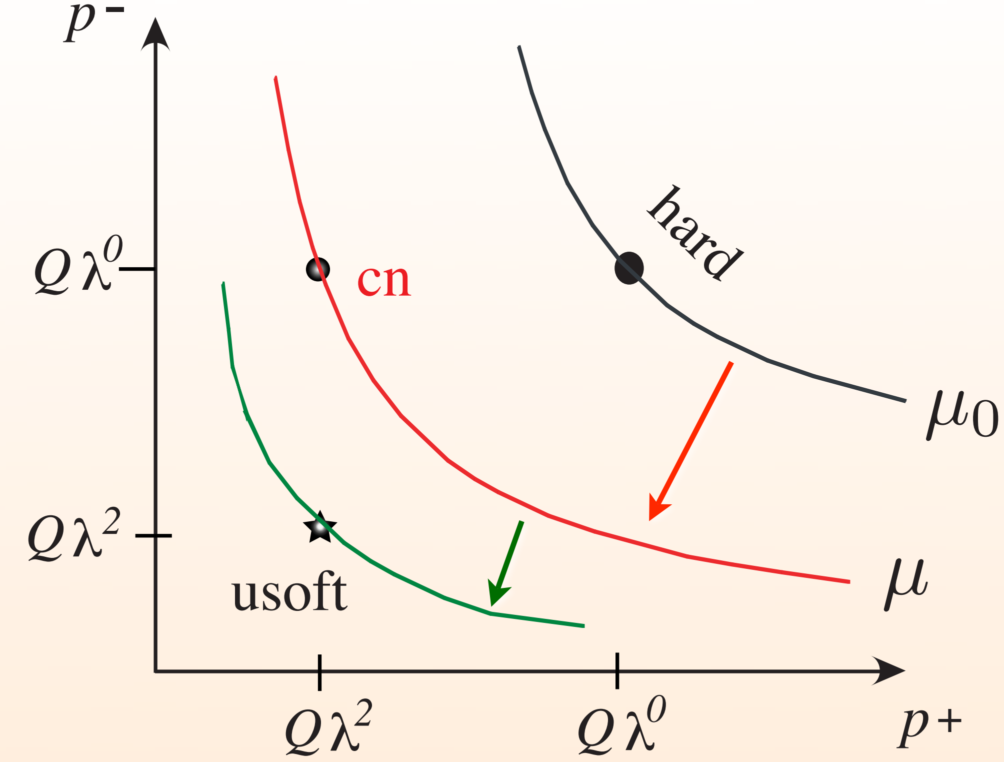
gives Sudakov dble. logs

one-loop $\quad Z = 1+\frac{\alpha_s(\mu)C_F}{4\pi}\left[\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\ln\left(\frac{\mu}{\bar{n}\cdot P}\right)+\frac{5}{2\epsilon}\right]$

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu}+(-2\epsilon\alpha_s+\beta)\frac{\partial}{\partial\alpha_s}$$

$$\Gamma^{\rm cusp}(\alpha_s) = -\frac{\alpha_s C_F}{\pi} \quad , \quad \gamma(\alpha_s) = -\frac{5\alpha_s C_F}{4\pi}$$

LL $\Gamma^{\text{cusp}}(\alpha_s) = -\frac{\alpha_s C_F}{\pi}$ $z = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$



this result shape fn. rge

solution

$$C(\mu) = C(\mu_0)\exp\left[-\frac{4\pi C_F}{b_0^2\alpha_s(\mu_0)}\left(\frac{1}{z} - 1 + \ln z\right)\right]$$

NLL $\gamma(\alpha_s) = -\frac{5\alpha_s C_F}{4\pi}$

need 2 loop Γ^{cusp}

at this level

Think of the log resummation for $\ln C(\mu)$

$$\ln C(\mu) = \frac{f_{\text{LL}}(z)}{\alpha_s(\mu_0)} + f_{\text{NLL}}(z, p^-) + \alpha_s(\mu_0) f_{\text{NNLL}}(z, p^-) + \ldots$$

Need:

series in $\ln C(\mu)$		one-loop	two-loops	three-loops
LL	$\alpha_s^n \ln^{n+1}$	$1/\epsilon^2$	–	–
NLL	$\alpha_s^n \ln^n$	$1/\epsilon$	$1/\epsilon^2$	–
NNLL	$\alpha_s^n \ln^{n-1}$	matching	$1/\epsilon$	$1/\epsilon^2$

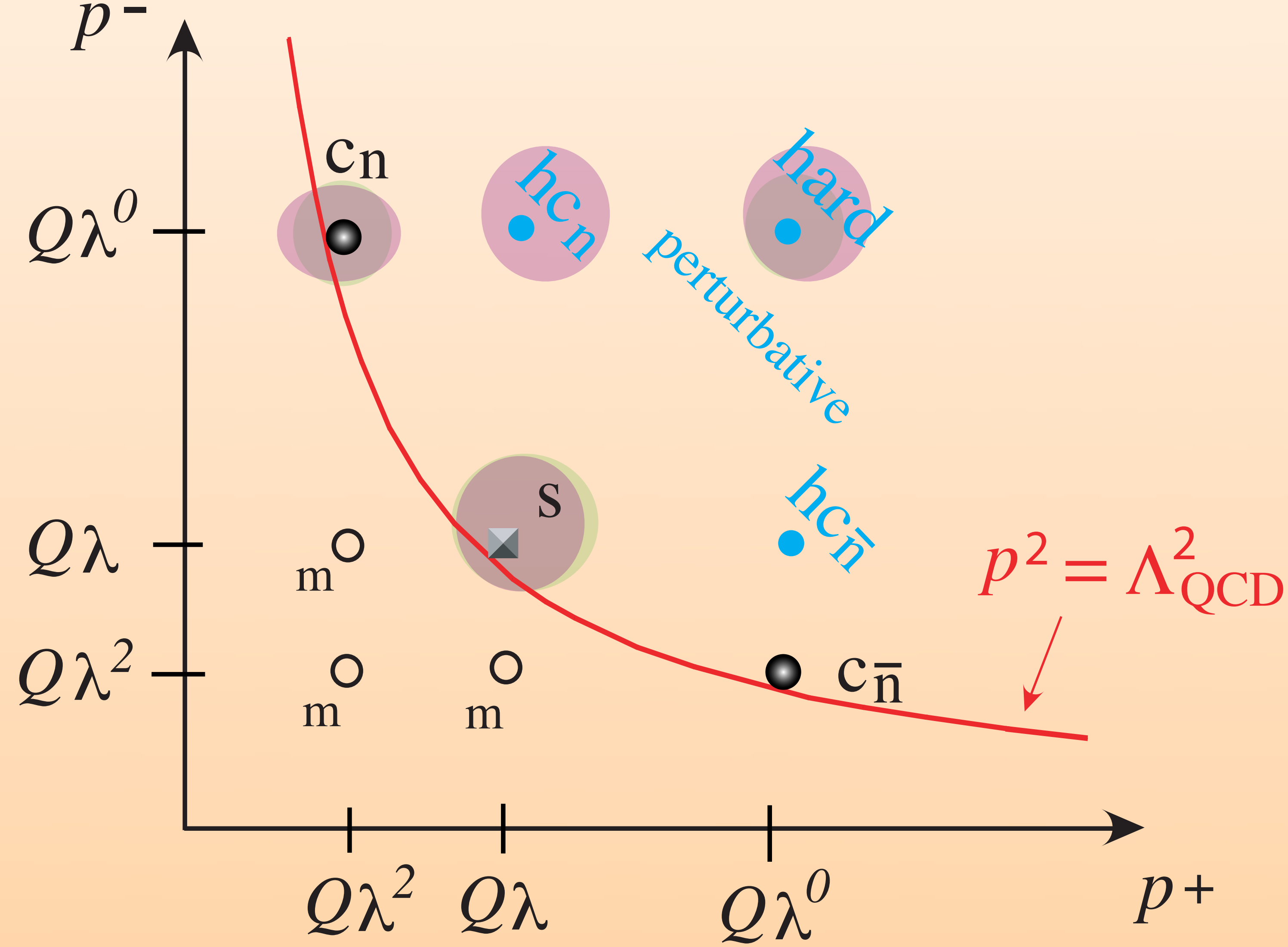
$\mathrm{SCET_{II}}$

$$\lambda = \frac{\Lambda}{Q}$$

- all known examples of endpoint singularities have > one hadron
- $\mathrm{SCET_{II}}$ allows us to treat cases with two or more hadrons

  eg. $B \to D\pi$, $B \to \pi \ell \bar{\nu}$, $e^- p \to e^- X \pi$

modes



SCET$_{\text{II}}$

$$\lambda = \frac{\Lambda}{Q}$$

- all known examples of endpoint singularities have > one hadron
- SCET$_{\text{II}}$ allows us to treat cases with two or more hadrons

  eg. $B \to D\pi\,,\; B \to \pi\ell\bar\nu\;,\quad e^- p \to e^- X\,\pi$

- $c_n\,, s\,, c_{\bar n}$ are definitely required as low energy modes
- "messenger" scales ○

