Position Auctions with Budgets: Existence and Uniqueness

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Abstract

We design a Generalized Position Auction for players with private values and private budget constraints. Our Mechanism is a careful modification of the Generalized English Auction of Edelman, Ostrovsky and Schwarz (2007). By enabling multiple price trajectories that ascent concurrently we are able to retrieve all the desired propertied of the Generalized English Auction, that was not originally designed for players with budgets. In particular, the unique ex-post equilibrium outcome of our auction is Pareto-efficient and envy-free. Moreover, we show that any other position auction that satisfies these properties and does not make positive transfers must obtain in ex-post equilibrium the same outcome of our mechanism, for every tuple of distinct types. This uniqueness result holds even if the players' values are fixed and known to the seller, and only the budgets are private.

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1 Introduction

Online advertisements via auction mechanisms are by now a major source of income for many Internet companies. Whenever an Internet user performs a search on Google, an automatic "position auction" is being conducted among several different potential advertisers, and Google places the winning ads next to the search results it outputs. Google and Yahoo! generate a revenue of several cents per each such auction, and these numbers add up to billions of dollars every year. The importance of correctly designing and analyzing the different economic properties of these auctions is clear.

Indeed, in this electronic setting, the interplay between theoretical models and practical implementations is rich. Many actual auction implementations were based on early theoretical insights, and the actual auction formats that have evolved over time motivated deep theoretical studies. Two examples are the papers of Varian (2007) and of Edelman et al. (2007), that analyze Google's "generalized second price" (GSP) auction, and show that it has many attractive properties. In particular, the GSP auction obtains in equilibrium an efficient (welfare-maximizing) allocation, with envy-free prices. Moreover, Edelman et al. (2007) extend these results to the incomplete-information setting via an elegant generalization of the English auction. Several other variants of position-auctions models have been studied, see e.g. Athey and Ellison (2008) and Kuminov and Tennenholtz (2009) and the references therein.

Almost all the works on position auctions completely ignore the issue of budgets, and focus on the bidder's value from winning one of the slots. In sharp contrast, all actual position auctions allow bidders to specify both a value and a budget, and the latter parameter serves an important role in the strategic considerations of the bidders. In fact, budgets are a weak point of the more general auction theory as well, with relatively few works that study the subject. The several works that do study the effect of budgets indicate that, because the existence of budgets changes the quasi-linear nature of utilities, properly inserting budgets into the model usually results in significant modifications to the theory, both technically and conceptually. Therefore the importance of studying position auctions with budgets is two-fold: to align realistic auction systems with theoretical auction models, and to enrich the theoretical understanding of the effects of budgets on auction design.

In this paper we design a position auction for players with budgets in an incomplete information setting, where both the bidders' values and budgets are private information. As could be expected, we observe that previous analysis, and in particular the analysis of the generalized English auction, fails when budget limits exist. Our main result is positive: we design a "generalized position auction" that retrieves all the nice properties of the generalized English auction, while taking budgets into account. In particular, our auction has a unique ex-post equilibrium,[1] and this equilibrium outcome is envy-free and Pareto-efficient. Moreover, we show that any other mechanism that al-

[1]For this we need to assume that budgets are distinct, see a discussion in the body of the paper.

ways obtains envy-freeness and Pareto-efficiency in ex-post equilibrium must choose the same slot assignment and the same payments as our mechanism, at least whenever all true budgets are distinct. This uniqueness result holds even if players' values are fixed and known, and the only private information of the players is their budgets. This last property is especially interesting given the argument of Edelman et al. (2007), that a complete-information assumption regarding the players' values is reasonable. Our uniqueness result shows that, in our context, such a relaxation will not make the problem easier.

A very recent paper by Aggarwal, Muthukrishnan, Pal and Pal (2009) describes a different mechanism for essentially the same setting as ours. Their mechanism extends the Hungarian method, and outputs a bidder-optimal matching for players with budgets. While our uniqueness result implies that both mechanisms always output the same outcome, the formulation of both auctions is completely different, where our formulation has its origins in the generalized English auction.[2]

To highlight the main properties of our auction, recall the basic setting of position auctions: there are k slots and n players, and player i obtains a value of $\alpha_l v_i$ from receiving slot l, where the constants $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ are given as an input to the mechanism (they are public knowledge), and each player is interested in at most one slot. In the generalized English auction, a single price gradually ascends, and players need to decide when to drop. Rename the players such that player 1 is the last to remain, player 2 is the second to last to remain, and so on. When the l'th player drops, she is allocated slot l for a payment that is equal to the price-point at which the $l+1$ player dropped. Thus, when l players remain, each one sees a fixed price for slot l and a gradually increasing price for the better slots, and should decide whether to drop and take slot l, or to remain and receive a better slot. The key observation in the analysis of Edelman et al. (2007) is that, if each player plans to drop at the price that makes her indifferent between slot l and slot $l-1$, the winner of slot l will not regret in retrospect the fact that she did not win a higher slot. This is immediate regarding slot $l-1$, but more subtle regarding the slots that are better than $l-1$, and follows from the fact that the first to drop among the remaining l players is the one with the lowest value.

With budgets, however, this key observation fails. A player that becomes indifferent between slot l and slot $l-1$ because she has the lowest value may later be able to offer a higher price than her competitors for the slots better than $l-1$, if the competitors are limited by a low budget. For this reason, a single price trajectory fails to reach an equilibrium. The other extreme, of performing k completely separate auctions sequentially, will also not yield an ex-post equilibrium since, intuitively, the competition on slot k depends on the identity of the winners of better slots, and

[2] Our work was done independently of the paper of Aggarwal et al. (2009). On top of the uniqueness result which is new, our analysis is completely different, and enables to improve some problematic technical aspects of the mechanism of Aggarwal et al. (2009). Most importantly, their mechanism requires players' types in a "general position", that is somewhat restrictive, while our formulation enables an analysis that encapsulates all types with distinct budgets.

vice-versa. Our solution is a hybrid between these two extremes. We maintain k price trajectories, one for each slot, that ascend in a carefully-designed concurrent way. Enabling low-valued players with high budgets to "join the race" at the later stages is the main high-level conclusion that stems from our technical analysis.

In addition to the straight-forward importance of a positive existence result, which illuminates some of the effects of budgets on auction design, in a more general context our analysis contributes another layer to the currently small literature on auctions with budgets. In particular, we wish to point out two conceptual aspects of the positive result: First, it is "detail-free"/"robust" (Wilson, 1987; Bergemann and Morris, 2005), while most previous works on auctions with budgets assume a Bayesian setting. Second, it should be contrasted with the recent interesting impossibility of Dobzinski, Lavi and Nisan (2008). They show, quite surprisingly, that there do not exist dominant-strategy incentive-compatible and Pareto-efficient multi-item auctions, even in the very restrictive setting of two identical items and two players with additive private valuations (and a private budget constraint). The existence result for position auctions demonstrates the importance of their assumption that players wish to receive *multiple* items. With unit-demand, a possibility (though unique) still exists, as demonstrated here.

2 Preliminaries

Basic model of position auctions. In a position auction there is a set $K = \{1, ..., k\}$ of items ("slots") and a set $N = \{1, ..., n\}$ of bidders, where each bidder is interested in receiving one of the slots. Each slot $l \in K$ is characterized by a known constant $\alpha_l > 0$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$, which is an input to the mechanism. Each bidder i obtains a monetary value of $\alpha_l v_i$ from receiving slot l, where v_i is a parameter that is privately known only to player i. We assume without loss of generality that $k \leq n$, since otherwise we can just ignore the $k - n$ lowest slots.

This model was studied in recent years (see e.g. Varian (2007) and Edelman et al. (2007)) in order to analyze the ad auctions that are conducted by search engines like e.g. Yahoo! and Google. In a nut-shell, search engines place paid online advertisements in proximity to search results that they output. Advertisers bid for the online placement of their advertisements, and the k winning bidders are positioned on the web-page according to their bids. The value v_i represents bidder i's expected profit given that the Internet user will click on her ad, and the constant α_l (the "click-through rate") represents the probability that the Internet user will indeed click on the ad, given that the ad is positioned at slot l. Slot 1 is the "best" position, i.e. has the highest click-through rate, slot 2 is the second-best, and so on and so forth. [3]

[3] We follow the exact model of Varian (2007) and of Edelman et al. (2007), in which the click-through rate depends only on the position of the slot, and not on other factors like the quality of the different ads. Few recent works have begun to study more complex click-through-rate models, see for example the work of Kuminov and Tennenholtz (2009) and the references therein.

Budget constraints and valid outcomes. All previous works on position auctions have assumed quasi-linear utilities, i.e. bidder i's utility from receiving slot l and paying P_i is equal to $\alpha_l v_i - P_i$. In this paper we analyze the effect of adding a hard budget limit that caps the maximal payment ability of a player. More precisely, each player i has a privately-known budget b_i, and cannot pay any price $P \geq b_i$. Thus the resulting utility of a player with type (v_i, b_i) that receives slot l and pays P is

$$u((v_i, b_i), l, P) = \begin{cases} \alpha_l \cdot v_i - P & P < b_i \\ \\ 0 & P \geq b_i \end{cases}$$

where the zero-utility for the case that $P \geq b_i$ captures the fact that if a player has to pay such P, she will default and will not complete the transaction. (our results continue to hold if this zero utility is replaced with any other negative utility). Note that the feasibility regime is any $P < b_i$, where the inequality is strict. This is more convenient due to some technical reasons that will be explained below. [4]

To summarize, we define a *valid outcome* of a position auction as a tuple $(s_i, p_i)_{i \in N}$, where every bidder i receives the slot $s_i \in K \cup \{k+1\}$ ($k+1$ is a dummy slot with $\alpha_{k+1} = 0$) and pays p_i. A valid outcome must additionally satisfy:

1. *(feasibility)* $s_i, s_j \in K, i \neq j$, implies $s_i \neq s_j$.
2. *(budget limit)* $p_i < b_i$.
3. *(ex-post Individual Rationality (IR))* $p_i \leq \alpha_{s_i} v_i$.

