1 Preliminaries

1.1 The General Model

In a unit-demand auction there is a finite set of k items and a finite set of n bidders N where each bidder is interested in receiving at most one item. We assume that $n \geq k \geq 1$. Every bidder i has a private valuation vector $v_i = (v_i(x))_{x \in K}$ where $v_i(x) \geq 0$ denotes bidder i's value for item x. In addition every bidder i has a private budget $b_i > 0$; bidder i cannot pay any amount equal or more than b_i.[1] A pair $t_i = (v_i, b_i)$ is called a *type*. It is convenient to add a null item, denoted by ϕ, in which its value for each bidder is zero. We assume that any bidder that does not get an item in K gets the null item and pays zero.

With the absence of budget constraints, bidder's i utility from receiving item x and paying p_i is equal to $v_i(x) - p_i$. However as budgets are incorporated in our model, we assume the utility function for bidder i with type $t_i = (v_i, b_i)$ is given by

$$u_i((v_i, b_i), x, p_i) = \begin{cases} v_i(x) - p_i & b_i > p_i \\ -1 & b_i \leq p_i \end{cases} \tag{1}$$

where the -1 utility for the case $p_i \geq b_i$ can be thought of as bidder i will not complete the transaction if she is required to pay b_i or more[2].

An *assignment* is a tuple $\mathbf{s} = (s_i)_{i \in N}$ where $s_i \in K \cup \{\phi\}$ such that for every pair of bidders $i, j \in N$ if $s_i, s_j \in K$ then $s_i \neq s_j$. An *outcome* in the auction is a tuple $(s_i, p_i)_{i \in N}$ where $(s_i)_{i \in N}$ is an assignment and p_i is the payment for bidder i.

For simplicity we assume the seller has a reserve price 0 for each item. Note that at this point nothing has been said about the rules of the auction, e.g. what are the possible strategies and how the outcome is determined.

Throughout this paper we assume that all values and budgets are integers; similar results can be obtained for the general case.

2 The DGS Ascending Auction

In this section we describe and analyze the ascending auction described by Demange et. al [**?**][3] generalized to budget constraints bidders. At each stage in the auction the auctioneer holds a vector of prices $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{R}_+^K$ where q_x is the price for item x at stage r.

At the first stage the prices are $\mathbf{q} = (0, \ldots, 0)$, and every bidder submits a subset of items which she is interested in. We refer to this subset as a *demand set*.[4] We say that a subset of items (of K) is *overdemanded* if the number of bidders interested in/demanding **only** items in this set is greater than the number of items in the set. If there is no overdemanded set (with respect to the submitted demand sets) then it is possible to assign each item to a bidder who demands it the auction is over; in this case if item $x \in K$ is assigned to bidder i she pays q_x and if i is assigned the null object she pays zero. Otherwise the auctioneer computes a minimal overdemanded set (with respect to the submitted subsets of the bidders) and for each item in this set it raises the price by one unit. Again, each bidder announces a demand set at the new prices and the auctioneer either

[1] We do not include allow b_i to be a feasible payment just for convenience.

[2] Replacing this with any other non positive utility does not alter the results.

[3] In [**?**] this auction is referred to as the *the exact auction mechanism.*

[4] Importantly, we do not assume here that the demand set i submits, necessarily maximizes i's utility.

can allocate the items (in the current prices), or otherwise computes a minimal overdemanded set in which the prices of the items in this set are again raised by one unit, and so forth. Note that the auctioneer will find a possible assignment eventually, since the prices are raised by a unit at each stage.