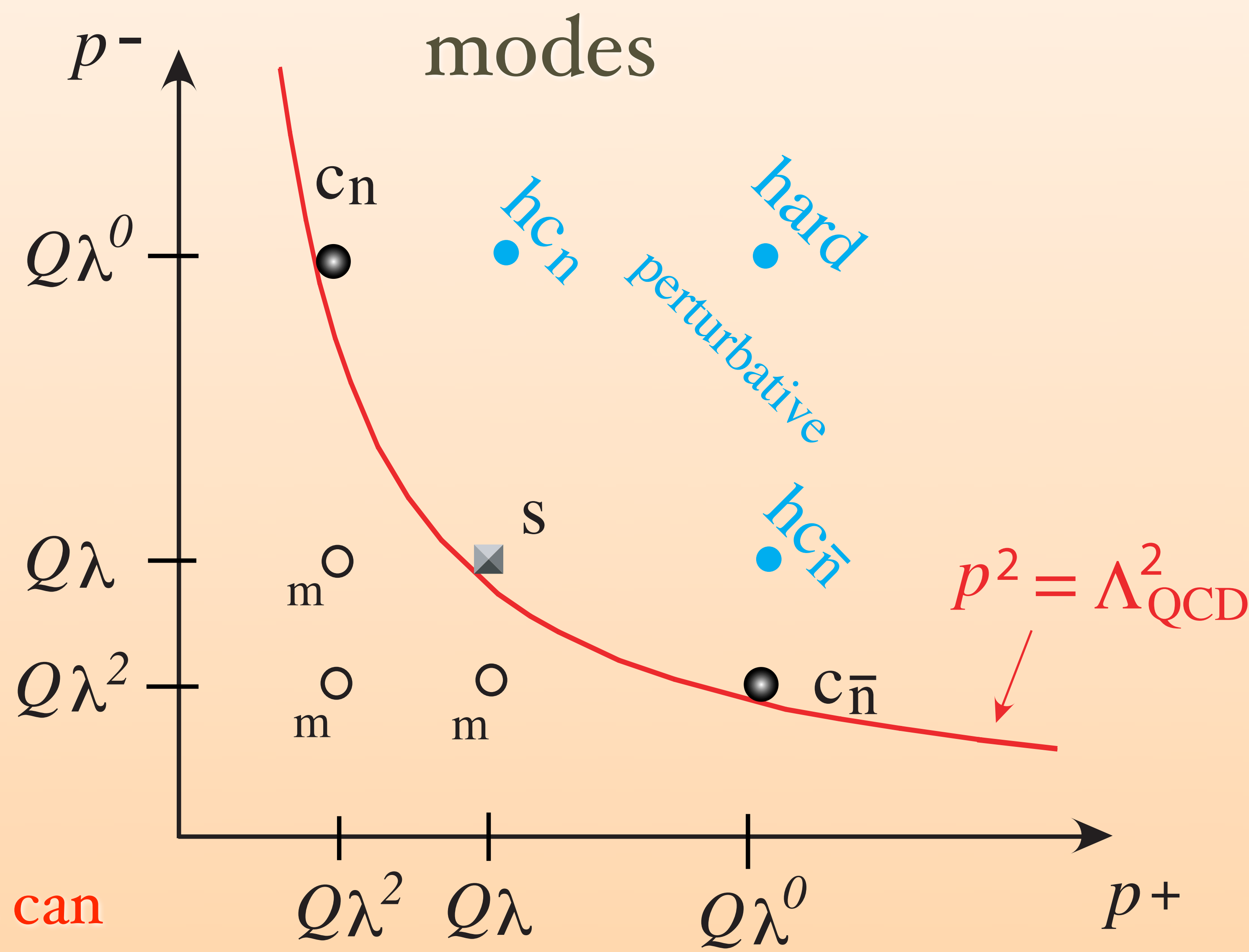
show up in perturbation theory

Becher, Hill, Neubert

but only for special choices of the IR regulators

Beneke, Feldmann; Bauer, Dorsten, Salem

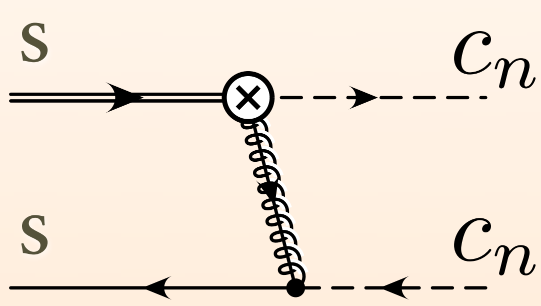
Must consider effect of confinement.
We will see shortly that the ○ modes can be absorbed into the other d.o.f.



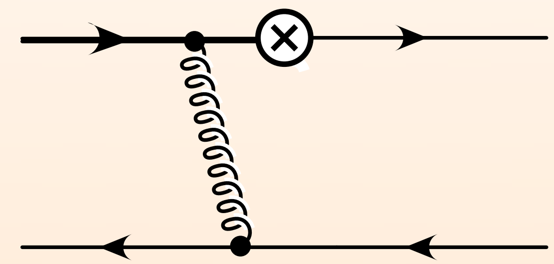
For our endpoint divergence

$$\int_0^1 \mathrm{d}x \, \frac{\phi_\pi(x)}{x^2} \, ,$$ the singularity comes from taking a double limit:

collinear $k^- \gg k^\perp, k^+$, then $k^- \to 0$



in QCD was



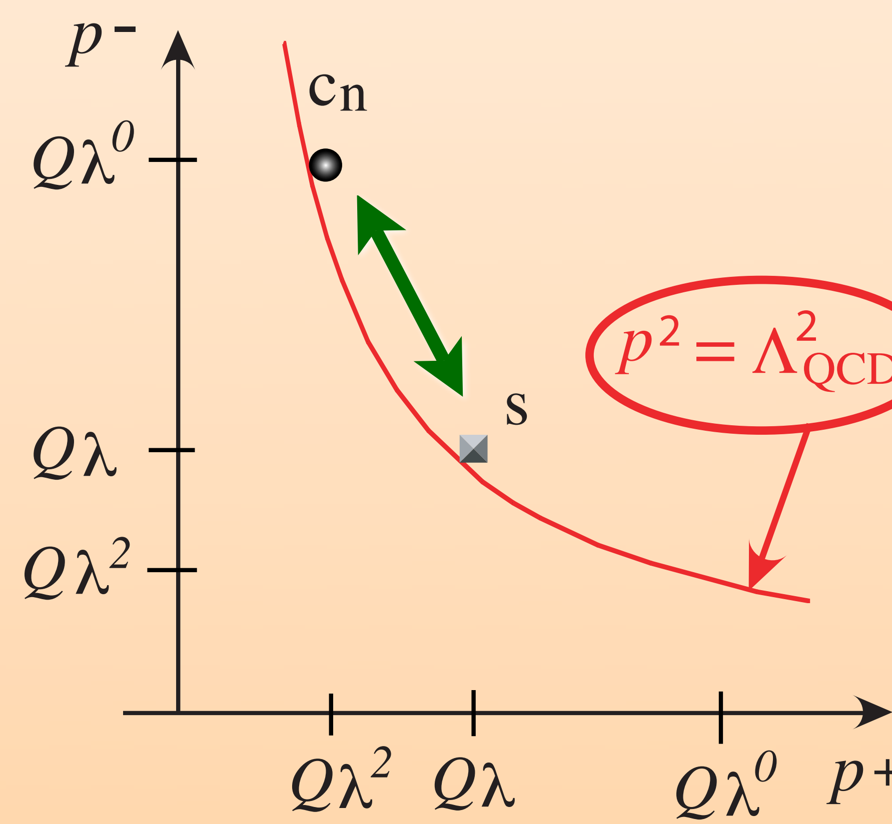
and $k^- \to 0$ encounters the soft region where there is another mode