Desired solution properties. Since there are many possible valid outcomes, one may wish to focus attention on those outcomes that are "efficient" and "fair", as captured by the following two classic properties:

1. *(Pareto-efficiency)* A valid outcome $o = (s_i, p_i)_{i \in N}$ is Pareto efficient if there is no other valid outcome $o' = (s'_i, p'_i)_{i \in N}$ such that $\alpha_{s'_i} v_i - p'_i \geq \alpha_{s_i} v_i - p_i$ (players weakly prefer o' to o) and $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ (the seller weakly prefers o' to o), with at least one strict inequality.
2. *(Envy-freeness)* A valid outcome $(s_i, p_i)_{i \in N}$ is envy-free if for every two distinct players $i, j \in N$ such that $p_j < b_i$, $\alpha_{s_i} v_i - p_i \geq \alpha_{s_j} v_i - p_j$. [5]

[4] When budgets are real numbers then either the set of all infeasible payments includes its infimum or the set of all feasible payments includes its supremum, and this choice does not seem to have any conceptual meaning.

[5] One may also consider to weaken the definition of envy-freeness to a stability condition, such that for each pair of players, at least one of them would not like to exchange slots and prices with the other. This turns out to be weaker than Pareto-efficiency: one can show that any Pareto-efficient outcome satisfies this stability condition, but there may be stable outcomes that are not Pareto-efficient. We additionally note that envy-freeness in position auctions is related to stable matchings in a two-sided graph, as discussed by Edelman et al. (2007).

The generalized English auction of Edelman et al. (2007) is envy-free, and we will show that this strong fairness property can still be achieved even when budgets are considered. Pareto-efficiency is strictly weaker than envy-freeness:

Proposition 1. *Every envy-free outcome in which all slots are allocated is Pareto-efficient.*

Proof. Assume by contradiction that $o = (s_i, p_i)_{i \in N}$ is envy-free but not Pareto-efficient, and let $o' = (s'_i, p'_i)_{i \in N}$ be a valid outcome that Pareto improves o. Without loss of generality $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$, since if it is an equality then there exists a player i with a strict inequality $s'_i v_i - p'_i > s_i v_i - p_i$ and we can slightly increase p'_i to get a Pareto improving outcome in which the seller's payoff is strictly larger than her payoff in o.

For any slot l, let q_l, q'_l be the payment of the player who receives slot l is o, o' respectively. Let j be some slot such that $q'_j > q_j$, and suppose player P_j received slot j in o'. Since player P_j's utility in o' is not smaller than her utility in o it follows that P_j received a different slot in o, say $s_j = l \neq j$. We get $\alpha_l v_j - q_l \leq \alpha_j v_j - q'_j < \alpha_j v_j - q_j$. Thus player P_j envies the player who got slot j in o, contradicting the fact that o is envy-free. □

The opposite to this proposition is not true, and there exist valid outcomes that are Pareto-efficient but not envy-free. For example, the outcome that maximizes the social welfare (sum of players' values for the slots they receive) and charges no payments from the bidders is Pareto-efficient, but is not envy-free.

Since the type (v_i, b_i) of player i is private information, known only to the player herself, we study the design of *mechanisms* that output (in equilibrium) a valid outcome which is Pareto-efficient and envy-free. We focus on so called "detail-free" solutions concepts, and use the equilibrium notions of ex-post Nash, and dominant strategies. [6] We sometimes refer to a direct mechanism that is incentive compatible in dominant strategies as a "truthful mechanism" (as is common in the CS literature).

Assuming distinct budgets. It turns out that, in some subtle sense, it is impossible to construct truthful and Pareto-efficient auctions, even for the case of a single item. It is long known (Che and Gale (1998); see also Krishna (2002)) that the following single-item mechanism is truthful when the true budgets are distinct: the winner is the player i with the maximal "bid" $\min(b_i, v_i)$, and she pays the second largest bid. It is not hard to verify that it is a dominant-strategy of the players to declare their true value and budget, and that the outcome is Pareto-efficient and envy-free. Moreover, Dobzinski et al. (2008) have recently shown (in a more general context) that this mechanism is the unique truthful and Pareto-efficient mechanism, at least when all budgets are distinct.

[6] Our direct mechanism also exhibits and an ex-post equilibrium, and not dominant strategies, due to technical reasons we explain in the sequel.

To demonstrate that the assumption of distinct budgets is crucial, suppose two players with budgets $b_1 = b_2 = 1$ and values $v_1 = v_2 = 3$. The $\min(v, b)$ mechanism chooses w.l.o.g. player 1 as the winner, and she pays a price of 1, for a resulting utility $v_1 - P_1 = 2$. If player 2 is able to pay exactly her budget that she can gain from declaring a false budget $b' > b_2$: she becomes the winner, and pays a price of 1, for a resulting utility $v_2 - P_2 = 2 > 0$. If a player is not able to pay her exact budget parameter b_i, but only any strictly lower payment, then the outcome is infeasible and player 1 will prefer to lose and pay 0 over paying the infeasible price of 1. Either way, this mechanism is not truthful when budgets are identical.

We explicitly spell out the assumption of distinct budgets, which was implicit in the previous works (most probably since the event of having non-distinct budgets has zero probability). This assumption leads to the technical requirement that players can only pay any price which is strictly less than their budgets. In the zero-probability event that there exist two players with equal budgets, the auction may be canceled in order to avoid infeasible outcomes.

Alternatively, one can assume a discrete type space (in other words, assuming that all parameters are integers). With discrete types, the mechanism can avoid the infeasibility of the outcome when budgets are identical by artificially increasing the budget of each player i by an arbitrarily small $\epsilon_i < 1$ with $\epsilon_i \neq \epsilon_j$ for any two players i, j. This pre-processing step will make the mechanism truthful and Pareto-efficient even if budgets are identical (envy-freeness will still be violated with identical budgets, though). One can verify that all our results continue to hold under these modifications, we do not repeat all proofs for the discrete setting to keep the exposition concise.

3 The Generalized Position Auction

3.1 The effect of budgets on the Generalized English Auction

It is constructive to start with a short discussion on the generalized English auction of Edelman et al. (2007). This auction gradually increases a price parameter Q, and players decide whether to drop or stay. Rename the players according to the reverse order at which they dropped (player 1 never dropped, player 2 dropped last, etc.). When player $l \leq k$ drops she wins slot l and pays the price at which player $l + 1$ dropped. This continous-time description is made discrete and formal by the following definition:

Definition 2 (The Generalized English Auction (Edelman et al., 2007))**.** *Initialize* $Q = 0$ *(current price) and* $l = \min(k + 1, n)$ *(current slot). Then perform:*

1. *Each player* i *declares a bid* p_i^l *(this is the price at which player* i *plans to drop).*
2. *The* l*'th highest bidder wins slot* l *and pays* Q *(recall that slot* $k + 1$ *is a dummy slot with* $\alpha_{k+1} = 0$*).*

3. *If $l = 1$ then terminate. Otherwise raise Q to be the l'th bid, decrease l by one, and repeat from step 1.*

Informally, when the price increases and $l \leq k$ active bidders remain, each bidder i faces two alternatives: to drop and win slot l for a price that is already fixed and known (this is the price at which the $(l+1)$'th bidder dropped), or to stay in the auction. This decision represents a trade-off between winning slot l or winning one of the better slots $1, ...l-1$ (for a higher price). In the formal definition the price does not increase continuously but the same tradeoff has to be made when the player chooses her new bid at step 1. The equilibrium strategies are derived by looking closely at this tradeoff. Assuming infinite budgets, the price P at which player i becomes indifferent between winning slot l for a price Q and winning slot $l-1$ for a price P should satisfy $\alpha_l v_i - Q = \alpha_{l-1} v_i - P$, or alternatively $P = (\alpha_{l-1} - \alpha_l) v_i + Q$. If the player bids this P in step 1 and ends up winning slot l, she is guaranteed not to regret the fact that she did not win slot $l-1$. The twist in the analysis of Edelman et al. (2007) is to show that this bidding strategy ensures that the player will not regret winning any better slot, not just slot $l-1$. In other words, this bidding strategy forms an ex-post Nash equilibrium. A simple way to observe that these strategies are indeed an ex-post equilibrium is to note that they lead to the VCG outcome, which is well-known to be incentive compatible.

With budgets, however, the picture changes and this auction no longer admits an ex-post equilibrium. The main difficulty arises from the fact that a player that prefers slot l over slot $l-1$ may still prefer slots that are better than $l-1$. To demonstrate this, consider the following example, with three players and two slots, and parameters $\alpha_1 = 1.1, \alpha_2 = 1, v_1 = 20, b_1 = 7.5, v_2 = 10, b_2 = 7.6, v_3 = 7, b_3 = 100$. With the generalized English auction, when the price reaches 7, player 3 faces a dilemma: if she will not drop, she might end up winning slot 2 for a price higher than her value for that slot (if players 1 and 2 will have infinite budgets, a piece of information she does have at the time of the decision). If she drops, she will realize in retrospect that she could have won slot 1 for a profitable price of 7.6 (while her value for slot 1 is 7.7), since players 1 and 2 turn out to be limited by their budgets, and hence cannot continue to compete with player 3 on slot 1 after the price reaches 7.6. Thus, the introduction of budgets enables the possibility that players who drop when the current slot is l might want to join again for some slot $l' < l$. Of-course, simply allowing players to re-join will cause more problems, since this implies changing the entire price hierarchy that was formed.