2.1 Competitive Prices

Denote by $D(\mathbf{q},(v_i,b_i))$ the *true demand set* of a bidder at prices $\mathbf{q}$ when her type is (v_i,b_i), that is

$$D(\mathbf{q},(v_i,b_i)) = \{x \in K \cup \phi | x \in \arg\max_{y \in K \cup \{\phi\}} \{v_i(y) - q_y : q_y < b_i\}\}. \tag{2}$$

Let $\mathbf{t} = ((v_1,b_1),\ldots,(v_n,b_n))$ be a profile of types. A vector of prices $\mathbf{q}$ is *competitive* (with respect to $\mathbf{t}$) if there is an assignment $\mathbf{s} = (s_i)_{i \in N}$ such that $s_i \in D(\mathbf{q},(v_i,b_i))$. Such an assignment is said to be *valid* for $\mathbf{q}$. A tuple $(\mathbf{q},\mathbf{s})$ is called a *competitive equilibrium* if $\mathbf{s}$ is valid for $\mathbf{q}$, and in addition for any item $x \in K$, if $s_i \neq x$ for every every bidder i, then $q_x = 0$. In other words the price of non allocated items in equilibrium is zero.

The following theorem given in [**?**] (without budget constraints) shows that if all bidders always announce their true demand set, i.e. all items that maximize their utilities in the given prices, then the auction terminates at the minimal competitive price vector. Formally,

Theorem 2.1. *Let $\mathbf{t} = ((v_1,b_1),\ldots,(v_n,b_n))$ be the profile of types. Let $\mathbf{q}^r$ be the prices at stage r and let $\mathbf{q}$ be the price vector at the end of the auction. If at every stage r, each bidder submits her true demand set $D(\mathbf{q}^r,(v_i,b_i)$, then $\mathbf{q}$ is competitive and for any other competitive price vector $\tilde{\mathbf{q}}$, $\mathbf{q} \leq \tilde{\mathbf{q}}$.*

The proof of Theorem 2.1 is identical to the proof of Theorem 1 by Demange et. al in [**?**] and therefore is omitted. Their proof uses the celebrated Hall theorem which asserts that a possible allocation exists if and only if there is no overdemanded set. Roughly speaking their proof only remains in the "higher level" of demand sets, and therefore the presence of budgets do not change any of their arguments. For the exact details we refer the reader to [**?**].

Demange et. al also show that there exists an assignment such that the final prices together with the final price vector is an equilibrium. Interestingly, as we show in the following example this is not true in our context.

Example 2.2. *Consider one item x and two bidders 1 and 2. Let $b_1 = b_2 = 10$ and $v_1 = 15$, $v_2 = 20$. For any $q_x < 10$ both bidders' true demand contains x. Therefore any competitive price is at least 10, but in any such price the item is not allocated.*

Example 2.2 shows that if ties are allowed then there is no competitive equilibrium. The following example shows that a competitive equilibrium does not exist even with no ties.

Example 2.3. *Consider two items, x and y and three bidders $1,2$ and 3. Let $b_1 = 10$, $b_2 = 11$ and $b_3 = 1000$. Let $v_1(x) = 1000$, $v_2(y) = 1000$, $v_3(x) = 20$, $v_3(y) = 21$ and all other values are zero. Note that the final prices will be $q_x = 11$ and $q_y = 12$, bidder 3 will get either item x or item y, and the other item will not be allocated.*

Examples 2.2 and 2.3 motivate the following definition:

Definition 2.4 (Independence). *We say that n numbers $x_1, \dots, x_n$ are in* independent*, if it is not possible to find two different nonempty subsets containing positive numbers, that sum up to the same number. Alternatively, for every linear combination $\sum_{i=1}^{n} e_i x_i$ where $e_i \in \{-1, 0, 1\}$ then if $e_i \neq 0$ then $x_i = 0$.*

For any profile of types $\mathbf{t} = ((v_1, b_1), \dots, (v_n, b_n))$ we denote by $H(\mathbf{t})$ the set of numbers $b_1, \dots, b_n, v_1(1), v_1(2), \dots, v_1(k), v_2(1), \dots, v_2(k), \dots, \dots, v_n(1), \dots, v_n(k)$.

Independence Assumption: For every type profile $\mathbf{t}$, the numbers in $H(\mathbf{t})$ are independent[5].