Based on our experience the formula:

$$\sum_{p_1 \neq 0} \int dp_{1r} \, F^{(q_b)}(p_1) = \int dp_1 \left[F^{(q_b)}(p_1) - F^{(q_b \to q_a)}_{\text{subt}}(p_1) \right]$$

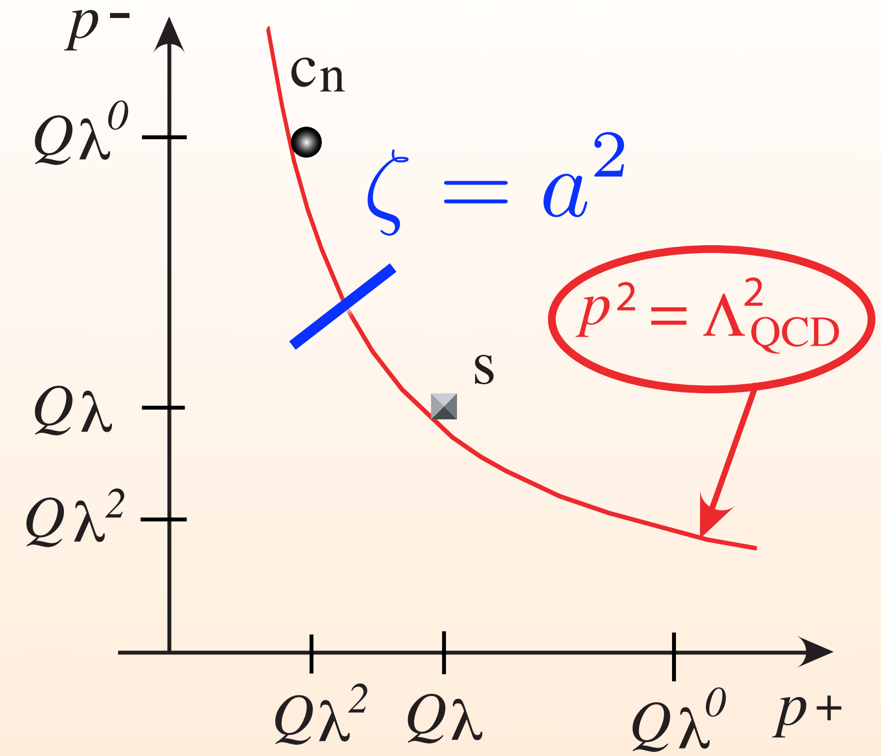
should avoid double counting the soft region, and thus remove the singularities here too.

Note: absence of onshell modes between c_n and s is due to a rapidity gap.



Rapidity distinguishes the d.o.f.

$$\zeta_k = k^- / k^+$$



n-collinear : $\quad \zeta_p \sim \lambda^{-2} \gg 1$

soft: $\quad \zeta_p \sim \lambda^0 \sim 1$

Check how this works with a rapidity cutoff

Wick rotated rapidity: soft: $\quad -a^2 \le \zeta'_k \le a^2\,,$

collinear: $\quad -a^2 \ge \zeta'_k \quad$ or $\quad \zeta'_k \ge a^2$

toy example

$$I^{\rm scalar}_{\rm full} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k-\ell)^2 + i0^+][k^2 + i0^+][(k-p)^2 + i0^+]} = \frac{-i}{16\pi^2 (p^- \ell^+)} \left[\frac{1}{\epsilon^2_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}} \ln\left(\frac{p^- \ell^+}{\mu^2}\right) + \frac{1}{2}\ln^2\left(\frac{p^- \ell^+}{\mu^2}\right) - \frac{\pi^2}{12} \right]$$

$$I^{\rm scalar}_{\rm soft} = \frac{-i}{16\pi^2 (p^- \ell^+)} \left[\frac{1}{2\epsilon^2_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}} \ln\left(\frac{\ell^+}{\mu_+}\right) + \ln^2\left(\frac{\ell^+}{\mu_+}\right) - \frac{\pi^2}{16} \right] \qquad \mu_+ = \mu/a.$$

$$I^{\rm scalar}_{\rm cn} = \frac{-i}{16\pi^2 (p^- \ell^+)} \left[\frac{1}{2\epsilon^2_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}} \ln\left(\frac{p^-}{\mu_-}\right) + \ln^2\left(\frac{p^-}{\mu_-}\right) - \frac{\pi^2}{16} \right] \qquad \mu_- = a\mu.$$

$$\boxed{\mu_+ \mu_- = \mu^2}$$

IR reproduced

$$I^{\rm scalar}_{\rm matching} = \frac{-i}{16\pi^2 (p^- \ell^+)} \left[-\frac{1}{2}\ln^2\left(\frac{p^- \mu_+}{\mu_- \ell^+}\right) + \frac{\pi^2}{24} \right]$$

The zero-bin minimal subtractions can be used to handle the overlaps. This ensure soft does not overlap collinear and visa versa.

However, doing our democratic subtractions produces problems with rapidity divergences in the UV, and standard dimensional regularization does not suffice for these.

Lets invent a gauge invariant dim.reg. like regulator that is formulated at the level of operators:

$$\boxed{\mu_+ \, \mu_- = \mu^2}$$

$$J(p_j^-, k_j^+) \big[(\bar{q}_s S)_{k_1^+} \Gamma_s (S^\dagger q_s)_{k_2^+} \big] \big[(\bar{\xi}_n W)_{p_1^-} \Gamma_n (W^\dagger \xi_n)_{p_2^-} \big]$$

$$\xrightarrow{\text{dim.reg.}} J(p_j^-, k_j^+, \mu_\pm)\, \mu^{2\epsilon} \left[(\bar{q}_s S)_{k_1^+} \frac{|\mathcal{P}^\dagger|^\epsilon}{\mu_+^\epsilon} \Gamma_s \frac{|\mathcal{P}|^\epsilon}{\mu_+^\epsilon} (S^\dagger q_s)_{k_2^+} \right] \left[(\bar{\xi}_n W)_{p_1^-} \frac{|\bar{\mathcal{P}}^\dagger|^\epsilon}{\mu_-^\epsilon} \Gamma_n \frac{|\bar{\mathcal{P}}|^\epsilon}{\mu_-^\epsilon} (W^\dagger \xi_n)_{p_2^-} \right]$$

$$= J(p_j^-, k_j^+, \mu_\pm, \mu^2)\, \mu^{2\epsilon} \left[\frac{|k_1^+ k_2^+|^\epsilon}{\mu_+^{2\epsilon}} (\bar{q}_s S)_{k_1^+} \Gamma_s (S^\dagger q_s)_{k_2^+} \right] \left[\frac{|p_1^- p_2^-|^\epsilon}{\mu_-^{2\epsilon}} (\bar{\xi}_n W)_{p_1^-} \Gamma_n (W^\dagger \xi_n)_{p_2^-} \right]$$

(Using absolute values preserves analyticity, it corresponds to positive mom. of particles and anti-particles.)