3.2 The Generalized Position Auction

In order to solve these difficulties, the Generalized Position Auction uses k price trajectories, one for each slot, that ascend concurrently as follows. Players first compete for the k'th slot, and each player decides when to suspend her participation in this slot's auction. The price ascent temporarily stops when exactly k players remain active. Let this price point be Q_k^1. The price ascent for slot $k-1$ starts from Q_k^1, all players (even those that suspended participation at the previous slot) may participate in the auction for slot $k-1$. Players again decide when to temporarily suspend participation, and when exactly $k-1$ players remain active the price ascent temporarily stops, and we move to slot $k-2$. This continues in a similar manner until we reach slot 1. In slot 1, the price ascent stops when exactly one player remains active. This player wins slot 1 and pays the last price that was reached (as in the English auction). At this point the auction of slot k resumes. There are now $k-1$ slots left, and so the auction continues until there remain $k-1$ active players, at this point the price ascent stops again, and the auction for slot $k-1$ resumes. This continues until the winner of slot 2 is determined. The auction of slot k is once again resumed, and this process continues in a similar manner until all slots are sold. As before, one is able to describe this process more formally via the following discrete-time mechanism.

Definition 3 (The Generalized Position Auction (indirect version))**.** *Initialize $t = 1$ (first round), $l = k$ (current slot is k), and $N_t = N$ (set of active players). Then perform:*

1. *Each player $i \in N_t$ declares a bid $p^t_{i,l}$. (this is the price at which player i will suspend participation at the auction for slot l at the current iteration t).*
2. *Let Q^t_l be the $(l+1)-(t-1)$ highest bid. (this is the price of slot l at round t – the point where the price ascent stops since the number of active players is equal to the number of remaining slots)*
3. *If $l > t$ then decrease l by 1 and repeat from step 1. Otherwise $l = t$ and,*
    - *The player i with the highest bid $p^t_{i,t}$ wins slot t and pays $P_t = Q^t_t$. (section 3.3 below describes the allowable tie-breaking rules).*
    - *If $t = k$ then terminate. Otherwise increase t by one, update N_t by removing the new winner, set $l = k$, and repeat from step 1.*

Suppose that player i had a bid $p^t_{i,l+1} > Q^t_{l+1}$ for slot $l+1 \leq k$, and is now required to choose her bid $p^t_{i,l}$ for slot l. If she were to assume that the alternative for her is to win slot $l+1$ for a price Q^t_{l+1} then her maximal willingness to pay for slot l, as explained in subsection 3.1, is $P = (\alpha_l - \alpha_{l+1})v_i + Q^t_{l+1}$. Since she cannot exceed her budget b_i, this myopic reasoning will therefore direct her to bid $\min(b_i, (\alpha_{l-1} - \alpha_l)v_i + Q^t_l)$. If player i had a bid $p^t_{i,l+1} \leq Q^t_{l+1}$ for slot $l+1$ she could simply increase her willingness to pay for slot l by the added value of slot l (compared to slot $l+1$), i.e. by $(\alpha_l - \alpha_{l+1})v_i$. This leads us to define:

Definition 4 (Myopic bidding in the Generalized Position Auction)**.** *The "myopic bidding strategy" is defined by:*

$$p^t_{i,l} = \begin{cases} \min(bi, (\alpha_l - \alpha_{l+1})v_i + \min(Q^t_{l+1}, p^t_{i,l+1})) & l < k \\ \min(b_i, \alpha_k v_i) & l = k \end{cases}$$

for any round t and any slot l.

Consider again the example given in section 3.1, with three players and two slots, and parameters $\alpha_1 = 1.1, \alpha_2 = 1, v_1 = 20, b_1 = 7.5, v_2 = 10, b_2 = 7.6, v_3 = 7, b_3 = 100$. In the Generalized Position Auction, when players are bidding myopically, the bids are as follows. In the first round, the slot-2-bids are $p^1_{1,2} = 7.5, p^1_{2,2} = 7.6, p^1_{3,2} = 7$. Therefore we get a cutoff price $Q^1_2 = 7$, and the slot-1-bids are $p^1_{1,1} = 7.5, p^1_{2,1} = 7.6, p^1_{3,1} = 7.7$. Hence player 3 wins slot 1 and pays 7.6. Players 1 and 2 continue to the second round, and the slot-2-bids remain as before $p^2_{1,2} = 7.5, p^2_{2,2} = 7.6$. Thus player 2 wins slot 2 and pays 7.5. One can easily verify that this is a valid outcome that is Pareto-efficient and envy-free. (recall that a player can pay only strictly less than her budget).

However, myopic bidding is not an ex-post equilibrium. To see this, consider again the example from the previous paragraph. If players 1 and 2 bid myopically then player 3 can decrease her price for slot 1 by bidding $p^1_{3,2} = 0$ and $p^1_{3,1} = 7.7$. The cutoff price for slot 2 will then be zero, and because of that the bids of players 1 and 2 for slot 1 will decrease to be 2 and 1, respectively. This problem is solved by forcing consistency, using a direct version of the above auction:

Definition 5 (The Generalized Position Auction (direct version))**.**

1. *Each player i reports a type (v_i, b_i). If two players report the same budget then the auction is canceled (slots are not allocated and no price is charged).*

2. *We simulate the indirect version of the Generalized Position Auction where each player i follows myopic bidding according to her declared type (v_i, b_i).*

The main results of this paper are summarized by the following theorem:

Theorem 6. *Assuming that all true budgets are distinct,*

1. *(ex-post equilibrium) For every player i, if all other players are truthful then it is a best response for player i to be truthful as well.*

2. *(desired properties hold) If all players are truthful then the Generalized Position Auction results in a valid outcome which is Pareto-efficient and envy-free.*

3. *(uniqueness) Fix any other mechanism that always results (in ex-post equilibrium) in a valid outcome which is Pareto-efficient and envy-free, and that never makes positive transfers. Then this mechanism must output the same outcome (slot assignments and payments) as our Generalized Position Auction for any tuple of types with distinct values and distinct budgets.*

Few remarks are in place. First, we note that the third result also implies that truthfulness is the *unique* ex-post equilibrium of the Generalized Position Auction. Second, it is interesting to note that when all budgets are sufficiently large the outcome of our auction is the same as the outcome of the generalized English auction, which in turn is equivalent to the outcome of VCG.

In many "real-life" settings one may prefer an indirect mechanism over a direct one. It is interesting to note that, while indeed most of the auctions being conducted in real settings are indirect, the electronic position auctions of Google and Yahoo! are actually direct mechanisms, where the advertisers are required to simply bid a value and a budget. However, if one insists on an indirect mechanism, a simple modification will "fix" our indirect version by forcing consistency – the indirect version should explicitly rule out a player that plays in an inconsistent way (i.e. the bids in the different phases contradict myopic bidding according to any fixed type). This does not require players to reveal their true type, and in this sense has the advantage of an indirect mechanism. A simple "revelation principle" argument shows that, when such a consistency enforcing rule is added to the indirect mechanism, myopic bidding according to the true type is an ex-post equilibrium.

As a last remark, one may notice that our theorem does not obey the usual rule of thumb in robust (detail-free) analysis for private values, that direct mechanisms exhibit dominant strategies, and the solution concept of ex-post equilibrium is used for indirect mechanisms. This is not the case here, although our mechanism is direct, because of our modeling of the budget constraint. Declaring the true budget is not a dominant strategy for player i since if another player j misreports and declares b_i instead of b_j, the auction will be canceled if player i reports truthfully. As remarked in section 2, this artifact of our definitions can be avoided if we assume a discrete type space. In that case the auction will never be canceled, and truthfulness will become a dominant-strategy of the direct mechanism.

3.3 Tie-breaking

The issue of tie-breaking requires some attention. In general, when there are several highest bids in step 3 of the generalized position auction, either all highest bidders have the same value, or at most one of them has a higher value, but her bid is cut at her budget. For example, suppose one item and two players that declare $(v_1, b_1) = (7, 10)$ and $(v_2, b_2) = (8, 7)$. Then at price 7 there will be a tie. Since player 2 cannot pay her budget, we must choose player 1 as the winner.

More formally, we prove the following intuitive property: if player i has larger value than player j, but her bid at some slot is smaller than j's bid, then it must be the case that i's bid was cut at her budget. This property will be extensively used throughout the analysis.

Claim 7. *Fix any round t, slot l, and $i, j \in N_t$. If $v_i \geq v_j$ and $p^t_{i,l} \leq p^t_{j,l}$, with at least one strict inequality, then $p^t_{i,l} = b_i$.*

Proof. We prove the claim by induction. For slot k the proof is immediate from the definition. Therefore we assume correctness for slot $l+1$ and prove for slot l. Assume by contradiction that $p^t_{i,l} \neq b_i$. If $p^t_{i,l+1} \geq Q^t_{l+1}$ then $p^t_{i,l} = (\alpha_l - \alpha_{l+1})v_i + Q^t_{l+1}) \geq (\alpha_l - \alpha_{l+1})v_j + Q^t_{l+1}) \geq p^t_{j,l}$, which is a contradiction since by assumption either $v_i > v_j$ or $p^t_{i,l} < p^t_{j,l}$. Otherwise $p^t_{i,l+1} < Q^t_{l+1}$. If $p^t_{i,l+1} = b_i$ we get a contradiction since $b_i \geq p^t_{i,l} \geq p^t_{i,l+1} = b_i$. Therefore by the induction assumption we must have $p^t_{j,l+1} \leq p^t_{i,l+1}$, and this inequality is strict if $v_i = v_j$. Thus $p^t_{i,l} = p^t_{i,l+1} + (\alpha_l - \alpha_{l+1})v_i > p^t_{j,l+1} + (\alpha_l - \alpha_{l+1})v_j \geq p^t_{j,l}$, a contradiction. □

Corollary 8. *Fix any round t, slot l, and $i, j \in N_t$. If $p^t_{i,l} = p^t_{j,l}$ then either $p^t_{i,l} = b_i$, or $p^t_{j,l} = b_j$, or $v_i = v_j$.*

Thus, if there exist two or more highest bidders in step 3 of the generalized position auction, we choose the winner to be some highest bidder i such that $b_i \neq Q^t_t$. Note that there exists at most one highest bidder with $b_i = Q^t_t$ since budgets are distinct. The tie-breaking among all players with equal value may be arbitrary, but consistent throughout the auction. We denote the tie-breaking order over the players by $\succ$, i.e. for two players i, j, $i \succ j$ implies that in case of a tie i will be chosen. This tie-breaking rule ensures that a player will always pay strictly less than her budget, and thus the outcome is ex-post individually rational.