We will show:

Theorem 2.5. *Under the independence assumption if $\mathbf{q}$ is the minimal competitive price vector then there exist an assignment $\mathbf{s}$ such that $(\mathbf{q}, \mathbf{s})$ is a competitive equilibrium.*

Before we prove Theorem 2.5 we provide a useful tool in the next lemma. First a definition is needed:

Definition 2.6. *Let $\mathbf{t} = ((v_1, b_1), \dots, (v_n, b_n))$ be a profile of types. Let $\mathbf{q}$ be a minimal competitive price vector and let $\mathbf{s}$ be a valid assignment for $\mathbf{q}$. In a $(\mathbf{q}, \mathbf{s})-$ graph or an* almost envy free graph *$T = (V, E)$, the set of nodes is N, and there exist a directed edge $(i, j) \in E$ if and only if decreasing the price by one unit the price of s_j will cause i to envy j, i.e. $u((v_i, b_i)), s_i, p_i) < u((v_i, b_i)), s_j, q_{s_j} - 1)$. An edge $(i, j) \in E$ is colored green if $q_{s_j} = b_i$ and red otherwise.*

Intuitively, a green edge from i to j capture the envy due the budget limit of bidder i, and a red edge (i, j) implies that i has the budget for getting j's item in her price but is indifferent to such an outcome, i.e. $v_i(s_i) - p_i = v_i(s_j) - p_j$.

Lemma 2.7. *Let $\mathbf{q}$ be a minimal competitive price vector and let $\mathbf{s}$ be a valid assignment for $\mathbf{q}$. Let T be the $(\mathbf{q}, \mathbf{s})$-graph and let $p_i = q_{s_i}$.*

1. *If $p_i > 0$ then the indegree of i is at least 1.*
2. *If $p_i > 0$, then on every simple predecessors path to i either there is a node j such that $p_j = 0$ or there exist a green edge (j, l) together with a path from l to i (l can be i).*
3. *T contains no cycles.*
4. *If $p_i > 0$ then $p_i \neq v_i(s_i)$.*

Proof.

1. Let i be such that $p_i > 0$ and suppose the indegree of i is zero. Then the price vector $\tilde{q}$ in which $\tilde{q}_{s_i} = q_i - 1$ and $\tilde{q}_j = q_j$ for all $j \neq i$ is competitive since $\mathbf{s}$ is valid for $\tilde{\mathbf{q}}$. This contradicts the minimality of $\mathbf{q}$.
2. Let i be such that $p_i > 0$ and assume the claim doesn't hold. This implies that there exists a cycle of red edges. Let $i_1, i_2, \dots, i_m, i_1$ be such cycle. Since for every $l = 1, \dots, m$ we have $p_{i_l} - p_{i_{l+1}} = v_i(s_{i_l}) - v_i(s_{i_{l+1}})$ ($l + 1$ is taken modulo m) we obtain that
$$0 = p_{i_1} - p_{i_2} + p_{i_2} - p_{i_3} + \cdots + p_{i_m} - p_{i_1} =$$
$$v_{i_1}(s_{i_1}) - v_{i_1}(s_{i_2}) + v_{i_2}(s_{i_2}) - v_{i_2}(s_{i_3}) + \cdots v_{i_m}(s_{i_m}) - v_{i_m}(s_{i_1}),$$
contradicting the independence assumption.

[5]Unless specified otherwise we will have this assumption through the entire paper.

3. We first show that no node has an indegree larger than 1. Suppose towards a contradiction that i has indegree $i > 0$. By part 2 of the lemma there exist two different predecessors paths $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_r$ where $i_m = j_r = i$ with satisfy the conditions in part 2. We choose these paths such that all edges are red perhaps but the first one, i.e. if the first is green then it is close as possible to i. Consider the first path; either (i_1, i_2) is green or $p_{i_1} = 0$. Note that on the first path for every $2 < l < m$ $p_{i_l+1} = v_{i_l}(s_{i_l}) - v_{i_l}(s_{i_{l+1}}) - p_{i_l}$ and for $l = 2$ either $p_{i_2} = v_{i_1}(s_{i_1}) - v_{i_1}(s_{i_2})$ or $p_{i_2} = b_1$. Since the payments of the second path can be written similarly, this implies that we can express p_i in two different ways only with budgets or values, contradicting the independence assumption.