Lets try it out

$\ell =$ soft momentum

$p =$ collinear momentum

Three IR Masses, $\mathbf{m_1}$, $\mathbf{m_2}$, $\mathbf{m_3}$

$$I^{\rm scalar}_{\rm full} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k-\ell)^2 - m_2^2 + i0^+][k^2 - m_1^2 + i0^+][(k-p)^2 - m_3^2 + i0^+]}$$
$$= \frac{-i}{16\pi^2 (p^-\ell^+)} \left[\frac{1}{2} \ln^2\left(\frac{m_1^2}{p^-\ell^+}\right) + {\rm Li}_2\left(1 - \frac{m_2^2}{m_1^2}\right) + {\rm Li}_2\left(1 - \frac{m_3^2}{m_1^2}\right) \right].$$

$$I^{\rm scalar}_{\rm soft} = \sum_{k^+\neq 0} \int \frac{d^D k_r}{(2\pi)^D} \frac{\mu^{2\epsilon}}{[k^2 - \ell^+ k^- - m_2^2 + i0^+][k^2 - m_1^2 + i0^+][-p^- k^+ + i0^+]} \boxed{\frac{|k^+|^\epsilon |k^+ - \ell^+|^\epsilon}{\mu_+^{2\epsilon}}}$$

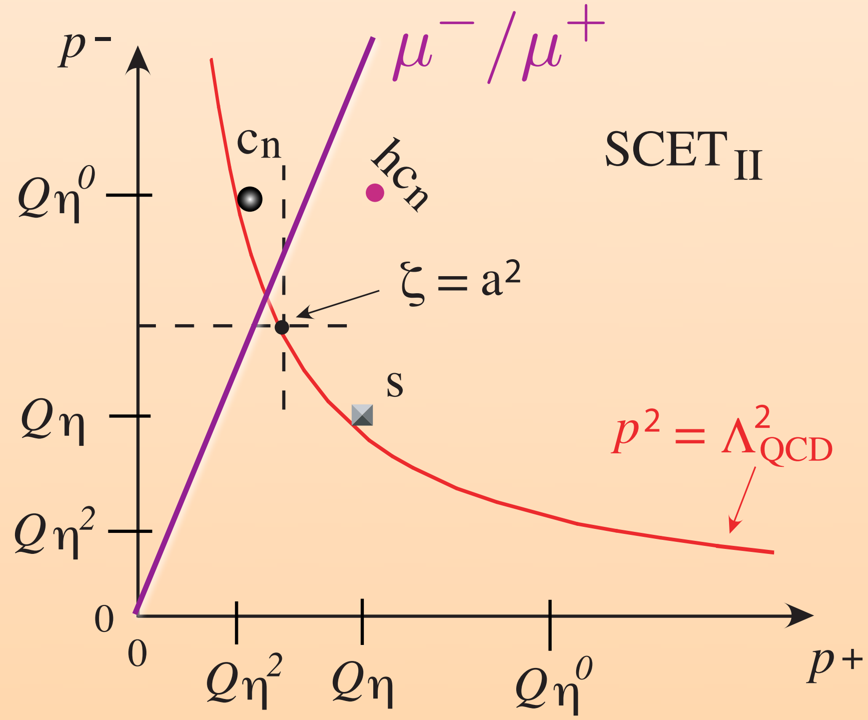
$$I^{\rm scalar}_{\rm cn} = \sum_{k^-\neq 0} \int \frac{d^D k'_r}{(2\pi)^D} \frac{\mu^{2\epsilon}}{[-\ell^+ k^- + i0^+][k^2 - m_1^2 + i0^+][k^2 - p^- k^+ - m_3^2 + i0^+]} \boxed{\frac{|k^-|^\epsilon |k^- - p^-|^\epsilon}{\mu_-^{2\epsilon}}}$$

regulate UV rapidity divergences

renormalized

$$I^{\rm scalar}_{\rm soft+cn} = \frac{-i}{16\pi^2 (p^-\ell^+)} \left[\frac{1}{2} \ln^2\left(\frac{m_1^2}{p^-\ell^+}\right) + {\rm Li}_2\left(1 - \frac{m_2^2}{m_1^2}\right) + {\rm Li}_2\left(1 - \frac{m_3^2}{m_1^2}\right) - \ln\left(\frac{m_1^2}{\mu^2}\right) \ln\left(\frac{\mu^2}{\mu_-\mu_+}\right) \right.$$
$$\left. + \ln^2\left(\frac{p^-}{\mu_-}\right) + \ln^2\left(\frac{\ell^+}{\mu_+}\right) - \frac{1}{2} \ln^2\left(\frac{p^-\ell^+}{\mu^2}\right) + \frac{5\pi^2}{12} \right].$$

vanishes for $\mu^+\mu^- = \mu^2$



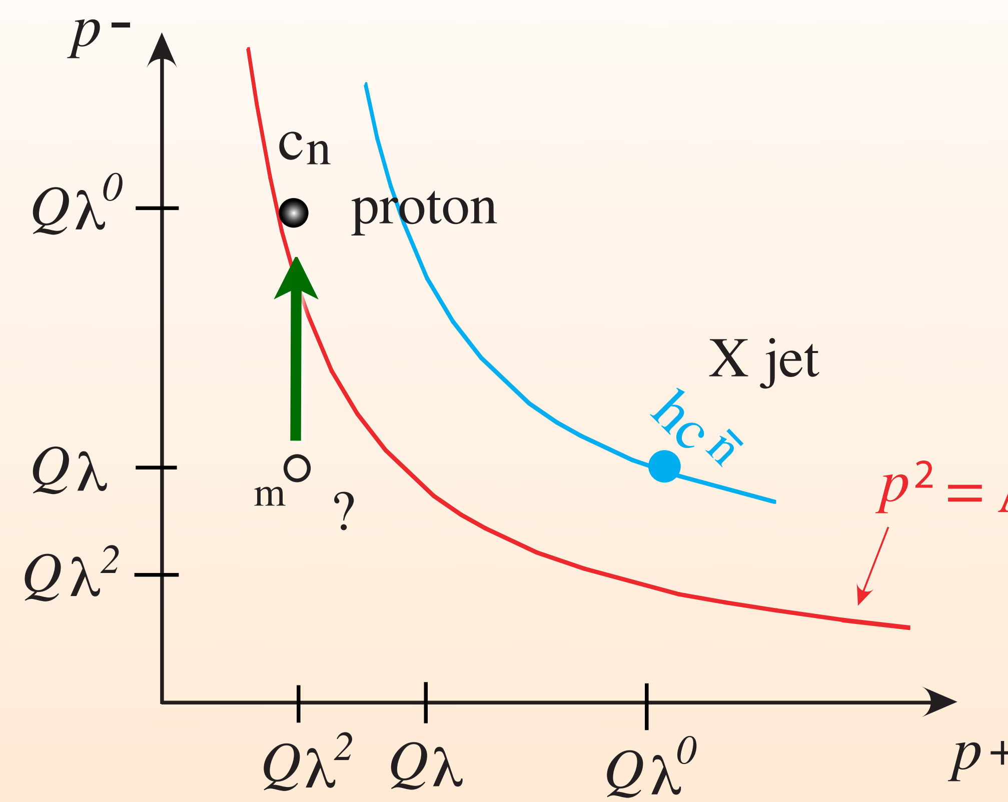
IR matches

What about the messenger modes?