4 Analysis

We use few additional terms and notations throughout the analysis: B^t_j denote the set of $j - t + 1$ highest bidders at slot j and iteration t. Ties for inclusion in B^t_j are settled the same way as described above, and in particular for any $i \in B^t_j$ we have $Q^t_j < b_i$. A player $i \in B^t_l$ is "strong" at slot l and iteration t, otherwise the player is "weak". We call P_t the "price of slot t". We say that slot l is *better* than slot $\tilde{l}$ if $l < \tilde{l}$ (and slot $\tilde{l}$ is *worse* than slot l). We sometimes use

$$q^t_{i,l} = \min(p^t_{i,l}, Q^t_l).$$

This gives $p^t_{i,l} = \min(b_i, q^t_{i,l+1} + (\alpha_l - \alpha_{l+1})v_i)$ for every player, slot, and round, which will simplify notation. Note that $p^t_{i,l} \geq p^t_{j,l}$ implies $q^t_{i,l} \geq q^t_{j,l}$.

One important observation that follow in a straight-forward way from the definition of the mechanism is that the outcome of round t depends only on the set of remaining players N_t, because the bids $p^t_{i,k}$ are fixed and identical in all rounds t. Thus a new round is simply a recursive call to the same auction, with a new set of players and a new set of slots.

Several monotonicity properties of the bids, for any round t, player i, and slot l, will turn out useful (the proof follows by simple algebra, and is omitted):

1. $q^t_{i,l} \geq q^t_{i,l+1}$. This implies $Q^t_l \geq Q^t_{l+1}$.
2. $p^{t+1}_{i,l} \geq p^t_{i,l}$ and therefore also $q^{t+1}_{i,l} \geq q^t_{i,l}$.[7] This implies $Q^{t+1}_l \geq Q^t_l$.
3. If $i \notin B^1_l$ and $p^t_{i,l} = q^t_{i,l} = b_i$, then player i will not win any slot $s \leq l$. (this follows from the previous two properties).

[7] One can prove this by induction on the slot $l = k, ..., 1$. For slot k the claim is immediate, this now implies the claim for slot $k - 1$, and so on.

4.1 Basic bid dynamics

We start with two rather subtle observations regarding the way that bids change from slot to slot and from round to round. While basic, these properties are at the heart of the proof, and will be repeatedly used to yield the more complex properties.

The first property states that, if player i wins slot t in round t, then every player j that bids lower than i in some slot $l > t$ in round t (i.e. $p^t_{j,l} \leq p^t_{i,l}$) will have the same bid $p^t_{j,l} = p^{t+1}_{j,l}$ for slot l in the next iteration $t + 1$. As an immediate implication we get that, if the winner i of slot t is strong in slot $l > t$ and round t then $p^t_{j,l} = p^{t+1}_{j,l}$ for any player $j \in N_{t+1}$, and therefore also $Q^t_l = Q^{t+1}_l$. Alternatively put, if $p^t_{i,l} \geq Q^t_l$ then $Q^t_l = Q^{t+1}_l$.

Claim 9. *Let player i be the winner of slot t, and fix some slot $l > t$ and some player $j \in N_{t+1}$ such that $p^t_{j,l} \leq p^t_{i,l}$. Then $p^t_{j,l} = p^{t+1}_{j,l}$.*

Proof. The proof is by induction on the slots. For $l = k$ the proof is immediate since $p^t_{j,k} = p^{t+1}_{j,k}$ for any player j. We assume correctness for slot $l+1$ and prove for l. If $i \in B^t_{l+1}$ then by induction $p^t_{j,l+1} = p^{t+1}_{j,l+1}$ for any player $j \notin B^t_{l+1}$, which implies $B^{t+1}_{l+1} = B^t_{l+1} \setminus \{i\}$, hence also $Q^{t+1}_{l+1} = Q^t_{l+1}$. This implies $p^t_{j,l} = p^{t+1}_{j,l}$ for all players in N_{t+1}. Otherwise assume that $i \notin B^t_{l+1}$. This implies $p^t_{j,l} = \min(b_j, (\alpha_l - \alpha_{l+1})v_j + p^t_{j,l+1})$. If $p^t_{j,l+1} \leq p^t_{i,l+1}$ then by induction $p^t_{j,l+1} = p^{t+1}_{j,l+1}$, hence $p^t_{j,l} = \min(b_i, (\alpha_l - \alpha_{l+1})v_j + p^t_{j,l+1}) = \min(b_i, (\alpha_l - \alpha_{l+1})v_j + p^{t+1}_{j,l+1}) \geq p^{t+1}_{j,l} \geq p^t_{j,l}$, and the claim follows. Otherwise suppose $p^t_{j,l+1} > p^t_{i,l+1}$. This implies $v_i < v_j$: if $v_i \geq v_j$, claim 7 implies that $p^t_{i,l+1} = b_i$, which implies that player i cannot win slot t, a contradiction. Thus $v_i < v_j$, and claim 7 implies $p^t_{j,l} = b_j \geq p^{t+1}_{j,l} \geq p^t_{j,l}$. □

The same claim can be made in terms of the q-bids:

Claim 10. *Let player i be the winner of slot t, and fix some slot $l > t$ and some player $j \in N_{t+1}$ such that $q^t_{j,l} \leq q^t_{i,l}$. Then $q^t_{j,l} = q^{t+1}_{j,l}$.*

Proof. If $p^t_{i,l} \geq Q^t_l$ then claim 9 implies $Q^{t+1}_l = Q^t_l$ and hence $q^t_{j,l} = q^{t+1}_{j,l}$. If $p^t_{i,l} < Q^t_l$ then $p^t_{j,l} = q^t_{j,l}$ and $p^t_{i,l} = q^t_{i,l}$ hence claim 9 implies $q^t_{j,l} = p^t_{j,l} = p^{t+1}_{j,l} \geq q^{t+1}_{j,l} \geq q^t_{j,l}$, and the claim follows. □

The second property states that, in the first iteration, the weakest player i in slot s among all players j that win some slot $s_j < s$ must be strong at slot s_i. This implies, for example, that if all weak players in slot s remain weak in all better slots $l' < l$ (in the first iteration) then the set of winners of slots $1...l$ is exactly B^1_l.

Claim 11. *Fix some slot s. Let $W_s = \{\ j \notin B^1_s$ and j wins some slot $s' \leq s\ \}$, suppose that W_s is not empty, and fix some $i \in argmin_{j \in W} p^1_{j,s}$. Let $s_i \leq s$ be the slot that i wins. Then $i \in B^1_{s_i}$.*

Proof. Assume by contradiction that $i \notin B^1_{s_i}$. Then there must exist a player j that wins some slot $s_j < s_i$ and $p^1_{j,s_i} < p^1_{i,s_i}$, otherwise by claim 9 we have $p^1_{i,s_i} = p^{s_i}_{i,s_i}$ which by bid monotonicity implies $i \notin B^{s_i}_{s_i}$, contradicting the fact that i wins s_i. Since j wins $s_j < s_i$ and $j \notin B^1_{s_i}$ we have $p^1_{j,s_i} \neq b_j$. By claim 7 we get $v_i > v_j$. However since j wins $s_j < s$, the minimality of i's bid at s implies $p^1_{j,s} \geq p^1_{i,s}$. Since $v_i > v_j$ we get $p^1_{i,s} = b_i$. Since $i \notin B^1_s$ this contradicts the fact that i wins $s_i \leq s$. □

Corollary 12. *Fix some slot s. Suppose that for any player $i \notin B^1_s$ we have that either $i \notin B^1_j$ for all slots $j < s$ or that i does not win any slot $j < s$. Then the set of players that win slots $1, ..., s$ is B^1_s.*

4.2 Envy-Freeness

The first property we prove is envy-freeness. For notational simplicity, throughout the subsection we rename the players such that player i wins slot i, for $i = 1, ..., k$, and every player $i > k$ does not win any slot.

Claim 13. *Fix any player $s > 1$ such that $q^1_{s,1} < b_s$. Then for any slot l, $q^1_{1,l} \geq q^1_{s,l}$, and if $s \in B^1_l$ then $1 \in B^1_l$.*

Proof. We first show that if $p^1_{1,l} < p^1_{s,l}$ then $1 \in B^1_l$. Otherwise it must follow that $p^1_{1,l} = q^1_{1,l} \neq b_1$, and claim 7 implies $v_1 < v_s$. Since $p^1_{1,1} \geq p^1_{s,1}$ it follows that $p^1_{s,1} = b_s$, a contradiction. This implies $q^1_{1,l} \geq q^1_{s,l}$. This also implies that if $s \in B^1_l$ but $1 \notin B^1_l$ then $p^1_{1,l} = p^1_{s,l} = Q^1_l$. Since $p^1_{1,l} \neq b_1$ (as player 1 wins slot 1) and $p^1_{s,l} \neq b_s$ (as $q^1_{s,l} = q^1_{s,1} < b_s$) then $v_1 = v_s$, and by corollary 8 also $p^1_{1,1} = p^1_{s,1}$. But this contradicts the consistency of the tie-breaking rule, since in slot l the tie-breaking preferred player 1 over player s and in slot 1 the tie-breaking preferred player s over player 1. □

Claim 14. *Fix any player $s > 1$ such that $q^1_{s,1} < b_s$, and any slot l. If $s \in B^1_l$ then $l' \in B^1_l$ for any $l' \leq \min(s, l)$.*