   Since by the first part of the lemma the maximum indegree is one, to complete the proof it is enough to show that there does not exist a cycle in which all it nodes have exactly outdegree 1. Suppose that there exists such a cycle $i_1, i_2, \ldots, i_m, i_1$. Let $\tilde{\mathbf{q}}$ be the price vector in which for every $j = 1, \ldots, m$ let $\tilde{q}_{s_{i_j}} = q_{s_{i_j}} - 1$. and all other prices remain the same. We claim that $\tilde{\mathbf{q}}$ is a competitive price vector contradicting the minimality of $\mathbf{q}$: let $\tilde{\mathbf{s}}$ be the assignment for every bidder $j = 1, \ldots, m$ $s_{i_j} = s_{i_{j+1}}$ (where $j+1$) is taken modulo m). Note that $\tilde{\mathbf{s}}$ is a valid for $\tilde{\mathbf{q}}$.

4. Suppose $p_i > 0$ and assume $p_i = v_i(s_i)$. By part 2 there exist a path $i_1, i_2, \ldots, i_m$ where $i_m = i$ where either $p_{i_1} = 0$, or $p_{i_1} = b_{i_1}$. Similarly to part 2 we can express p_i in a linear combination of values and/or budgets, except i's. But since p_i also equals $v_i(s_i)$ this contradicts the independence assumption.

□

Proof of Theorem 2.5: Let $\mathbf{q}$ be a minimal competitive price vector and $\mathbf{s}$ a valid assignment for $\mathbf{q}$. We show that $(\mathbf{q}, \mathbf{s})$ is an equilibrium. Suppose there exist an item $x \in K$ such that $q_x > 0$ and no bidder gets this item. Observe that there exist two different bidders $l, j \in N$ such that for each $i \in \{l, j\}$ either $q_x = b_i$ or $q_x = v_i(s_i) - v_i(x) - p_i$. We first show that for each $i \in \{l, j\}$ there is a linear combination of elements in $H(\mathbf{t})$ that sum up to q_j. Fix some arbitrary $i \in \{l, j\}$. If $q_j = b_i$ we are done. Suppose $q_x = v_i(s_i) - v_i(x) - p_i$. If $p_i = 0$ we are done. If $p_i > 0$ then by part 2 of Lemma 2.7 there exists a simple path $i_1, i_2, \ldots, i_m$ where $i_m = i$, such that either (i_1, i_2) is a green edge or $p_{i_1} = 0$. But this implies the existence of such a linear combination. It remains to show that for each i the linear combination is different as this contradicts the independence assumption. This follows since for each $i \in \{l, j\}$ either $q_x = b_i$ or $-v_i(x)$ appear only in the linear combination we found for i. □

2.2 Truthfulness

2.3 Incentive Compatibility

In the previous section we analyzed the outcome of the auction when bidders always announce their true demand sets. In this section we analyze the DGS auction when bidders can use different strategies.

For any stage r let H_r be the history of demand sets of the bidders up to stage r. A bidding strategy for i is a sequence $\tau_i^1, \tau_i^2, \ldots$, such that for each $r \geq 1$, $\tau_i^r : H_r \times \mathbb{R}_+^K \rightarrow 2^{K \cup \{\phi\}}$ maps a history in H_r and a vector of prices to a demand set. All our results do not depend on the histories' structure. Thus, with a slight abuse of notation we write $\tau_i^r(\mathbf{q})$ to denote the demand set i submits at round r under the strategy τ_i, when the price vector is $\mathbf{q}$.

We say that a strategy τ_i for i is *consistent with type* (v_i, b_i) if for every price vector $\mathbf{q}$ and every stage r, $\tau_i^r(\mathbf{q}) = D_i(\mathbf{q}, (v_i, b_i))$. A strategy is *consistent* if there exist a type for which it is consistent with it.[6]

Essentially, by limiting all bidders to use consistent strategies, the auction is a direct revelation mechanism in which each bidder only submit a type and the auctioneer computes the outcome (e.g. by simulating the whole process). We call this auction the *direct* DGS auction. Formally, the direct DGS auction is defined as follows:

- Every player i, submits a bid (v_i, b_i).
- Let $\mathbf{t} = ((v_1, b_1), \ldots, (v_n, b_n))$. If $H(\mathbf{t})$ do not satisfy the independence assumption the auction is terminated.
- The auctioneer computes a competitive equilibrium with $(\mathbf{s}, \mathbf{q})$ where $\mathbf{q}$ is a minimal price vector, assigns s_i to bidder i and charges her q_{s_i}.