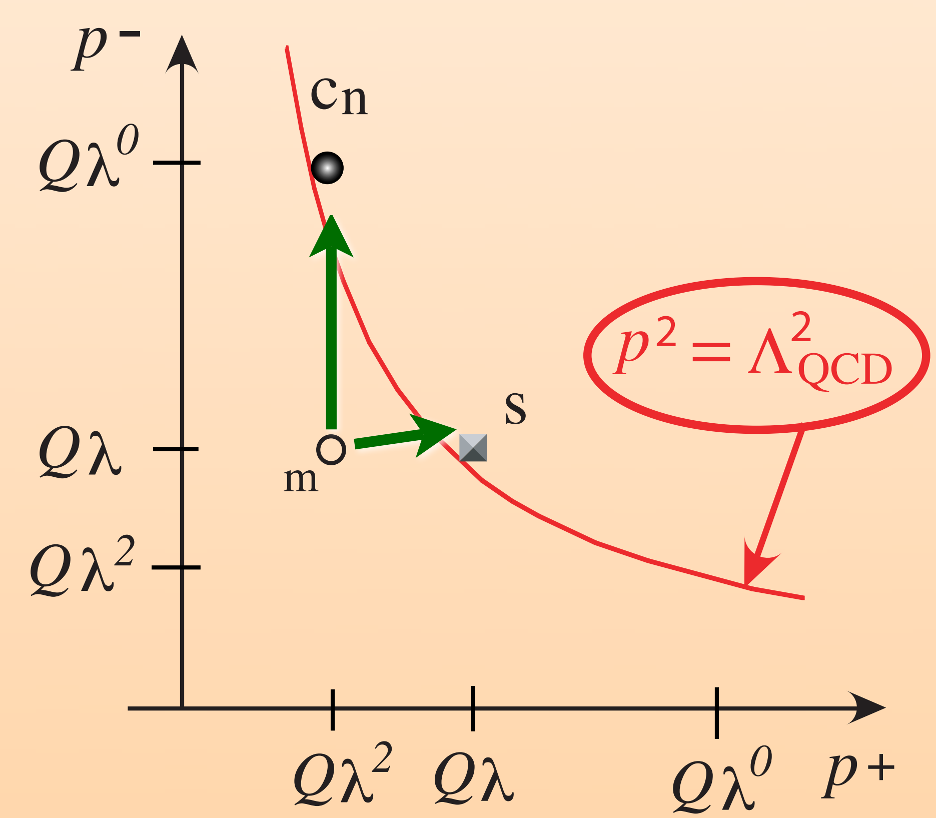
eg. DIS as $x \to 1$

confinement causes messenger to be absorbed into collinear proton

A.Manohar



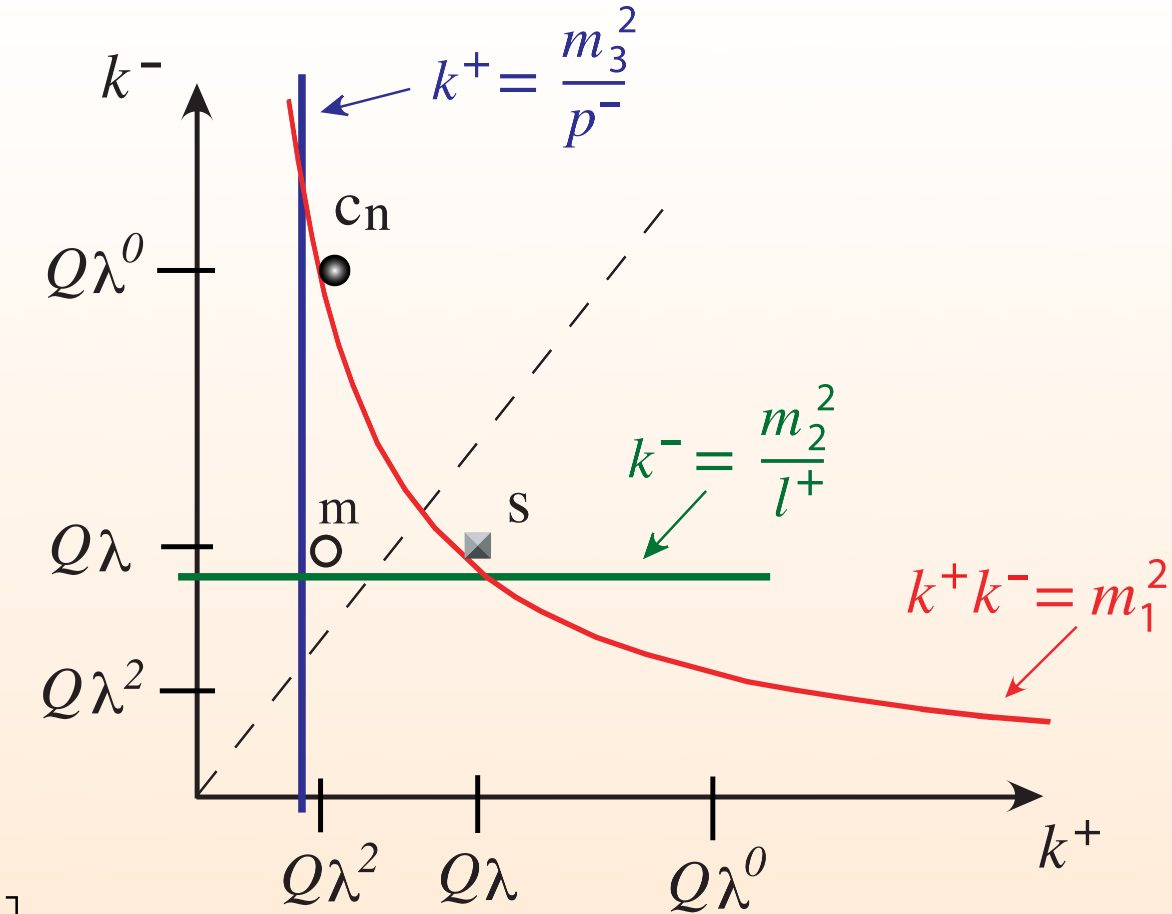
In the case with endpoint singularities, we expect confinement to cause the messenger to be absorbed into a combination of c_n and s



Lets find out

Three IR Masses, m_1, m_2, m_3

study double log



What if $m_1 = 0$?

$$I_{\rm full}^{\rm scalar} = \frac{-i}{16\pi^2(p^-\ell^+)}\Big\{\frac{1}{2}\ln^2\Big[\frac{\xi - i0^+}{Q^4}\Big] + {\rm Li}_2\Big[\frac{Q^2(m_1^2-m_2^2)}{\xi} - i0^+\Big] + {\rm Li}_2\Big[\frac{Q^2(m_1^2-m_3^2)}{\xi} - i0^+\Big] - {\rm Li}_2\Big[\frac{-(m_1^2-m_2^2)(m_1^2-m_3^2)}{\xi}\Big]\Big\}$$

$$\xi \equiv Q^2 m_1^2 - m_2^2 m_3^2$$

$$I_{\rm full}^{\rm scalar}(m_1=0) = \frac{-i}{16\pi^2(p^-\ell^+)}\Big[\ln\Big(\frac{m_2^2}{Q^2}\Big)\ln\Big(\frac{m_3^2}{Q^2}\Big)\Big]$$

If we set $m_1 = 0$ we become sensitive to "m" region.

In QCD we expect confinement to act like $m_1 \neq 0$, so that the s & c modes absorb "m", just as we saw in our calculations with rapidity regulators.

Implications for our singular Convolutions

$$A_\pi = \sum_{p^-_{1,2}\neq 0} \int dp^-_{1r} dp^-_{2r}\ J(p_1^-,p_2^-)\ \langle \pi_n(p_\pi) \big| (\bar\xi_n W)_{p_1^-} \bar{n}\!\!\!/ \gamma_5 (W^\dagger \xi_n)_{-p_2^-} \big| 0\rangle \left|\frac{p_1^- p_2^-}{\mu_-^2}\right|^\epsilon$$

$$J(x_1,x_2) = 1/(p_1^-)^2 = 1/[(\bar n\cdot p_\pi)^2\, x_1^2]$$

$$= -i \frac{f_\pi}{\bar n\cdot p_\pi}\Big(\frac{\bar n\cdot p_\pi}{\mu_-}\Big)^{2\epsilon} \sum_{x_1\neq 0} \int dx_{1r}\, dx_2\ \frac{1}{(x_1)^2}\ \delta(1-x_1-x_2)\ \phi_\pi(x_1,x_2)\ \big|x_1 x_2\big|^\epsilon$$

$$= -i \frac{f_\pi}{\bar n\cdot p_\pi}\Big(\frac{\bar n\cdot p_\pi}{\mu_-}\Big)^{2\epsilon} \sum_{x_1\neq 0} \int dx_{1r}\ \frac{1}{(x_1)^2}\ \theta(1-x_1)\theta(x_1)\ \hat\phi_\pi(x_1)\ \big|x_1(1-x_1)\big|^\epsilon$$

$$= \frac{-i f_\pi}{\bar n\cdot p_\pi}\Big(\frac{p_\pi^-}{\mu_-}\Big)^{2\epsilon} \int dx_1\ \frac{\theta(x_1)}{(x_1)^2}\ \Big[\theta(1-x_1)\ \hat\phi_\pi(x_1) - x_1 \hat\phi_\pi'(0)\Big] \big|x_1(1-x_1)\big|^{-\epsilon}$$

$$= -i \frac{f_\pi}{\bar n\cdot p_\pi}\Big(\frac{\bar n\cdot p_\pi}{\mu_-}\Big)^{2\epsilon} \Big\{ \int_0^1 dx_1\ \frac{\phi_\pi(x_1) - x_1\phi_\pi'(0)}{(x_1)^2} + \frac{1}{2\epsilon_{\rm UV}}\ \big[\phi_\pi'(0)\big] \Big\}$$

$$O_{ct}^{[1]} = -\frac{1}{2\epsilon_{\rm UV}} \int dp_2^-\ \Big[\frac{\partial}{\partial p_1^-}\Big] (\bar\xi_n W)_{p_1^-} \bar{n}\!\!\!/ \gamma_5 (W^\dagger \xi_n)_{-p_2^-} \Big|_{p_1^-\to 0} .$$

$$A_\pi + A_\pi^{ct} = -i \frac{f_\pi}{\bar n\cdot p_\pi} \Big\{ \int_0^1 dx_1\ \frac{\phi_\pi(x_1,\mu) - x_1\phi_\pi'(0,\mu)}{(x_1)^2} + \phi_\pi'(0,\mu)\ \ln\Big(\frac{\bar n\cdot p_\pi}{\mu_-}\Big) \Big\} \quad + D(\mu,\mu_-)\ \phi_\pi'(0,\mu)$$