Proof. We prove by induction on the number of slots k. For $k = 1$ the claim is empty. Assume correctness for any $k' < k$ slots and let us prove for k. We have $1 \in B^1_l$ by claim 13. Therefore $B^2_l = B^1_l \setminus \{1\}$. Claim 13 also implies $q^1_{1,2} \geq q^1_{s,2}$ which by using claim 10 implies $q^2_{s,2} = q^1_{s,2} \leq q^1_{s,1} < b_s$. Since $s \in B^2_l$ the induction assumption implies $l' \in B^2_l$ for any $1 < l' \leq \min(s, l)$, and the claim follows. □

Claim 15. *Fix any player $s > 1$ such that $q^1_{s,1} < b_s$. Then $B^1_s = \{1, ..., s\}$, and $s \notin B^1_l$ for any $l < s$. In particular, if $s > k$ then player s is always weak. In addition we have $P_s = Q^1_s = q^1_{s,s}$.*

Proof. If $s \in B^1_l$ for some $l < s$ then by claim 14 we have that $\{1, ..., l, s\} \subseteq B^1_l$ which contradicts the fact that $|B^1_l| = l$. If $s \notin B^1_s$ then by combining claims 13 and 9 we have $p^1_{s,s} = p^2_{s,s} = \cdots = p^s_{s,s}$, which implies that $s \notin B^s_s$, a contradiction. Thus using claim 14 again we have $B^1_s = \{1, ..., s\}$, and claim 9 then implies $P_s = Q^1_s = q^1_{s,s}$. □

Claim 16. *Fix any player i and any two slots l, s with $s < l \leq k+1$. Then $q^t_{i,s} \leq p^t_{i,s} \leq \min(b_i, q^t_{i,l} + (\alpha_s - \alpha_l)v_i)$ (where we define $q^t_{i,k+1} = \alpha_{k+1} = 0$). Furthermore, if $i \notin B^t_j$ for any $s \leq j < l$ then the two inequalities become equalities.*

Proof. We prove by induction on $s = k, ..., 1$. For $s = k$ the claim is by definition. Now fix $s < k$ and assume correctness for $s+1$ and any $l' > s+1$. We need to show correctness for s and any $l > s$. We have by definition $q^t_{i,s} \leq p^t_{i,s} = \min(b_i, (\alpha_s - \alpha_{s+1})v_i + q^t_{i,s+1})$. If $l = s+1$ we are done. Otherwise $l > s+1$ and we have by induction $q^t_{i,s+1}) \leq \min(b_i, q^t_{i,l} + (\alpha_{s+1} - \alpha_l)v_i)$. Combining the two equations, the first part of the claim follows. If $i \notin B^t_j$ for any $s \leq j < l$ then the first inequality is equality by definition, and the second inequality is equality by the induction assumption. Thus the second part of the claim follows as well. □

Claim 17. *For any slot $l = 1, ..., k$, $P_l = \max_{s>l} \min(b_s, (\alpha_l - \alpha_s)v_s + P_s)$, where we define $\alpha_s = 0$ and $P_s = 0$ for any player $s > k$.*

Proof. It is enough to prove the claim only for $l = 1$, since the price P_l for $l > 1$ is determined by a recursive auction for l slots and a set of players N_l, and in that auction slot l is the first slot. We will show that, for any player $s > 1$, $q^1_{s,1} = \min(b_s, (\alpha_1 - \alpha_s)v_s + P_s)$. Since $P_1 = \max_{s>1} q^1_{s,1}$, the claim will then immediately follow. If $q^1_{s,1} = b_s$ then by claim 16 we have $b_s = q^1_{s,1} \leq p^1_{s,1} \leq \min(b_s, q^1_{s,s} + (\alpha_1 - \alpha_s)v_s) \leq \min(b_s, (\alpha_l - \alpha_s)v_s + P_s) \leq b_s$, implying the claim. Otherwise $q^1_{s,1} < b_s$ and by claim 15 we have that $s \notin B^1_l$ for any $l < s$. By claims 16 and 15 we get $q^1_{s,1} = \min(b_s, q^1_{s,s} + (\alpha_1 - \alpha_s)v_s) = \min(b_s, P_s + (\alpha_1 - \alpha_s)v_s)$. □

Lemma 18. *The outcome of the Generalized Position Auction (direct version) with truthful bidding is envy-free. Furthermore, if $s < l$ and $p^s_{s,s} > P_s$ then one direction of envy-freeness holds with a strong inequality: $\alpha_s v_s - P_s > \alpha_l v_s - P_l$.*

Proof. Consider any two players s, l. We will show that s does not envy l. If $s < l$ then the only non-trivial possibility is $l \leq k$. In this case $(\alpha_s - \alpha_l)v_s + Q^s_l \geq (\alpha_s - \alpha_l)v_s + q^s_{s,l} \geq p^s_{s,s} \geq P_s$, where the second inequality follows by claim 16. This implies $\alpha_s v_s - P_s \geq \alpha_l v_s - Q^s_l \geq \alpha_l v_s - P_l$, and if $p^s_{s,s} > P_s$ then the first inequality is strict. If $s > l$ then, by claim 17, $P_l \geq \min(b_s, (\alpha_l - \alpha_s)v_s + P_s)$. Thus, if $P_l < b_s$ then $P_l \geq (\alpha_l - \alpha_s)v_s + P_s$ which again implies $\alpha_s v_s - P_s \geq \alpha_l v_s - P_l$. □

This finishes the proof of envy-freeness. For the proof of uniqueness in section 4.6 below it will be useful to state one more easy implication of the above:

Claim 19. *Fix any slot s, and suppose that there exists a player $j > s$ such that $P_s = (\alpha_s - \alpha_j)v_j + P_j < b_j$, and $v_j \neq v_s$. Then for any slot $l > s$ we have $\alpha_s v_s - P_s > \alpha_l v_s - P_l$.*

Proof. We show that $p^s_{s,s} > P_s$ which implies the claim by Lemma 18. By the proof of claim 17 we have $p^s_{j,s} = q^s_{j,s} = \min((\alpha_s - \alpha_j)v_j + P_j, b_j) = P_s$. Therefore if $p^s_{s,s} = P_s$ then $p^s_{s,s} = p^s_{j,s}$. Since $p^s_{s,s} \neq b_s$ and $p^s_{j,s} \neq b_j$ we get by corollary 8 that $v_s = v_j$, a contradiction. Thus $p^s_{s,s} > P_s$, and the claim follows. □

4.3 An inductive tool

The proof of truthfulness is significantly more involved than the envy-freeness proof, and we proceed via several steps. We first construct a helpful inductive argument. Truthfulness requires to analyze situations where some player, say l, changes her declaration (while the other declarations are fixed). We therefore fix two declarations of player l, (v, b) and $(\tilde{v}, \tilde{b})$, where $\tilde{v} \geq v$ and $\tilde{b} \geq b$. Suppose player l wins slot l and pays P_l when declaring (v, b), and wins slot $\tilde{l}$ and pays $\tilde{P}_{\tilde{l}}$ when declaring $(\tilde{v}, \tilde{b})$. (l and/or $\tilde{l}$ can take the value $k+1$ to denote that player l loses). We use $\tilde{x}$ to describe the variable x in the execution for $(\tilde{v}, \tilde{b})$, for example $\tilde{B}_l^1, \tilde{q}_{i,l}^1$, and so on.

Definition 20. *A slot s is an "anchor" (for the two declarations (v, b) and $(\tilde{v}, \tilde{b})$) if:*

1. *$s \in \{l, \tilde{l}\}$, and if $s = \tilde{l}$ then $p_{l,s}^1 = \tilde{p}_{l,s}^1$.*
2. *If $l \in B_s^1$ then $p_{i,s}^1 = \tilde{p}_{i,s}^1$ for any player $i \notin B_s^1$ (and hence $\tilde{B}_s^1 = B_s^1$).*
3. *If $l \notin B_s^1$ then $p_{i,s}^1 = \tilde{p}_{i,s}^1$ for any player $i \neq l$, such that $i \notin B_s^1$ and $p_{i,s}^1 \leq p_{l,s}^1$.*

We will specify in the next subsection conditions for such an anchor to exist. Using this anchor slot, we will construct a simple but powerful inductive argument:

Claim 21. *Fix an anchor slot s^*, and suppose that either $b = \tilde{b}$, or $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$. Then there exists a slot $1 \leq j^* \leq s^*$ such that the set of winners of slots $1, ..., j^*$ in both declarations is the same set.*

Proof. We first note that $\tilde{q}_{i,s}^1 \geq q_{i,s}^1$ and $\tilde{p}_{i,s}^1 \geq p_{i,s}^1$ for any player i and any slot s (this follows by a simple induction on the slot $s = k, ..., 1$). We say that a player $i \notin B_{s^*}^1$ "jumped" if $p_{l,s^*}^1 \geq p_{i,s^*}^1$ and there exists a slot $j \leq s^* - 1$ such that $i \in B_j^1$.