In the next theorem we show that bidding the direct DGS auction is incentive compatible. This result has been proved independently by [**?**]. Our proof is simpler and provides an alternative approach for understanding the problem.

Theorem 2.8. *The direct DGS auction is truthful, that is for every bidder it is a dominant strategy to report her true type.*

We will assume w.l.o.g. that all bid profiles discussed in the proof satisfy the independence assumption. Through out the proof we fix some bidder i and fix the submitted types of all bidders but i, these are $\mathbf{t}_{-i} = (t_j)_{j \in N \setminus \{i\}}$. For any type t_i let $\mu(t_i) = (\mathbf{q}(t_i), \mathbf{s}(t_i))$ be the competitive equilibrium when i bids t_i, and let $p_i(t_i) = q_{s_i}(t_i)$ be her payment.

Lemma 2.9. *For any t_i, t_i' in which $s_i(t_i) = s_i(t_i')$ (i is assigned the same item), $p_i(t_i) = p_i(t_i')$ (i pays the same price).*

Proof. Fix some type $t_i = (v_i, b_i)$ in which player i is assigned an item $x \in K$ and let $\hat{t}_i = (\tilde{v}_i, \tilde{b}_i)$ be the type obtained by t_i by letting $\tilde{v}_i(x) = v_i(x)$, $\tilde{v}_i(y) = 0$ for all other items, and $\tilde{b}_i = b_i$. It is enough to show that $s_i(\tilde{t}_i) = s_i(t_i)$ and $p_i(t_i) = p_i(\tilde{t}_i)$: Suppose this is true. Let t_i and t_i' be two different types in which bidder i obtains the same item x but $p_i(t_i) < p_i(t_i')$. Therefore $p_i(\tilde{t}_i) < p_i(\tilde{t}_i')$. But $\mathbf{q}(\tilde{t}_i)$ are competitive prices with respect to $(\tilde{t}_i, t_{-i})$ (as $\mathbf{s}(\tilde{t}_i, t_{-i})$ is a valid assignment) contradicting the minimality of $\mathbf{q}(\tilde{t}_i')$.

In the following sequence of claims we prove that $s_i(\tilde{t}_i) = s_i(t_i)$ and $p_i(t_i) = p_i(\tilde{t}_i)$.

Claim 2.10. *For every item y, $q_y(\tilde{t}_i) \le q_y(t_i)$.*

Proof. Since $\mathbf{q}(t_i)$ are competitive with respect to $(\tilde{t}_i), t_{-i})$, and the auction outputs the minimal competitive prices this follows. □

Claim 2.11. *$s_i(\tilde{t}_i) = x$, i.e. i is also assigned x when she reports $\tilde{t}_i$.*

Proof. Assume that this is not the case and let $s_i(\tilde{t}_i) = y \neq x$. Since $\tilde{v}_i(y) = 0$ and since by the previous claim $q_x(\tilde{t}_i) \le q_x(t_i)$ it must be that $q_x(\tilde{t}_i) = q_x(t_i) = v_i(x)$ otherwise this contradicts that $\mathbf{q}(\tilde{t}_i)$ are competitive. But $q_x(t_i) = v_i(x)$ contradicts part 4 of Lemma 2.7. □

[6] Consistent strategies can be thought of bidding through a proxy bidder (see e.g. [**?**]).

For every item $x \in K$ with $q_x > 0$ denote by $w(x)$ the winner of item x, and let $z(x)$ be a bidder such that $(z(x), w(x))$ is an edge in the μ-graph.