= finite

can think of this as a new parameter

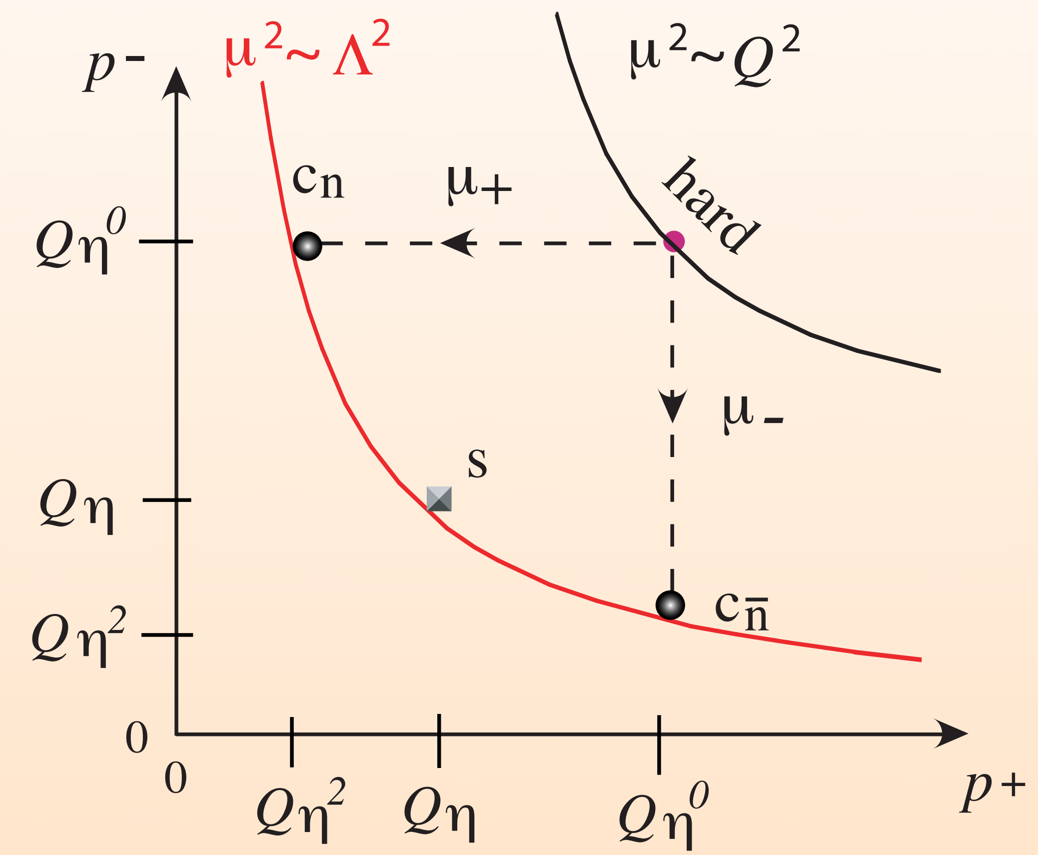
$$= -i \frac{f_\pi}{\bar n\cdot p_\pi}\ \int_0^1 dx_1\ \frac{\phi_\pi(x_1)}{(x_1^2)_\varnothing}$$

Associated RGE flow?

Bjorn Lange, Aneesh Manohar, & I.S. in progress

$$\mu_+ \mu_- = \mu^2$$

$$\mu \frac{d}{d\mu} \mathcal{O}^{\rm ren}(\omega') = \int d\omega \, \gamma(\omega', \omega) \, \mathcal{O}^{\rm ren}(\omega)$$



$$A_\pi^{\rm ren} \;=\; d(\mu)\phi'_\pi(0,\mu) + \int_0^1 dx \, \big[C(x,\mu)\big]_+ \phi_\pi(x,\mu)$$

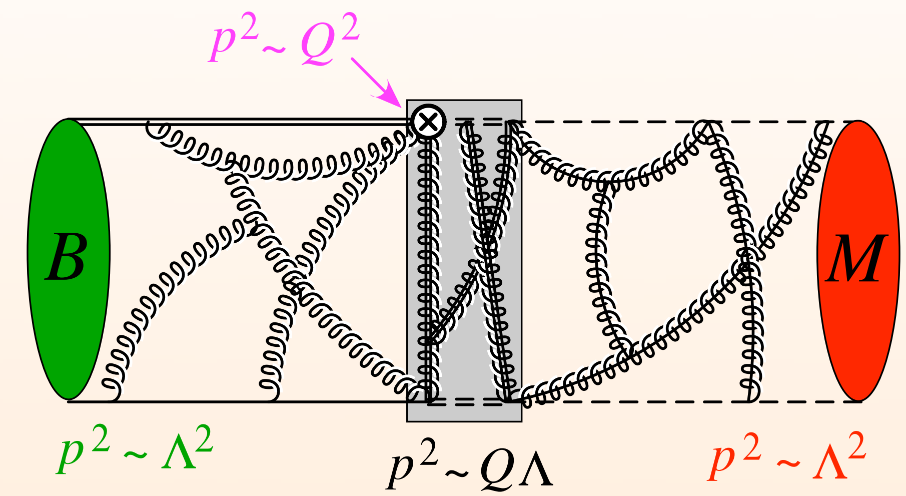
these two terms mix, and close under the flow

This generates an interesting series:

$$\Big[C(x)\Big]_+ \sim \Big[\frac{1}{x^2}\Big]_+ + \Big[\alpha_s \ln(\mu) \frac{\ln x}{x^2} + \dots\Big]_+ + \Big[\alpha_s^2 \ln^2(\mu) \frac{\ln^2 x}{x^2} + \dots\Big]_+ + \dots$$

Application to Singular Cases

$$B \to \pi \ell \bar{\nu}$$



$$f(E) = \int dz\, T(z,E)\ \zeta_J^{BM}(z,E)$$

$$+\ C(E)\ \zeta^{BM}(E)$$

Factorize $\zeta^{BM}(E)$:

$$\zeta^{B\pi}(E) = \frac{f_\pi f_B m_B}{4E^2}\ \pi\alpha_s(\mu)\int_0^1 du\, dv\, dw \int dk_1\, dk_2 \Bigg\{ \frac{4}{9}\ \delta_{k_1k_2}\ \delta_{uv} \frac{(1+v)\phi_\pi(u,v)}{(v^2)_\varnothing}\ \frac{\phi_B^-(k_1,k_2)}{(k_1)_\varnothing}$$