Claim 22. *If a player $i \neq l$ with $i \notin B_{s^*}^1$ and $p_{l,s^*}^1 \geq p_{i,s^*}^1$ did not jump then $p_{i,j}^1 = \tilde{p}_{i,j}^1$ and $i \notin \tilde{B}_j^1$ for any slot $j \leq s^*$.*

Proof. Since $i \notin B_{s^*}^1$ and $p_{l,s^*}^1 \geq p_{i,s^*}^1$ but i did not jump we have $i \notin B_j^1$ for any slot $j \leq s^*$. We show the claim by induction on $j = s^*, s^* - 1, ..., 1$. The base case $j = s^*$ follows since slot s^* is an anchor. Assume that $\tilde{p}_{i,j+1}^1 = p_{i,j+1}^1$ and $i \notin \tilde{B}_{j+1}^1$ for some $j < s^*$. Then $\tilde{p}_{i,j} = \min(b_i, (\alpha_j - \alpha_{j+1})v_i + \tilde{p}_{i,j+1}) = \min(b_i, (\alpha_j - \alpha_{j+1})v_i + p_{i,j+1}) = p_{i,j}$, completing the first part of the inductive step. Since $\tilde{p}_{i',j} \geq p_{i',j}$ for any player i', $i \notin B_j^1$ implies $i \notin \tilde{B}_j^1$. □

Claim 23. *If $l \in B_{s^*}^1$ and there does not exist a player that jumped then the set of players that win slots $1, ..., s^*$ is identical in both declarations (v, b) and $(\tilde{v}, \tilde{b})$.*

Proof. Every player $i \notin B_{s^*}^1$ satisfies $p_{l,s^*}^1 \geq p_{i,s^*}^1$, and, since no such player jumped, corollary 12 implies that the players in $B_{s^*}^1$ win slots $1, ..., s^*$ in declaration (v, b). We will show that the players in $\tilde{B}_{s^*}^1$ win slots $1, ..., s^*$ in declaration $(\tilde{v}, \tilde{b})$, which will imply the claim since $B_{s^*}^1 = \tilde{B}_{s^*}^1$. Assume by contradiction that some player $i \notin \tilde{B}^1_{s^*}$ wins slot $s_i \leq s^*$ (w.l.o.g. i has a minimal bid in slot s^* among all such players). By claim 11 it follows that $i \in \tilde{B}^1_{s_i}$. On the other hand since $B^1_{s^*} = \tilde{B}^1_{s^*}$ we have $i \notin B^1_{s^*}$, and $i \neq l$. Thus claim 22 implies $i \notin \tilde{B}^1_{s_i}$, a contradiction. $\square$

Using this, if $l \in B^1_{s^*}$ then we can conclude the proof of claim 21 by choosing $j^* = s^*$.

Claim 24. *If $l \notin B^1_{s^*}$ then there exists a player i that jumped such that $p^1_{i,s^*} < p^1_{l,s^*}$.*

Proof. If $s^* = l$ then by claim 9 there is a player i' that wins slot $s_{i'} < s^*$ and $p^1_{i',s^*} < p^1_{l,s^*}$, and by claim 11 there exists a player i with $p^1_{i,s^*} \leq p^1_{i',s^*}$ that wins slot $s_i < s^*$ and $i \in B^1_{s_i}$ (i may be i'). Therefore i jumped. If $s^* = \tilde{l}$ then by assumption $p^1_{l,s^*} = \tilde{p}^1_{l,s^*}$ and therefore $l \notin \tilde{B}^1_{s^*}$. As above this implies that there exists a player $i \neq l$ that wins slot $s_i < s^*$ such that $\tilde{p}^1_{i,s^*} < \tilde{p}^1_{l,s^*}$ and $i \in \tilde{B}^1_{s_i}$. Player i also satisfies $i \notin B^1_{s^*}$ and $p^1_{l,s^*} > p^1_{i,s^*}$. We argue that i jumped: otherwise claim 22 implies $i \notin \tilde{B}^1_j$ for any slot $j \leq s^*$, a contradiction. $\square$

Therefore we assume that there exists a player that jumps. For two players i, j and a slot s, we denote $p^1_{i,s} \succ p^1_{j,s}$ if $p^1_{i,s} > p^1_{j,s}$, or $p^1_{i,s} = p^1_{j,s}$ and $i \succ j$. Let i^* be a player with minimal bid p^1_{i,s^*} w.r.t. $\succ$ among all players that jumped. Let $j^* \leq s^* - 1$ be some slot such that $i^* \in B^1_{j^*}$.

Claim 25. *For any player $i \neq l$ such that $i \notin B^1_{j^*}$, and for any slot $j \leq j^*$, we have: (1) $i \notin B^1_j$, (2) $i \notin \tilde{B}^1_j$, and (3) $p_{i,j} = \tilde{p}_{i,j}$. In addition, if $l \notin B^1_{j^*}$ then $\tilde{l} > j^*$ and $l > j^*$.*

Proof. Consider a player $i \notin B^1_{j^*}$. Assume first that $p^1_{i^*,s^*} \succ p^1_{i,s^*}$ (note that this implies that $i \neq l$ since $p^1_{i^*,s^*} < p^1_{l,s^*}$). By the minimality assumption on i^* we have that $i \notin B^1_j$ for any slot $j < s^*$. By claim 22 we also have $p_{i,j} = \tilde{p}_{i,j}$ and $i \notin \tilde{B}^1_j$ for any slot $j < s^*$. If $p_{i,s^*} = p_{i^*,s^*}$ and $i \succ i^*$ (note that this still implies $i \neq l$) then since $i \notin B^1_{j^*}$ and $i^* \in B^1_{j^*}$ we must have $p^1_{i,j^*} = b_i$, which implies the three properties.

Otherwise $p_{i,s^*} > p_{i^*,s^*}$. We must have $v_i > v_{i^*}$, otherwise we get by claim 7 that $p_{i^*,s^*} = b_{i^*}$ which is a contradiction since $i^* \notin B^1_{s^*}$ and $i^* \in B^1_{j^*}$. Since $p_{i,j^*} \leq p_{i^*,j^*}$ we get $p^1_{i,j^*} = b_i$, and since $i \notin B^1_{j^*}$ then $i \notin B^1_j$ for any $j < j^*$. In addition, if $i \neq l$ or $i = l$ and $b = \tilde{b}$ then $b_i = \tilde{p}^1_{i,j^*}$, implying $i \notin \tilde{B}^1_j$ and $p_{i,j} = \tilde{p}_{i,j}$ for any $j \leq j^*$.

This establishes the three properties for $i \neq l$, and that, if $b = \tilde{b}$ and $l \notin B^1_{j^*}$ then player l does not win any slot $j \leq j^*$. If $l \notin B^1_{j^*}$ and $b < \tilde{b}$ then we get $p^1_{l,j^*} = b$ from the above paragraph. Thus player l cannot win any slot $j \leq j^*$ in declaration (v, b), hence $l > j^*$. It remains to show $\tilde{l} > j^*$. Since $b < \tilde{b}$ we have by assumption $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$. If $b > P_{\tilde{l}}$ then since $P_j \geq p^1_{l,j} = b$ for any $j \leq j^*$ we get $\tilde{l} > j^*$. Similarly, if $b > \tilde{P}_{\tilde{l}}$, then since $\tilde{P}_j \geq \tilde{p}^1_{l,j} \geq p^1_{l,j} = b$ for any $j \leq j^*$ we again get $\tilde{l} > j^*$. $\square$

By claim 25, the conditions of corollary 12 hold for slot j^* and declaration (v, b) (note that by claim 25 player l wins slot $l > j^*$ in declaration (v, b)). Therefore the players in $B^1_{j^*}$ win slots

$1, ..., j^*$ in this declaration. To finish the proof of claim 21 we argue that these players are the winners of slots $1, ..., j^*$ in declaration $(\tilde{v}, \tilde{b})$ as well.

Assume by contradiction that there exists a player $x \notin B^1_{j^*}$ that wins a slot $s_x \leq j^*$ in declaration $(\tilde{v}, \tilde{b})$. By claim 25 we have $x \neq l$ since $\tilde{l} > j^*$. Assume without loss of generality that x has a minimal bid $\tilde{p}^1_{x,j^*}$ among all players $x \notin B^1_{j^*}$ that win some slot $s \leq j^*$ in declaration $(\tilde{v}, \tilde{b})$. By claim 25, $p^1_{x,j^*} = \tilde{p}^1_{x,j^*}$. Since $p^1_{i,j^*} \leq \tilde{p}^1_{i,j^*}$ for any player i it follows that $x \notin \tilde{B}^1_{j^*}$ as well. By claim 25 we have $x \notin \tilde{B}^1_{s_x}$, and therefore by claim 11 there must exist $y \notin \tilde{B}^1_{j^*}$ such that $\tilde{p}^1_{y,j^*} < \tilde{p}^1_{x,j^*}$ and y wins some slot $s_y \leq j^*$. By the minimality assumption on the choice of x we must have $y \in B^1_{j^*}$. Therefore $p^1_{y,j^*} \geq p^1_{x,j^*}$. But we also have $p^1_{y,j^*} \leq \tilde{p}^1_{y,j^*} < \tilde{p}^1_{x,j^*} = p^1_{x,j^*}$, a contradiction. □

4.4 The effect of declaration changes on slot prices

We first show that slot l is a suitable anchor:

Claim 26. *Slot l is an anchor.*

Proof. We prove by induction on the slot $s = k...l$. By definition $p^1_{i,k} = \tilde{p}^1_{i,k}$ for any player $i \neq l$. Assume correctness for slot $s+1$ and let us prove for s. If $l \in B^1_{s+1}$ then $p^1_{i,s} = \min(b_i, (\alpha_s - \alpha_{s+1})v_i + \min(Q^1_{s+1}, p^1_{i,s+1})) = \tilde{p}^1_{i,s}$ for every player $i \neq l$, since by the induction assumption $Q^1_{s+1} = \tilde{Q}^1_{s+1}$ and $p^1_{i,s+1} = \tilde{p}^1_{i,s+1}$. Otherwise assume $l \notin B^1_{s+1}$. For every player i with $p^1_{i,s+1} \leq p^1_{l,s+1}$ we again get by definition $p^1_{i,s} = \tilde{p}^1_{i,s}$.