For the next claim we need some definitions. Let A denote the set of items in which their prices decreased from the case that i bids t_i to the case that i bids $\tilde{t}_i$. That is

$$A = \{y \in K : q_y(\tilde{t}_i) < q_y(t_i)\}.$$

We also define two functions from K to $K \cup \{\phi\}$.

We let $\delta(y) = z$ if $s_j(t_i) = y$ and $s_j(\tilde{t}_i) = z$. We let $\gamma(y) = z$ if there exist a pair of players j, l and an item w, such that $s_j(t_i) = w$, $s_j(\tilde{t}_i) = z$ $s_l(t_i) = l$ and (j, l) is an edge in the $\mu(t_i)$-graph.

Claim 2.12. *If $y \in A$, then $\delta(y) \in A$. Moreover, if $q_y(t_i) > 0$, and $\gamma(y) \neq x$, then $\gamma(y) \in A$.*

Proof. Assume $s_j(t_i) = y$ and $z = \delta(y)$. Since $\mathbf{q}$ are competitive with respect to (t_i, t_{-i}), j does not prefer z at $q_z(t_i)$ to y at $q_y(t_i)$. But this means that if $q_y(\tilde{t}_i) < q_y(t_i)$, and $q_z(\tilde{t}_i) = q_z(t_i)$, then j would prefer y when i submits $\tilde{i}$ contradicting the competitiveness of $\mathbf{q}(\tilde{t}_i)$.

To prove the second part suppose $q_y(t_i) > 0$ and $z = \gamma(y) \neq x$. Let j be such that $s_j(t_i) = w$ and $s_j(\tilde{t}_i) = z$, and l be such that $s_l(t_i) = y$. If (j, l) is red. Such a configuration exists by part 1 of Lemma 2.7. Thus, j is indifferent between getting w in $q_w(t_i)$ and y in $q_y(t_i)$. Therefore if $y \in A$ then j is strictly better off getting y in $q_y(\tilde{t}_i)$ then getting any other item x' in $q_{x'}(t_i)$, implying that $w \in A$. If (j, l) is green, then obtaining y in $q_y(\tilde{t}_i) - 1$ or less is strictly better for j than obtaining any item x' in $q_{x'}(t_i)$, again implying that $y \in A$. □

To finish the proof we need to show that $p_i(t_i) = p_i(\tilde{t}_i)$. Suppose that $p_i(t_i) > p_i(\tilde{t}_i)$. Hence $x \in A$. Therefore since there are no cycles in $\mu(t_i)$-graph, by the last claim it must be that some item whose price is zero when i submits t_i belongs to A - a contradiction. □

We now show that i gets the item x which maximizes her welfare. Assume this is not the case, and by bidding truthfully i gets another item. Let G denote the game where i bids truthfully, and let $y \neq x$ denote the item that i gets in G, for p_y. Assume that there is another game $\tilde{G}$, in which i lies, and gets x for $\tilde{p}_x$. As i is not envious in G, we must have that $p_x > \tilde{p}_x$, where p_x is the price of x in G.

We define another game G', in which i bids the same values for x, y, the same budget, but all the other valuations are zero (the other players bid the same).

Claim 2.13. *For every item z, $p'_z \leq p_z$*

Proof. Stems from minimality (every envy free allocation in G is envy free in G'). □

Claim 2.14. *i gets y in G'.*

Proof. If i gets something which is not x and not y, he is envious in the player who gets y for $p'_y < p_y$ (if there is equality i paid her valuation in G).

If i gets x, note that $p_x - p'_x \geq 0$. If $p_x - p'_x > 0$, then let

$$A = \{y \in K : q_y(\tilde{t}_i) < q_y(t_i)\}.$$

As $b \in A$, this gets to a contradiction.

If $p_x - p'_x = 0$, then there is a red edge between b and a. In this case, the player before i in the DAG (which we denote by j) got b, and we have $p'_b = p_b$. Iterating this argument for j and b, we get that a player which paid zero before now pays her valuation (which contradicts independence), or we are stuck in a green edge (and can't continue). □

Note that as a menu exists, $p'_y = p_y$. Moreover, as i is not envious in G', $p'_x > \tilde{p}_x$.