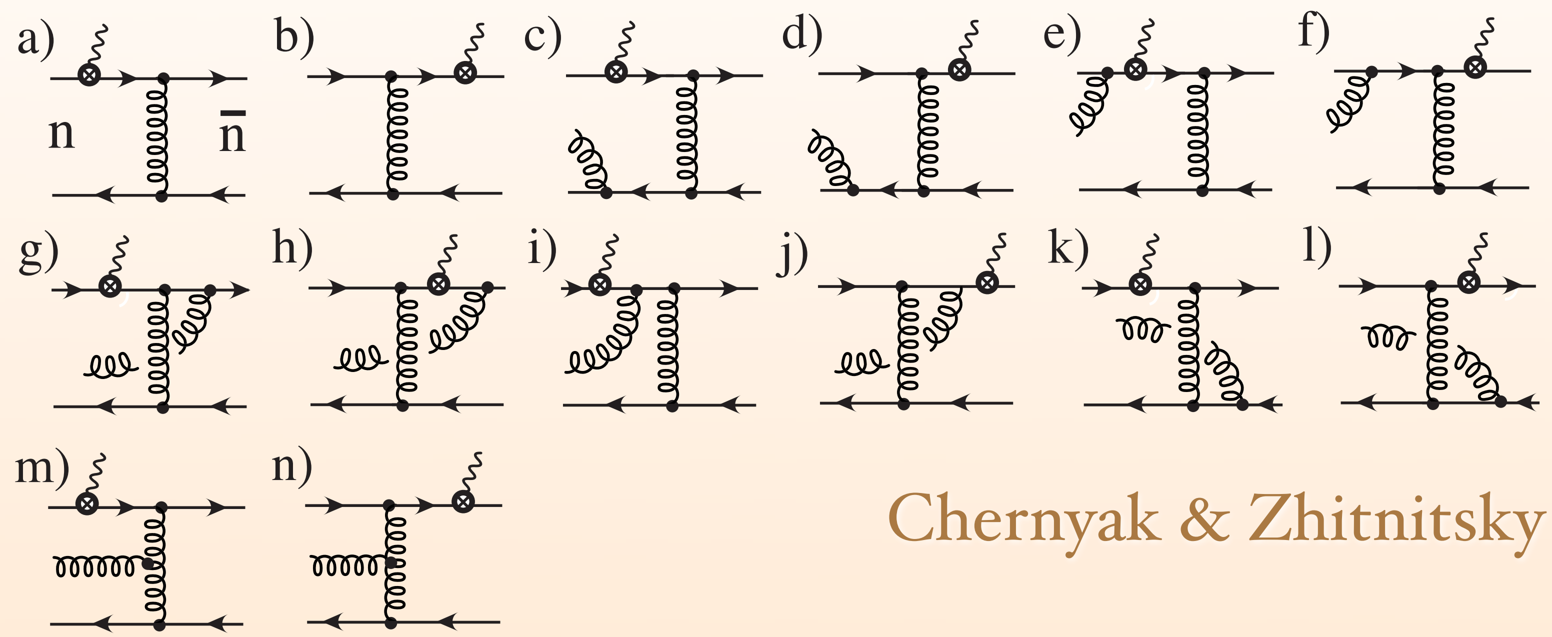
$$+ \frac{4\mu_\pi}{9}\ \delta_{k_1k_2}\ \delta_{uv} \frac{(\phi_\pi^p + \frac{1}{6}\,\phi_\pi'^{\sigma})(u,v)}{(v^2)_\varnothing}\ \frac{\phi_B^+(k_1,k_2)}{(k_1^2)_\varnothing} + \frac{f_{3B}}{f_B}\int dk_3\ \delta_{k_1k_2k_3}\ \delta_{uv}\Bigg[\frac{\phi_\pi(u,v)}{(v^2)_\varnothing}$$

$$\times\ \phi_{3B}(k_1,k_2,k_3)\frac{9k_3+k_1}{9[(k_1+k_3)^2k_1]_\varnothing} - \frac{\phi_\pi(u,v)}{v_\varnothing}\frac{8k_3\,\phi_{3B}(k_1,k_2,k_3)}{9[(k_1+k_3)^2k_1]_\varnothing}\Bigg]$$

$$+ \frac{f_{3\pi}}{f_\pi}\ \delta_{k_1k_2}\ \delta_{uvw}\Bigg[\frac{\phi_{3\pi}(u,v,w)}{[(v+w)^2\,v]_\varnothing} - \frac{7\,\phi_{3\pi}(u,v,w)}{9[w(v+w)^2]_\varnothing} + \frac{8\,\bar{v}\,\phi_{3\pi}(u,v,w)}{9[v^2w(u+w)]_\varnothing}\Bigg]\frac{\phi_B^+(k_1,k_2)}{(k_1^2)_\varnothing}\ \Bigg\}$$

$\boxed{\gamma^* \rho \longrightarrow \pi}$

$$\langle \pi^+(p')|\bar{q}\gamma^\nu q|\rho^+(p,\varepsilon^\perp)\rangle \;=\; i\epsilon^{\nu\alpha\beta\lambda} p_\alpha p'_\beta \varepsilon^\perp_\lambda \; F_{\rho\pi}(q^2)$$



Chernyak & Zhitnitsky

$$F_{\rho\pi}(Q^2) = \frac{4\pi\alpha_s(\mu)}{27Q^4}\int dx \int dy \int dz \int du \int dv \int dw \left\{ 4 f_\rho^T f_\pi \mu_\pi \, \frac{\delta_{xy}\delta_{uv}\, \phi_{\rho_\perp}(x,y)\phi_\pi^p(u,v)}{(y^2)_\emptyset v_\emptyset} \right.$$

$$+ f_\rho^V m_\rho f_\pi \, \delta_{xy}\delta_{uv} \left[\frac{g_{\rho_\perp}^{(v)}(x,y)\phi_\pi(u,v)}{x_\emptyset y_\emptyset (v^2)_\emptyset} + \frac{g_{\rho_\perp}^{(A)}(x,y)\phi_\pi(u,v)}{4x_\emptyset y_\emptyset (u^2)_\emptyset (v^2)_\emptyset} \right] + \frac{f_\rho^{3A} f_\pi}{4} \delta_{uv}\delta_{xyz}\, \phi_\pi(u,v)\phi_{3\rho}(x,y,z)$$

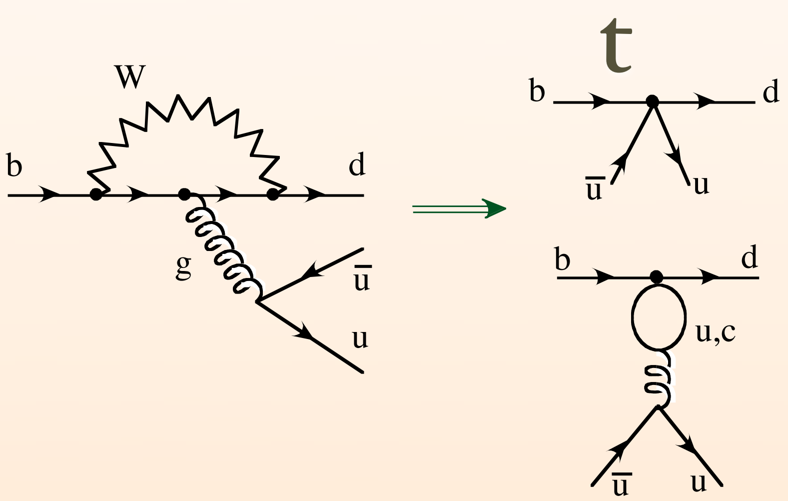
$$\times \left[\frac{8}{(\bar{y}^2 x)_\emptyset v_\emptyset} + \frac{2}{(\bar{z}zy)_\emptyset v_\emptyset} - \frac{9}{(\bar{y}^2 x)_\emptyset (v^2)_\emptyset} - \frac{1}{(\bar{z}zx)_\emptyset (v^2)_\emptyset} - \frac{1}{(z\,\bar{y}^2)_\emptyset (v^2)_\emptyset} \right]$$

$$\left. - f_\rho^T f_{3\pi}\, \phi_{3\pi}(u,v,w)\phi_{\rho_\perp}(x,y)\, \delta_{uvw}\delta_{xy} \left[\frac{9}{2(\bar{u}^2 v)_\emptyset (y^2)_\emptyset} + \frac{1}{2(\bar{u}^2 w)_\emptyset (y^2)_\emptyset} + \frac{1}{(\bar{u}vw)_\emptyset y_\emptyset} \right] \right\}$$

Annihilation in B-Decays

new physics in penguins?

$$B \to \pi\pi, \quad B \to K\pi$$



$$A = (V_{ub}V^*_{ud})\ T + (V_{cb}V^*_{cd})\ P$$
$$\qquad us \qquad\qquad\qquad cs$$

fix phase convention, use isospin and data

$$10^3\,\hat{P}_{\pi\pi} = -0.58 + 2.5i$$
$$10^3\,\hat{P}_{K\pi} = \pm 4.4 + 3.2i$$