Otherwise $p^1_{i,s+1} > p^1_{l,s+1}$. Since $l \notin B^1_{s+1}$ and l wins slot $l < s+1$ we have $p^1_{l,s+1} \neq b_l$, which implies by claim 7 that $v_i > v_l$. Therefore for any i with $p^1_{i,s} \leq p^1_{l,s}$ we have $p^1_{i,s} = b_i \geq \tilde{p}^1_{i,s} \geq p^1_{i,s}$. Hence $\tilde{p}^1_{i,s} = p^1_{i,s} = b_i$, implying $\tilde{p}^1_{i,s} = p^1_{i,s} = b_i$. If $l \in B^1_s$ then all players $i \notin B^1_s$ have $p^1_{i,s} \leq p^1_{l,s}$, and the claim follows. □

This inductive tool enables a very simple analysis of the effect of a declaration change from (v, b) to $(\tilde{v}, \tilde{b})$, as detailed above:

Claim 27. *If $b = \tilde{b}$ or $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$ then:*

1. *$\tilde{l} \leq l$.*

2. *The winner of every slot $s > l$ is the same player in both declarations, and the losing players are the same.*

3. *For any slot $s \geq l$, $\tilde{P}_s = P_s$.*

Proof. We prove by induction on the number of slots k. If $k = 1$ then the claim is immediate from the definition of the mechanism. We assume correctness for $k' < k$ slots and prove for k slots. By lemma 21 there exists a slot $j^* \leq l$ such that the winners of slots $1, ..., j^*$ are the same in both declarations. If $j^* < l$ then at iteration $j^* + 1$ in both declarations we are left with the same set of players, and a mechanism for $k - j^* < k$ slots, and the induction assumption implies the claim. If $j^* = l$ then clearly the first property holds since player l wins a slot $1, ..., j^*$ in both declarations. In addition the set of players at iteration $j^* + 1$ is the same for both declarations, hence each slot $j > j^*$ has the same winner in both declarations, which implies by claim 17 that $\tilde{P}_s = P_s$ for any slot $s \geq l$, as claimed. □

The third property of this claim is especially misleading, as it might appear true even without the requirement that $b = \tilde{b}$ or $b > \tilde{P}_{\tilde{l}}$. It is interesting to see a counter example to this:

Example 1. *Consider a setting of two slots with $\alpha_1 = 1000$ and $\alpha_2 = 1$, and three players with types $\theta_1 = (1, 1000)$, $\theta_2 = (10, 10)$ and $\theta_3 = (11, 11)$ (recall that the first number is the value and the second number is the budget). Suppose player 3 changes her type to $\tilde{\theta}_3 = (11, 1001)$. Then the price of slot 2 strictly decreases.*

We continue to prove few more useful properties.

Claim 28. *If $b = \tilde{b}$ or $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$ then $\tilde{P}_{\tilde{l}} \geq P_{\tilde{l}}$.*

Proof. If $\tilde{l} = l$ then the claim is immediate from the above. Otherwise some other player l_1 wins slot l in the declaration $(\tilde{v}, \tilde{b})$, and suppose l_1 won slot s_1 in declaration (v, b). Note that $s_1 < l$ since by claim 27 the winners of slots $l + 1, ..., k$ plus all losers are the same in both declarations. Let l_2 be the player that wins slot s_1 in declaration $(\tilde{v}, \tilde{b})$, and suppose l_2 won slot s_2 in declaration (v, b). We again have $s_2 < l$. When this terminates we must reach a player l_r that won slot $s_r = \tilde{l}$ in declaration (v, b). Denote $s_0 = l$. We argue by induction on $i = 0, ..., r$ that $\tilde{P}_{s_i} \geq P_{s_i}$. The base case of $i = 0$ follows from claim 27. We now assume by induction that $\tilde{P}_{s_i} \geq P_{s_i}$ and prove that $\tilde{P}_{s_{i+1}} \geq P_{s_{i+1}}$. Note that $b_{l_{i+1}} > \tilde{P}_{s_i} \geq P_{s_i}$. If $\tilde{P}_{s_{i+1}} \geq b_{l_{i+1}} > P_{s_{i+1}}$ then we immediately get the inductive claim. Otherwise assume $\tilde{P}_{s_{i+1}} < b_{l_{i+1}}$. Player l_{i+1} wins slot s_{i+1} in (v, b), hence $\alpha_{s_{i+1}} v_{l_{i+1}} - P_{s_{i+1}} \geq \alpha_{s_i} v_{l_{i+1}} - P_{s_i}$. On the other hand player l_{i+1} wins slot s_i in $(\tilde{v}, \tilde{b})$, hence $\alpha_{s_i} v_{l_{i+1}} - \tilde{P}_{s_i} \geq \alpha_{s_{i+1}} v_{l_{i+1}} - \tilde{P}_{s_{i+1}}$. Since $\tilde{P}_{s_i} \geq P_{s_i}$ it follows that $\tilde{P}_{s_{i+1}} \geq P_{s_{i+1}}$, as claimed. □

Claim 29. *If $v = \tilde{v}$ and $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$ then slot $\tilde{l}$ is an anchor.*

Proof. By claim 28 we have $\tilde{P}_{\tilde{l}} > P_{\tilde{l}}$. Since $\tilde{b} > b > P_{\tilde{l}} > Q^1_{\tilde{l}}$ and $v = \tilde{v}$ we have $\tilde{p}^1_{i,s} = p^1_{i,s}$ for any player i and any slot $s \geq \tilde{l}$. □

Claim 30. *If $v = \tilde{v}$ and $b > \tilde{P}_{\tilde{l}}$ then $\tilde{l} = l$.*

Proof. We first note that by claim 27 we have $\tilde{l} \leq l$. We prove by induction on the number of slots k. If $k = 1$ then the claim is immediate from the definition of the mechanism. We assume correctness for $k' < k$ slots and prove for k slots. By lemma 21, using slot $\tilde{l}$ as an anchor, there exists a slot $j^* \leq \tilde{l}$ such that the winners of slots $1, ..., j^*$ are the same in both declarations. If $j^* < \tilde{l}$ then at iteration $j^* + 1$ in both declarations we are left with the same set of players, and a mechanism for $k - j^* < k$ slots, and the induction assumption implies the claim. If $j^* = \tilde{l} \leq l$ then since player l wins one of the slots $1, ..., j^*$ in both declarations it must follow that $\tilde{l} = l$. □

Claim 31. *If $v = \tilde{v}$ and $\tilde{b} > b$ then $\tilde{l} \leq l$.*

Proof. Suppose by contradiction that $\tilde{l} > l$. Then we have $b > P_l \geq P_{\tilde{l}}$, where the second inequality follows from envy-freeness (claim 17). But then according to claim 27 we get $\tilde{l} \leq l$, a contradiction. □

4.5 Truthfulness in equilibrium

We now have enough tools to show that truthfulness is an ex-post equilibrium of our auction. Throughout, we fix the true type of player i to be (v_i, b_i), and denote by $u_i(v, b)$ player i's utility when declaring some type (v, b) (the declaration of all other players is fixed throughout). We need to show that $u_i(v_i, b_i) \geq u_i(v, b)$, for any other type (v, b).

Claim 32. *For any $b > b_i$ and any v, $u_i(v, b) \leq u_i(v, b_i)$.*

Proof. Suppose player i wins slot s and pays P_s when declaring (v, b). If $b_i \leq P_s$ then i has a resulting non-positive utility, while, by declaring (v, b_i) i guarantees a non-negative utility. If $b_i > P_s$ then by claim 30, when declaring (v, b_i) player i still wins slot s and still pays P_s. Therefore in any case $u_i(v, b) \leq u_i(v, b_i)$, and the claim follows. □

Claim 33. *For any $b \leq b_i$ and any v, $u_i(v, b) \leq u_i(v_i, b)$.*

Proof. Suppose player i wins slot s and pays P_s when declaring (v_i, b) and wins slot $\tilde{s}$ and pays $\tilde{P}_{\tilde{s}}$ when declaring (v, b) (s and/or $\tilde{s}$ can take the value $k+1$ to denote that i loses). Since $b \leq b_i$, i's payment is at most her budget, and so she has a non-negative utility from both declarations. By envy-freeness, $\alpha_s v_i - P_s \geq \alpha_{\tilde{s}} v_i - P_{\tilde{s}}$, where $P_{\tilde{s}}$ denotes the price of slot $\tilde{s}$ when player i declares (v_i, b). If $v > v_i$ then $\tilde{P}_{\tilde{s}} \geq P_{\tilde{s}}$ by claim 28. If $v < v_i$ then $\tilde{P}_{\tilde{s}} = P_{\tilde{s}}$ by claim 27. In any case, we have $\alpha_{\tilde{s}} v_i - P_{\tilde{s}} \geq \alpha_{\tilde{s}} v_i - \tilde{P}_{\tilde{s}}$. We get $u(v_i, b) = \alpha_s v_i - P_s \geq \alpha_{\tilde{s}} v_i - \tilde{P}_{\tilde{s}} = u_i(v, b)$, as claimed. □

Claim 34. *For any $b \leq b_i$, $u_i(v_i, b) \leq u_i(v_i, b_i)$.*

Proof. Let $f(v, b)$ denote the slot assigned to player i when declaring (v, b), and $P(v, b)$ be i's payment when declaring (v, b). Define $g(v, b) = \alpha_{f(v,b)} \cdot v - P(v, b)$, i.e. this is i's utility if she declares (v, b) and if her true value is indeed v. We will argue that $g(v, b) = \int_0^v f(x, b)dx$. For $v' > v$ we have by claim 27 that $f(v', b) \leq f(v, b)$, and that, if $f(v', b) = f(v, b)$ then $P(v', b) = P(v, b)$. Let $v_1^*, ..., v_L^*$ be the discontinuity points of $f(\cdot, b)$ (i.e. when b is fixed and v increases from 0 to ∞). In other words, for any index $1 \leq l \leq L-1$ and any $v_l^* < x_1 < x_2 < v_{l+1}^*$ we have $f(x_1, b) = f(x_2, b)$ and $P(x_1, b) = P(x_2, b)$. Therefore $\frac{\partial g(v,b)}{\partial v}|_{v=x_1} = \frac{\partial g(v,b)}{\partial v}|_{v=x_2} = \alpha_{f(x_1,b)}$. Since there is a finite number $L \leq k$ of such discontinuity points we get $g(v, b) = \int_0^v f(x, b)dx$. By claim 31 we have

$f(x,b) \leq f(x,b')$ for any $b \leq b'$, implying using the above that $g(v,b) \leq g(v,b')$. Since $b \leq b_i$ we get $u_i(v_i,b) = g(v_i,b) \leq g(v_i,b_i) = u_i(v_i,b_i)$, and the claim follows. □