The following claim is a simple corollary of the way we run each stage in the mechanism

Claim 2.15. *If a player increases all her valuations by a constant, and she doesn't get to the budget limit, then she gets the same set of items for the same price*

We know that $v_x - \tilde{p}_x > v_y - \tilde{p}_y = v_y - p_y$. Thus, the constant $\delta = v_x - v_y - p_x + p_y > 0$. Let $c > 1$ be any constant such that $\delta/c < p_y$ Now consider a game G'', in which i bids $v''_x = v_x - v_y + p_y - \delta/c$, $v''_y = v_y - v_y + p_y - \delta/c = p_y - \delta/c$.

Since G'' and G' are off by a constant, the mechanism is run in the same way, and i should get y for p_y. But this is more than he values the item, and is thus impossible.

Lemma 2.16. *Player i gets the best deal.*

Proof. Suppose player i prefers b but gets a.

Claims: (i) If i drops all values to 0 but $v_i(a)$ and $v_i(b)$ still gets a.

Proof - suppose i gets b. the price of b cannot increase because of minimal competitive prices.

Next we show it cannot be that the price of b didn't change and you got b. Suppose in contradiction that this is the case. This means that i had an edge to b before and the price of a didn't drop otherwise in the new instance i envies the player that got a. Therefore the player, say j, that had a red edge to i in the original graph got a (there is one player since the graph is a dag). But this means that the player that pointed to j get what j got and so on. We will end in a player that paid zero (since it is a dag). But in the new graph since everything is shifted this player that paid zero now pays something positive which is exactly his value (since all edges were red). But this means that some player points to this player in the new graph implying a contradiction to the independence assumption. (one cannot pay his value and budget because of the independence assumptions - make it a separate lemma).

Next we show it cannot be that the price of b didn't drop and you got b. Let A be the set of items in which their prices dropped..... get a contradiction by showing that A can grow (it can't be that A is everything).

(ii) After dropping if you add a constant to both you get the same item. (the proof is similar to the previous one).

(iii) in $(v(a)', v(b)') = (v(a) - v(b) + p_b + \frac{\epsilon}{2}, \infty)$ and budget ∞ i gets b and pays p_b because of the menu. If i drops b to $p_b + \frac{\epsilon}{2}$. Now add both values in to $v(a)$ and $v(b)$ by the same constant. Thus you get b (contradicting that you ever got a).

$v(a) - v(b) + p_b + \frac{\epsilon}{2}$ can be negative but it is not good for us....

□

Theorem 2.17. *If all players but i are restricted to be consistent then it is a dominant strategy for i to be truthful (even allowing him non consistent strategies).*

Proof. All players but i are not envy. Raise the value of the i for the slot he got to ∞. Thus i is also not envy. .... □

3 Uniqueness

Let G denote our mechanism, and let M denote another truthful envy free mechanism (assuming such mechanism exists). Assume that if the players bid b_i, v_i then the outcome (or prices) are different between M, and G. We let $p_1, \dots p_k$ denote the prices in G, and $q_1, \dots q_k$ denote the prices in M. As G is point wise minimal, we have that for every item x, $p_x \leq q_x$.

Lemma 3.1. *The same set of players paid zero in G,M.*

Proof. Assume the converse. Let i_1 be a player which paid zero in G, but received some item x_1 and paid $q_{x_1} > 0$ for it in M. Let i_2 be the player which i_1 points to at the envy graph of G. As M as envy free, it must be that $p_{x_1} = q_{x_1}$, and i_2 received x_1 in G (there was a red edge from i_1 to i_2). If i_2 doesn't receive any item in M, then since he is not envious at i_1 it must be that $q_{x_1} = v_{i_2}(x_1)$ which is a contradiction since $p_{x_1} = q_{x_1}$. Otherwise, i_2 received another item x_2 in M. A similar argument shows that $p_{x_2} = q_{x_2}$, and that if i_3 received x_2 in G, then i_2 has a red edge to i_3. As the number of players is finite, some player i_j which received $x_{i_j - 1}$ in G will now get no item and pay nothing, which is a contradiction. □