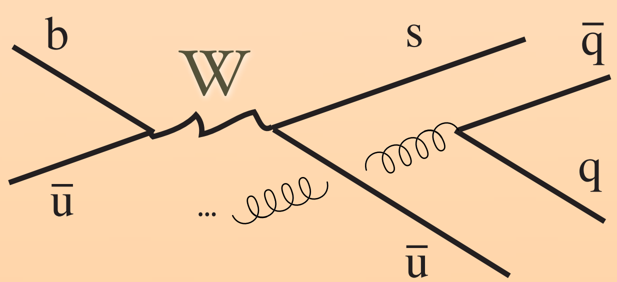
$$10^3\,\hat{P} = (C_3 \sim 0.01)A_1^{(0)} + (C_1 \sim 1)\frac{\alpha_s(m_b)}{\pi}A_2^{(0)} + (C_1 \sim 1)\frac{\alpha_s(m_b)\Lambda}{m_b}A_3^{(1)}$$

$$+A^{\mathrm{new}}_{\mathrm{physics}}$$
$$+A^{\mathrm{charm}}_{\mathrm{non.pert.}}$$

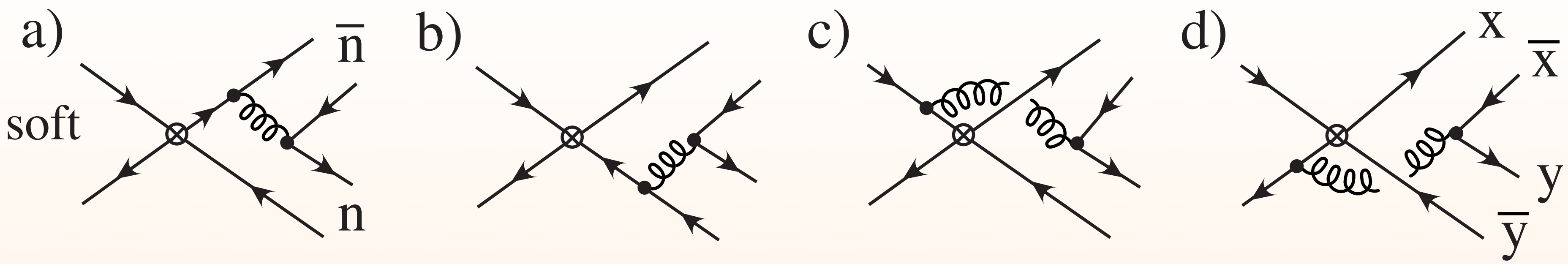
short-distance part too small to explain the data

$\sim 0.3i$

many people "explain" it with SM annihilation



Keum, Li, Sanda



Keum, Li, Sanda

regulate singularity with $k_\perp$

$$\mathrm{Im}\,[xm_b^2 - k_\perp^2 + i\epsilon]^{-1} = -\pi\delta(xm_b^2 - k_\perp^2)$$

Beneke, Buchalla, Neubert, Sachrajda

model singularity as non-factorizable

$$X_A = \int_0^1 dy/y = (1+\rho_A e^{i\varphi_A})\ln(m_B/\Lambda)$$

Apply rapidity factorization:

Arnesen, Ligeti, Rothstein, I.S. (hep-ph/0607001)

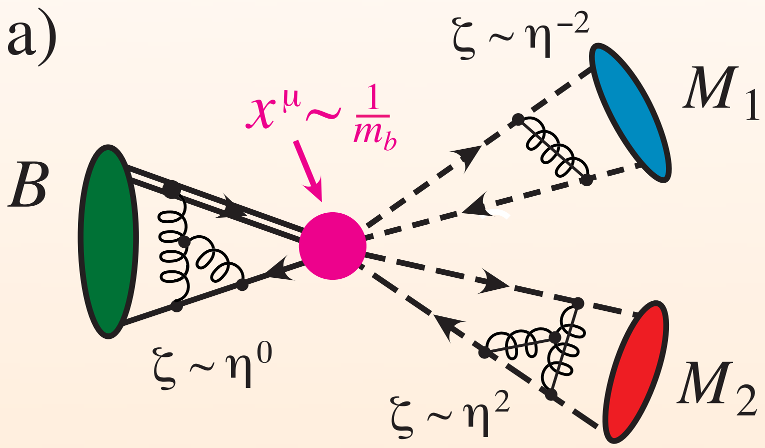
$$A^{(1)}_{Lann}(\bar B \to M_1 M_2) = \frac{G_F f_B f_{M_1} f_{M_2}}{\sqrt{2}} \int_0^1 dx\, dy\, H(x,y)\,\phi^{M_1}(y)\phi^{M_2}(x)$$

well defined and REAL

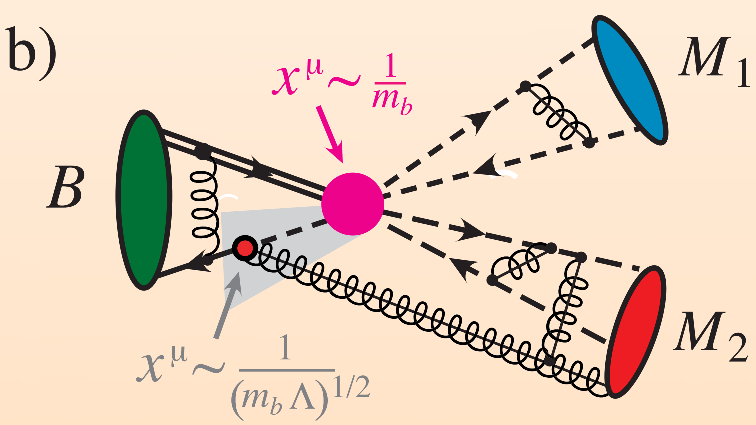
$$\propto \int_0^1 dy \left[\frac{\phi_{M_1}(y)}{y}\right]\left[\int_0^1 dx\,\frac{\phi_{M_2}(x)+\bar x\phi'_{M_2}(1)}{\bar x^2} + \phi'_{M_2}(1)\left\{\ln\left(\frac{\bar p^-_{M_2}}{\mu_-}\right) + D(\mu_-)\right\}\right]$$

$$-\int_0^1 dy \int_0^1 dx\, \frac{\phi_{M_1}(y)\phi_{M_2}(x)}{y(x-xy-1)}$$

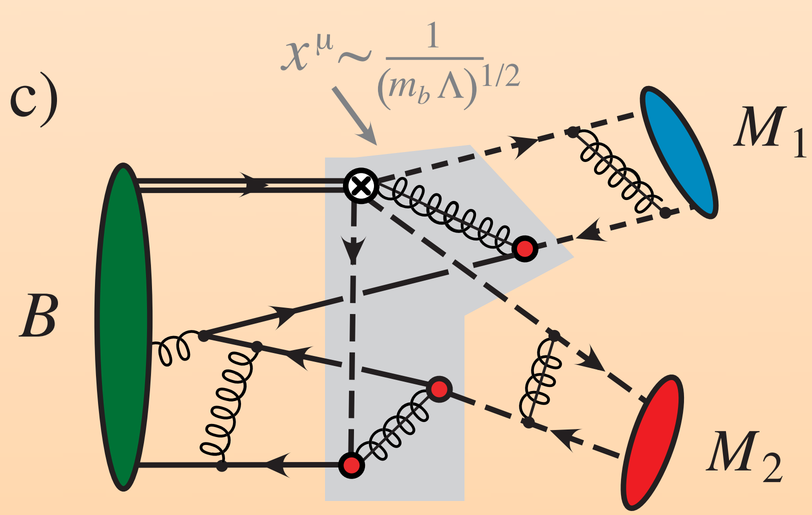
Annihilation is real at lowest order in α_s expansion



Suffers from endpoint divergences.
But they do not introduce a phase.



Leading order but small.
No endpoint divergences here.
No imaginary part here either.



A soft rescattering annihilation contribution DOES have a strong phase, but is one higher order in α_s

Summary

- Differential formulation of continuum EFT

  new tools for thinking about field theory modes

- Resolves singularities.

- Interesting applications in B-physics and to processes with hard scattering

Future

- $k_\perp$-dependent parton distribution functions

- Lots of phenomenology to examine.

- Application to other EFT's

THE END