Lemma 35. *Truthfulness is an ex-post equilibrium of the Generalized Position Auction.*

Proof. We need to show that any false declaration (v,b) yields weakly smaller utility than the true declaration (v_i,b_i). If $b > b_i$ we have $u_i(v,b) \leq u_i(v,b_i) \leq u_i(v_i,b_i)$, where the first inequality follows from claim 32 and the second inequality follows from claim 33. If $b \leq b_i$ we have $u_i(v,b) \leq u_i(v_i,b) \leq u_i(v_i,b_i)$, where the first inequality follows from claim 33 and the second inequality follows from claim 34. □

4.6 Uniqueness

We finish the analysis by showing that the Generalized Position Auction is the unique mechanism that satisfies all the desirable properties discussed at the beginning. We need one additional natural requirement:

Definition 36 (No Positive Transfers (NPT))**.** *A mechanism has the "No Positive Transfers" (NPT) property if no player receives a positive payment from the mechanism.*

This property is necessary for the uniqueness result. Consider for example a setting of one item and two players, with $b_1 = 1, b_2 = 2$, and $v_1 = 5, v_2 = 3$. The Generalized Position Auction sells the item to player 2 for a price of 1. A different mechanism that violates NPT is: first pay each player a subsidy of 4 dollars (this increases the bidders' budgets). Then run our mechanism using the updated budgets. It is not hard to verify that this is truthful, individually rational, and envy-free. However the result will now be different: player 1 will receive the item and will pay 3 dollars. It is interesting to note that the usual quasi-linear setting does not exhibit such a phenomena, and it is well-known that one can normalize the payment of a losing player to be 0 without affecting the outcomes of the mechanism being considered. As this simple example shows, when budgets limits are a real constraint this is not quite the case.

Together with ex-post IR, NPT implies that the payment of a losing player is exactly zero. This is in fact the only use of the NPT property, and one can replace the NPT requirement with a "zero payment for losers" requirement. This seems like a natural and common property.

A second issue that requires some attention is ruling out ties. Clearly, if the Generalized Position Auction encounters a tie during its execution, it can be decided in several ways, affecting the outcome. Thus, the uniqueness result can only hold when there are no ties, i.e. when all types are distinct w.r.t. both the value and the budget.

Let M denote the Generalized Position Auction, and fix any other truthful mechanism M' that satisfies NPT, envy freeness, pareto optimality, and ex-post individual rationality.

Lemma 37. *For any tuple of types $(\vec{v}, \vec{b})$ such that $v_i \neq v_j$ and $b_i \neq b_j$, M and M' output the same slot assignment and the same payments. Moreover, this holds even if the values of the players are fixed and are publically known, and only the budgets are private information.*[8]

Proof. Fix any tuple of types $(\vec{v}, \vec{b})$ in T^*. Define $w(s), w'(s)$ as the winners of slot s in mechanisms M, M', respectively, and let P_l, P'_l be the payment of the winner of slot l in mechansims M, M', respectively. We start with two claim and then prove the lemma by induction.

Claim 38. *$P'_s \geq P_s$ for any slot $1 \leq s \leq k$.*

Proof. Let A contain all slots $1 \leq s \leq k$ such that $P_s > P'_s$, and suppose by contradiction that A is not empty. For any $s \in A$, let l be the slot that $i = w(s)$ wins in M' (i.e. $w(s) = w'(l) = i$). We claim that $l \in A$: if $P'_l \geq P_l$ then we get $\alpha_s v_i - P_s \geq \alpha_l v_i - P_l \geq \alpha_l v_i - P'_l \geq \alpha_s v_i - P'_s > \alpha_s v_i - P_s$, where the first inequality follows from envy-freeness of M since $P_l \leq P'_l < b_i$, and the third inequality follows from envy-freeness of M' since $P'_s < P_s < b_i$, and we get a contradiction. Thus, a player wins a slot in A in M if and only if she wins wins a slot in A in M'. We will show that there exists at least one player that does not receive a slot in A in M but must win a slot in A in M', and will thus get a contradiction.

Let $s^* = \max(s \in A)$. By claim 17 let $i = w(l)$ for $l > s^*$ be a player such that $P_{s^*} = \min(b_i, P_l + (\alpha_s - \alpha_l)v_i)$ (we may choose $l = k+1$ to denote the fact that i loses in M). We have $P'_{s^*} < P_{s^*} \leq b_i$, and $\alpha_l v_i - P_l < \alpha_{s^*} v_i - P'_{s^*}$. Note that i wins a slot $l \notin A$ in M (since $l > s^*$). We will show that i must win a slot in A in M', which will be a contradiction. For any slot $j \notin A$ (including $j = k+1$ to consider the possibility that i loses in M'), either $P'_j \geq b_i$, or $P_j \leq P'_j < b_i$, in which case $\alpha_j v_i - P'_j \leq \alpha_j v_i - P_j \leq \alpha_l v_i - P_l < \alpha_{s^*} v_i - P'_{s^*}$, where the second inequality follows by the envy-freeness of M since $P_j < b_i$. Since $P'_{s^*} < b_i$ and M' is envy-free it follows that i cannot win slot j in M'. Thus player i must win some slot in A, a contradiction. □

Claim 39. *Define the set B to contain all slots $1 \leq l \leq k$ such that $P_l = P'_l$. Then the set of players that win a slot in B is identical in both M and M', i.e. $\{ w(s) \mid s \in B \} = \{ w'(s) \mid s \in B \}$.*

Proof. Assume by contradiction that there exists a player i that wins a slot $s \in B$ in M, and a slot $l \notin B$ in M' (as before we can have $s = k+1$). Note that by claim 38 and since $l \notin B$ we have $P'_l > P_l$. We get $\alpha_s v_i - P'_s = \alpha_s v_i - P_s \geq \alpha_l v_i - P_l > \alpha_l v_i - P'_l$, where the first inequality follows by envy-freeness of M, since $P_l < P'_l < b_i$. Since $P'_s = P_s < b_i$ this contradicts the envy-freeness of M'. □

Claim 40. *Let B be as defined in claim 39. Then for any $s \in B$ we have $w(s) = w'(s)$.*

[8] Alternatively, it can be stated that the budgets are public knowledge and the values are private information. For simplicity we restrict attention to just one version.

Proof. Fix a slot $s \in B$. We assume that for any $l \in B$ with $l < s$ we have $w(l) = w'(l)$ and prove $w(s) = w'(s)$, which implies the claim by induction. Let $i = w(s)$. Suppose by contradiction that $w'(s) = j \neq i$. Suppose player j wins slot s_j in M. By claim 39 we have $s_j \in B$ and by assumption we have $s_j > s$. By claim 17 we have $P_s \geq \min(b_j, (\alpha_s - \alpha_{s_j})v_j + P_{s_j})$. Since $P'_s = P_s$ and $P'_{s_j} = P_{s_j}$, envy-freeness of M' implies $P_s = (\alpha_s - \alpha_{s_j})v_j + P_{s_j} < b_j$. Claim 19 then implies that for any slot $l > s$ we have $\alpha_s v_i - P_s > \alpha_l v_i - P_l$. Now suppose player i wins slot s_i in M. By claim 39 we have $s_i \in B$ and by assumption we have $s_i > s$. Since $P'_s = P_s$ and $P'_{s_i} = P_{s_i}$ we get $\alpha_s v_i - P'_s > \alpha_{s_i} v_i - P_{s_i}$. Since $P'_s = P_s < b_i$ we get a contradiction to the envy-freeness of M'. Thus $w(s) = w'(s)$ and the claim follows. □

We now prove by induction on $l = k, ..., 0$ that, for all type declarations: (1) the set of players that win slots $1, ..., l$ is the same in both mechanisms (they do not necessarily win the same slots), and (2) for any slot $k \geq s > l$, the same player wins slot s in both mechanisms, and $P'_s = P_s$. The lemma will then follow by taking $l = 0$.

To prove the base case of $l = k$ we need to argue that the same set of players lose in both mechanisms: for any slot $s \leq k$, if $s \in B$ (as defined in claim 39 above) then a losing player i in M cannot win s in M' by claim 40. If $s \notin B$ then by claim 38 we have $P'_s > P_s \geq \min(b_i, \alpha_s v_i)$ and since M' is ex-post IR it follows that $w'(s) \neq i$. Hence i must lose in M' as well.

We now assume correctness for some index $l \leq k$ and prove the inductive claim for $l-1$. All we need to show is that $w(l) = w'(l)$, and $P_l = P'_l$. Let $i = w(l)$ be the winner of slot l in mechanism M, and suppose that $i = w'(l')$. Note that $l' \leq l$ by the induction assumption. We first prove that $P_{l'} = P'_{l'}$. By claim 38 we have $P_{l'} \leq P'_{l'}$, and assume by contradiction that the inequality is strict. Since $b_i > P'_{l'}$ we have by envy-freeness that $\alpha_l v_i - P_l \geq \alpha_{l'} v_i - P_{l'}$. Since $P_{l'} < P'_{l'}$ we can pick a small enough $\epsilon > 0$ such that $\alpha_l v_i - (P_l + \epsilon) > \alpha_{l'} v_i - P'_{l'}$. Now if player i declares a different type $(v_i, P_l + \epsilon)$ (i.e. the same value and a budget just above her price in M) then by claim 30 we have that player i wins slot l in M in the new type declaration as well. By the induction assumption player i wins some slot $l'' \leq l$ in M' in the new declaration, and her new payment P'' is at most her new budget $P_l + \epsilon$. We get $\alpha_{l''} v_i - P'' \geq \alpha_l v_i - (P_l + \epsilon) > \alpha_{l'} v_i - P'_{l'}$. Thus player i strictly increased her utility be misreporting her type, contradicting the truthfulness of M'. Thus $P_{l'} = P'_{l'}$. Therefore $l' \in B$, and by claim 40 we have $w'(l') = w(l')$. Since $w(l) = w'(l')$ by assumption we get $l' = l$, and the claim follows. □

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