Let x_1 be an item which maximizes $q_{x_1} - p_{x_1}$, and i_1 the player which received x_1 in G. We show that i_1 still gets x_1 in M:

Lemma 3.2. *i_1 gets x_1 in M.*

Proof. Denote $d = q_{x_1} - p_{x_1}$ Assume that i_1 doesn't get x_1. According to Lemma 3.1, i_1 gets an item x_2 in M. As i is not envious in M, we have $q_{x_2} - p_{x_2} = d$, and there is a red edge from x_1 to x_2 in the envy graph of G. Let i_j denote the player which got x_j in G. An inductive argument shows that i_j must get an item in M, and that if that item is x_{j+1}, then $q_{x_j+1} - p_{x_j+1} = d$, and x_j points at $x_j + 1$ in the envy graph of G. Since the set of players is finite, one of the players which received an item in G will not receive an item in M, which is a contradiction. □

We now define a new input to our mechanism and to M, and consider the behavior of both mechanisms on the new input. Let x be the item which maximized $q_x - p_x$, and i be the player which received x in G, M, we define

$$\forall j \neq i,\ \tilde{v}_j(y) = v_j(y), \tilde{b}_j = b_j$$

$$\tilde{v}_i(x) = \frac{p_x + q_x}{2},\ \forall y \neq x,\ \tilde{v}_i(y) = 0, \tilde{b}_i = b_i$$

that is, all the players bid the same, except player i which only wants x, and wants it for $(p_x + q_x)/2$. Denote the allocation and prices of our mechanism on the new bids by $\tilde{G}$, and the allocation and prices of the new mechanism by $\tilde{M}$.

Claim 3.3. *Player i gets x in $\tilde{M}$.*

Proof. Since every allocation in G is also envy free in $\tilde{G}$, for every item y, $\tilde{p}_y \leq p_y$. Thus by the envy free on $\tilde{G}$, i must get x in $\tilde{G}$. As our mechanism is truthful, $\tilde{p}_x = p_x > 0$, and i pays in $\tilde{G}$. According to Lemma 3.1, i also pays in $\tilde{M}$. However, as i is only interested in x, it must be that i gets x in $\tilde{M}$. □

Claim 3.3 is a contradiction. If i pays strictly less than q_x, then it is better for her to lie and report $\tilde{v}_i$ and not v_i. If she pays q_x or more, than she pays more than her value for the item, as $(q_x + p_x)/2 < q_x$.

4 Uniqueness

Lemma 4.1. *In $\tilde{M}$, the set of players that pay a positive price are the same players as in M.*

Proof. Suppose this is not the case. Start from some player p that doesn't get an item in M and gets in $\tilde{M}$. Therefore p got an item a slot that he pointed at in our graph and so on. It will cause that someone will point to an empty slot which means his price is v_i contradicting the independence assumption. Note that by minimality all prices didn't drop. From this player there is an edge to a some another player (their items). A cycle will close... □

Lemma 4.2. *Suppose an item s sold for $\tilde{Q}_s > Q_s$ in $\tilde{M}$. Playerp gets it in $\tilde{M}$ (same that was in M).*

Proof. Let s be the item that increased the most. □

Example 4.3. *Bad Example (not every slot is sold, or 5 slots and 5 players and every one pays).*
$b_2 = 10$, $b_4 = 11$, $b_i = \infty$ *for others.*
All values but the following are zero:
$v_1^A = 30$, $v_1^C = 29$, $v_2^A = \infty$, $v_2^B = 20$, $v_3^B = 13$, $v_2^C = 14$, $v_3^D = 12$,, $v_4^D = 20$, $v_4^D = \infty$, $v_5^A = 20$, $v_5^E = 21$.
The prices at the end will be $p_A = 10$, $p_B = 8$, $p_C = 9$, $p_D = 7$, $p_E = 11$.
Another bad example: If only players $5, 4, 2$ *exists and the only slots are* A *and* E *then only one of the slots will be sold.*

References